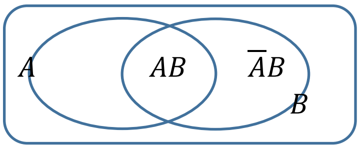
Solutions exercises chapter 3



1. **a.** $P(A|B) = \frac{P(AB)}{P(B)} = \frac{0.30}{0.75} = \frac{2}{5}$ (= 40%), since

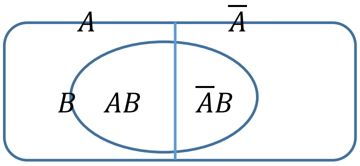
$P(A \cup B) = P(A) + P(\bar{A}B)$,

so $P(\bar{A}B) = P(A \cup B) - P(A) = 0.8 - 0.35 = 0.45$

And $P(B) = P(AB) + P(\bar{A}B) = 0.3 + 0.45 = 0.75$

b. Use the product rule of independent events (interchange the order of the events)

$$P(ABC) = P(CAB) = P(C)P(A|C)P(B|AC) = \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{10}$$



2.

a. Given probabilities (the events are readily defined):

$P(A) = 0.98$, $P(B\,|\,A) = 0.97$ en $P(B\,|\,\bar{A}) = 0.05$

It immediately follows that: $P(\bar{B}\,|\,A) = 0.03$, $P(\bar{B}\,|\,\bar{A}) = 0.95$ and $P(\bar{A}) = 1 - P(A) = 0.02$

Requested is $P(B)$, which we can compute with the law of total probability:

$P(B) = P(B\,|\,A)P(A) + P(B\,|\,\bar{A})P(\bar{A})$.

Substitute the known probabilities: $P(B) = 0.97 \cdot 0.98 + 0.05 \cdot 0.02 = 0.9516$

b. Requested: $P(\bar{A}\,|\,\bar{B})$.

We will use: $P(\bar{A}\,|\,\bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$, $P(\bar{A} \cap \bar{B}) = P(\bar{B}\,|\,\bar{A})P(\bar{A})$, $P(\bar{B}) = 1 - P(B)$

$P(\bar{B}) = 1 - P(B) = 1 - 0.9516 = 0.0484$,

$P(\bar{A} \cap \bar{B}) = P(\bar{B}\,|\,\bar{A})P(\bar{A}) = 0.95 * 0.02 = 0.019$.

$$P(\bar{A}\,|\,\bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \frac{0.95 * 0.02}{0.0484} = 0.393$$

c. If A and B are independent, then $P(A \cap B) = P(A)P(B)$ or $P(B|A) = P(B)$ should be true.
We computed $P((B) = 0.9516$, which is not the same as $P(B|A) = 0.97$.
So A and B are **not independent**: A and B are dependent.

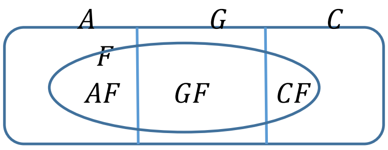
3. Define the events $U =$ "Driver is Under influence" and $B =$ "Blood test is positive"
Then the described probabilities are: $P(B|U) = 0.75$, $P(B|\bar{U}) = 0.02$ en $P(U) = 0.05$
Requested is $P(U|B)$

$$\text{Solution: } P(U|B) = \frac{P(UB)}{P(B)} = \frac{P(B|U)P(U)}{P(B|U)P(U) + P(B|\bar{U})P(\bar{U})}$$

$$= \frac{0.75 \cdot 0.05}{0.75 \cdot 0.05 + 0.02 \cdot 0.95} \approx 0.6673$$

4. 1. We have a partition of the Sim-cards of mobile phones into the events
$A =$ "Africel"" , $G =$ "Gamcel" and $C =$ "Comium".
Let $F =$ "Fixed subscription" (the complement of F is "prepaid").

