Solutions exercises chapter 2

1. **a.** 7! = 5040 (permutation rule)

b. the number of permutations (variations) of

7 out of 10: $10 \cdot 9 \cdot 8 \cdot \ldots \cdot 5 \cdot 4 = \frac{10!}{3!} = 604800$

c. In this case we have combinations of 7 out of 10: $\binom{10}{7} = \frac{10!}{7!3!} = 120.$

Note that $\binom{10}{7} = \binom{10}{3}$.

(after every choice of 3 out of 10 causes one remainder of 7 out of 10)

d. $\text{P(at least 2 face cards)} = 1 - \text{P(at most 1 face card)}$ (*complement rule*)

$= 1 - [\text{P(no face cards)} + \text{P(one face card)}]$

$= 1 - \left[\frac{\binom{36}{13}}{\binom{52}{13}} + \frac{\binom{16}{1}\cdot\binom{36}{12}}{\binom{52}{13}}\right] \approx 1 - 0.0352 = 96.48\%$

Use the approach described in section 2.2. (sketch a vase and a sample drawn from it!)

e. $\binom{30}{6} \cdot \binom{24}{7} \cdot \binom{17}{8} \cdot \binom{9}{9} = \frac{30!}{6!7!8!9!}$

First choose a group of 6 out of 30 persons: $\binom{30}{6}$ possible combinations of 6 out of 30; continue by choosing 7 from the remaining 24 to fill the second group: $\binom{24}{7}$ possibilities, etc.

The right-hand formula is found by expanding the multiplied binomial coefficients, or by direct reasoning: if the 30 names are put in an arbitrary order and the first 6 are group 1, the following 7 group 2, etc., then we can determine the number of distributions as the division of 30! and the number of orders of group 1 (6!), of group 2 (7!) , etc., so in total 6! · 7! · 8! · 9!.

2. $\frac{169}{1000}$: There are169 numbers with lowest digit 2, which can be verified in several ways:

- Distinguish the number of 2`s in the 3 digit number:: $3 \cdot 7^2 + 3 \cdot 7 + 1 = 169$.
- Consider the number of 3 digits numbers with all digits **at least 2** and cancel all numbers with digits at least 3: $8^3 - 7^3 = 169$.

3. $\frac{3\cdot9\cdot8\cdot7}{10\cdot9\cdot8\cdot7} = \frac{3}{10}$

The reasoning is as follows:

Suppose the balls are numbered: the red balls are 1, 2, 3 and the white are 4, 5, …, 10.).

Drawing 4 times at random without replacement leads to a symmetric sample space of $10 \cdot 9 \cdot 8 \cdot 7$ ordered foursomes (property 2.1.11b). The number of these foursomes with a red ball at the fourth position can be counted as follows: there are 3 possibilities to first (!) put a red ball in position 4. Consequently there are 9 possibilities to choose another ball in position 3, 8 for position 2 and 7 for position 1.

The product rule allows us to multiply and find $3 \cdot (9 \cdot 8 \cdot 7)$ such ordered permutations of 4 out of 10 with a red one on position 4.

In **conclusion:** the probability that the fourth drawn ball is red is the same as the probability that the first is red.

This would **not be the case if information about the first three draws** is available (see chapter 3, example 3.1.4).

Generalization (we will use this in future, e.g. example 5.4.9): If we have N elements in a population, of which R has a specific property, and n are drawn randomly and without replacement, then the probability if the event A_k = "the k^{th}-draw is an element with the property" is:

$$P(A_k) = \frac{R \cdot \frac{(N-1)!}{(N-n)!}}{\frac{N!}{(N-n)!}} = \frac{R}{N}, \qquad voor\ k = 1,2,3,\ldots\ldots, n \leq N$$

4. a. $\frac{\binom{16}{5}}{\binom{52}{5}} = \frac{16!/11!}{52!/47!} \approx 0.0017$

- The simplest approach is to use combinations (the order is not important for the event at hand): compute the proportion of combinations with only face cards, $\binom{16}{5}$, within the total number of combinations of 5 out of 52: $\frac{\binom{16}{5}}{\binom{52}{5}}$
- But reasoning in ordered outcomes (permutations) one can compute the proportion of the number of permutations of 5 out of 16 face cards within the total number of permutations of 5 out of 52 cards: $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \frac{52!}{47!}$.

b. $\frac{4 \cdot 4 \cdot (50 \cdot 49 \cdot 48)}{\frac{52!}{47!}} \approx 0.0060$

The order is important here: count the number of favorable outcomes in the numerator, by first placing an ace on position 1 (4 possibilities),
then a king on position 5 and
finally any of the remaining 50 cards on position 2, 3 and 4.