2. Then we know: $P(F|A) = 0.1$, $P(F|G) = 0.2$ and $P(F|C) = 0.3$
Furthermore $P(A) = 2 \cdot P(G)$.
And finally: $P(V) = 0.15$



3. Requested is $P(C)$.
Solution: Set $P(G) = x$, then $P(A) = 2x$ and $P(C) = 1 - 3x$

Now apply the law of total probability:

$$P(F) = P(AF) \quad + \quad P(GF) \qquad + \; P(CF)$$
$$= P(F|A)P(A) + P(F|G)P(G) + P(F|C)P(C)$$
So: $$0.15 \; = \; 0.1 \times 2x \; + \; 0.2 \times x \; + 0.3 \times (1-3x)$$
Solving the equation for x: $x = P(G) = 0.3$. So $P(C) = 1 - 3x = 0.1$

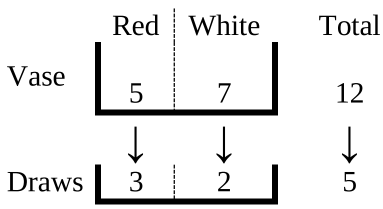
5. 1. Define the events $L_1 =$ “The drawer with two golden coins is chosen" ,
$L_2 =$ “The drawer with two silver coins is chosen”.
$L_3 =$ “The drawer with one golden and one silver coins is chosen”, and
$A =$ “The first draw results in a golden coin” .

2. Implicitly the following probabilities are given: $P(L_1) = P(L_2) = P(L_3) = \frac{1}{3}$ and
$P(A|L_1) = 1, P(A|L_2) = 0$ and $P(A|L_3) = \frac{1}{2}$

3. Requested is the probability: $P(L_1|A)$, since if the first and the second are golden then drawer 1 is opened.
$P(L_1|A) = \frac{P(AL_1)}{P(A)}$ (definition of conditional probability).
$P(AL_1)$ is computed with the product rule: $P(AL_1) = P(L_1)P(A|L_1) = \frac{1}{3} \cdot 1$
And en $P(A)$ is computed with the law of total probability:
$$P(A) = P(A|L_1)P(L_1) + PA|L_2)P(L_2) + P(A|L_3)P(L_3) = \frac{1}{3} \cdot 1 + 0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2}$$
So $P(L_1|A) = \frac{P(AL_1)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}$

(*We can explain this result intuitively: each of the 6 coins in the drawers has the same probability* $\frac{1}{6}$ *to be chosen first. If the first one is golden there are 3 equally likely possibilities: two in drawer 1 and one in drawer 3 (one golden en one silver). If it is one of the 2 golden coins in drawer 1, the other one is golden as well. Only in case of the golden coin of drawer 3 the second coin is solver.*
So the probability of a second golden one (given that the first golden is) is $\frac{2}{3}$.)

6. **a.** See the sketch of the 5 draws from 5 red and 7 white marbles without replacement.
We can define the event “3 red marbles in 5 draws”$= R_3$ and “face up number 5” $= F_5$:



$$P(R_3|F_5) = \frac{\binom{5}{3} \cdot \binom{7}{2}}{\binom{12}{5}} \approx 0.2652$$

b. If we define $F_i =$ “Face up number $= i$” and $R_3 =$ “3 Red” (like in a.), we know that $P(F_i) = \frac{1}{6}$, $i = 1, .., 6$ and the conditional probabilities $P(R_3|F_i)$ can be determined similarly as $P(R_3|F_5)$ in a. and $P(R_3)$ with the law of total probability:
$$P(R_3) = P(R_3F_3) + P(R_3F_4) + P(R_3F_5) + P(R_3F_6)$$
$$= P(R_3|F_3)P(F_3) + P(R_3|F_4)P(F_4) + P(R_3|F_5)P(F_5) + P(R_3|F_6)P(F_5)$$
$$= \sum_{i=3}^{6} \frac{\binom{5}{3} \cdot \binom{7}{i-3}}{\binom{12}{i}} \cdot \frac{1}{6} \approx 0.1385$$

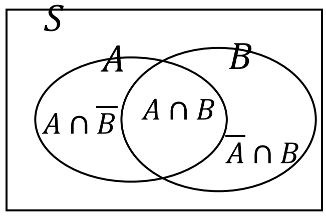
7.