c. $\frac{\binom{4}{1}\binom{4}{1}\binom{44}{3}}{\binom{52}{5}} = \frac{5 \cdot 4 \cdot 4 \cdot 4 \cdot (44 \cdot 43 \cdot 42)}{52!/47!} \approx 0.0815$

5. $\frac{\binom{6}{4} + 1}{6!} \approx 0.0222$

6. a. Assume there are (exactly) 10 substandard rings and 90 good ones in the packet, then we can apply the complement rule to find:

$$P(\text{"at least one bad}) = 1 - P(\text{"alle good"}) = 1 - \frac{\binom{90}{10}}{\binom{100}{10}} \approx 0.6695$$

b. $P(\text{"all good"}) = \frac{\binom{80}{10}}{\binom{100}{10}} \approx 0.0951$

7. Assume in this exercise 3 groups of tickets: 4 main prizes, 10 consolation prizes and 86 “no prizes”. Buying tickets can be seen as choosing one of the hundred tickets, at random and without replacement. The order in which the tickets are bought is not essential for winning a prize. Buying 5 tickets is choosing at random a combination of 5 out of 100. (We discussed a similar situation in section 2.2, but then we had 2 types of things, now 3.)

a. $\frac{\binom{4}{1} \cdot \binom{10}{1} \cdot \binom{86}{3}}{\binom{100}{5}} \approx 0.0544$

see diagram →

Main	Consolation	No prize	Total
4	10	86	100
↓	↓	↓	↓
1	1	3	5

b. $\frac{\binom{14}{1} \cdot \binom{86}{4}}{\binom{100}{5}} \approx 0.3949$

c. $\frac{\binom{86}{5}}{\binom{100}{5}} \approx 0.4626$

d. 0.5374

8. $\dfrac{\binom{N_1}{n_1}\cdot\binom{N_2}{n_2}\cdot\ldots\cdot\binom{N_k}{n_k}}{\binom{N}{n}}$

9. a. $2\cdot\frac{\binom{6}{4}\cdot\binom{20}{9}}{\binom{26}{13}}\approx 0.4845$ (Take the distinguishable distributions 4+2 and 2+4 into account)

b. $\dfrac{\binom{6}{3}\cdot\binom{20}{10}}{\binom{26}{13}}\approx 0.3553$

10. a. $\binom{13}{6}$

b. $\binom{30}{6}\cdot\binom{24}{7}\cdot\binom{17}{8}\cdot\binom{9}{9}=\frac{30!}{6!7!8!9!}$

c. $\binom{n}{n_1}\cdot\binom{n-n_1}{n_2}\cdot\ldots\cdot\binom{n_k}{n_k}=\dfrac{n!}{n_1!\,n_2!\ldots n_k!}$.

11. a. $\dfrac{\binom{7}{2}+\binom{7}{3}}{\binom{11}{5}}\approx 0.1212$

The total number of sequences equals the number of choices of the positions of the 5 zero\`s (or the 6 one\`s): $\binom{11}{5}=\binom{11}{6}$.

The favorable number is the number of sequences starting with 0001… or 1110…
In the first case there are 2 zero\`s and 5 one\`s for the remaining 7 positions, etc.

b. The number of sequences with 5 zero\`s, 6 one\`s and 5 runs can be determined by distinguishing the "start-run": if the first run (and the last) consists of zero\`s, then the 5 zero\`s will be split up into 3 runs, e.g. after the second and the fourth zero: 00 | 00 | 0. These breaks can be chosen in $\binom{4}{2}$ ways (choose 2 breaks out of 4 positions (after the first, second, third or fourth zero).

The number of one\`s in the first chosen break can be chosen in $5=\binom{5}{1}$ ways (1, 2, 3, 4 or 5 one\`s). So there are $\binom{4}{2}\cdot\binom{5}{1}$ sequences with 5 runs, starting with a run of zero\`s.

The number of sequences with a start-un of 1's is computed similarly.

b. $\dfrac{\binom{4}{2}\binom{5}{1}+\binom{5}{2}\binom{4}{1}}{\binom{11}{5}}=\dfrac{70}{462}\approx 0.1515$

12. $\dfrac{\binom{10}{4}}{10!/6!}=\dfrac{1}{4!}=\dfrac{1}{24}$

Analyzing the problem leads to the following:

- The order is important, since we are interested in increasing orders in the draws.
- The total number of outcomes of 4 draws (without replacements) is the number of permutations of 4 out of 10: $\frac{10!}{6!}$.
- The number of increasing permutations is more difficult to count: one could choose to start with 1, 2 … etc. but this gets complicated with many cases to count.
- If you would consider a combination of 4 numbers (e.g. 1, 2, 3 and 4), then you can construct only one increasing permutation (in the example (1, 2, 3, 4)). So, **for each**

combination there is one increasing order, and reverse: the number of increasing permutations equals the number of combinations, $\binom{10}{4}$!