1. We define the events $K =$ “student knows the answer” (and we assume the answer is correct in that case) and $C =$ “correct answer”.
2. Given probabilities: $P(K) = \frac{3}{5}$, $P(C|K) = 1$ and $P(C|\overline{K}) = \frac{1}{2}$
3. Requested: $P(K|C)$

Bayes` Rule: $P(K|C) = \frac{P(KC)}{P(C)} = \frac{P(C|K)P(K)}{P(C|K)P(K) + P(C|\overline{K})P(\overline{K})} = \frac{1 \cdot \frac{3}{5}}{1 \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{2}{5}} = \frac{3}{4}$

8. **a**. We know that A and B are independent, so $P(AB) = P(A) \cdot P(B)$.

We have to prove that $P(A\overline{B}) = P(A) \cdot P(\overline{B})$ (A and $\overline{B}$ are independent as well)..



The link between AB and $A\overline{B}$ is visible in the Venn diagram:

$A = AB \cup A\overline{B}$, where AB and $A\overline{B}$ are mutually exclusive.

$P(A) = P(AB \cup A\overline{B}) = P(AB) + P(A\overline{B})$

So $P(A\overline{B}) = P(A) - P(AB) = P(A) - P(A)P(B) = P(A)[1 - P(B)] = P(A) \cdot P(\overline{B})$

Conclusion: A and $\overline{B}$ are independent as well, from which it follows that $\overline{A}$ and $\overline{B}$ are independent.

b. A, B and C are independent if each pair is independent **and** $P(ABC) = P(A)P(B)P(C)$

A and BC are independent as well, because:

$P(A \cap BC) = P(ABC) = P(A)P(B)P(C) = P(A) \cdot P(BC)$.

9. If A and B are independent, then $P(AB) = P(A) \cdot P(B)$

If A and B are mutually exclusive, then $P(AB) = 0$.

So mutually exclusive events can only be independent if

$P(AB) = P(A) \cdot P(B) = 0$, so if $P(A) = 0$ and/or if $P(B) = 0$.

In general mutually exclusive events will be dependent (which is intuitively clear: if A occurs, B cannot occur, and reverse. Information about the occurrence of A is informative for the occurrence of B).

10. This is the probability of 6 (independent) trials in a row without a 6: $\left(\frac{5}{6}\right)^6 \approx 33.49\%$

11. $X =$ "The number of correct predictions for 12 matches": 12 Bernoulli trials with success probability $p = \frac{1}{3}$. Applying the binomial formula: $\sum_{k=10}^{12} P(X = k) = \sum_{k=10}^{12} \binom{12}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{12-k} \approx 0.054\%$

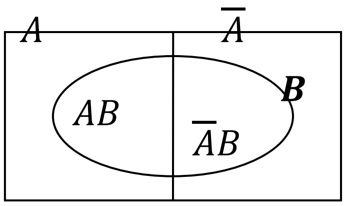
12. **a.** Requested: $P(B)$, where B is the event that the ELISA test is positive.

We define furthermore A as the event that a person is really having Antibodies (HIV)

Given probabilities: $P(B|A) = 0.997$ and $P(\overline{B}|A) = 0.003$,

$P(B|\overline{A}) = 0.015$ and $P(\overline{B}|\overline{A}) = 0.985$,

$P(A) = 0.01$ and $P(\overline{A}) = 0.99$



Law of total probability: $P(B) = P(B|A)P(A) + P(B|\overline{A})P(\overline{A})$

Substitution: $P(B) = 0.997 \times 0.01 + 0.015 \times 0.99 = 0.02482$

b. Requested: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{0.997 \times 0.01}{0.02482} \approx 40.2\%$

So: only 40% of the persons with a positve test result is really infected, the others are "false positives".

(*In b. we implicitly applied Bayes` Rule: the denominator P(B) was computed in a.: without the result in a. we would compute:* $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\overline{A})P(\overline{A})}$)