Solutions exercises part 1

1. a. $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1}e^{-u}du = -u^{\alpha-1}e^{-u}|_{u=0}^{u\to\infty} + \int_0^\infty (\alpha-1)u^{\alpha-2}e^{-u}du = 0 + (\alpha-1)\Gamma(\alpha-1)$
(integration by parts has been applied)

b. $\Gamma(1) = \int_0^\infty e^{-u}du = -e^{-u}|_{u=0}^{u\to\infty} = 0 — 1 = 1$

$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \cdots =$ (n-1)×(n-2)×…..×1× Γ(1) = (n-1)!

c. $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty u^{-\frac{1}{2}}e^{-u}du = \int_0^\infty \sqrt{2}x^{-1}e^{-\frac{1}{2}x^2}xdx = \int_0^\infty \sqrt{2}e^{-\frac{1}{2}x^2}dx = 2\sqrt{\pi}\int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx$

$= 2\sqrt{\pi} \times \frac{1}{2} = \sqrt{\pi}$

2. **a.** $F_Y(y) = P(Y \le y) = P(-3X + 5 \le y) = P\left(X \ge \frac{y-5}{-3}\right) = 1 - P\left(X \le \frac{y-5}{-3}\right) = 1 - F_X\left(\frac{y-5}{-3}\right)$

$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{3}f_X\left(\frac{y-5}{-3}\right)$

b. $f_Y(y) = \frac{1}{3}f_X\left(\frac{y-5}{-3}\right) = \frac{1}{3} \times \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{y-5}{-3}\right)^2} = \frac{1}{\sqrt{2\pi \times 9}}e^{-\frac{1}{2}\times\frac{(y-5)^2}{9}}$,

in which we recognize the normal density function with μ = 5 and $\sigma^2 = 9$.

3. $F_Z(z) = P(Z \le z) = P(\min(X,Y) \le z) = 1 - P(\min(X,Y) \ge z) = 1 - P(X \ge z \; en \; Y \ge z)$

$= 1 - P(X \ge z\,) \times PY \ge z) = 1 - [1 - F_X(z)]^2$

$f_Z(z) = \frac{d}{dz}F_Z(z) = f_X(z) \times 2[1 - F_X(z)] = \lambda e^{-\lambda z} \times 2e^{-\lambda z} = 2\lambda e^{-2\lambda z}, for\ z \ge 0$ (elsewhere $f_Z(z) = 0$)

$Z = min(X,Y)$ is exponentially distributed as well, with parameter 2λ (EZ = ½ E(X))

4. **a.** $E(X_i) = \frac{1}{2}$ (symmetry), $E(X_i^2) = \int_0^1 x^2dx = \left[\frac{1}{3}x^3\right]_{x=0}^{x=1} = \frac{1}{3}$, so $var(X) = E(X^2) - (EX)^2 = \frac{1}{12}$.

b. $F_{X_1}(x) = \int_0^x 1du = [u]_{u=0}^{u=x} = x$ (if $0 \le x \le 1$). $F_{X_1}(x) = 0$ for $x < 0$ and $F_{X_1}(x) = 1$ for $x > 1$.

c. $F_Y(y) = P(Y \le y) = P(-2\ln(X_1) \le y) = P\left(\ln(X_1) \ge -\frac{1}{2}y\right)$

$= P\left(X_1 \ge e^{-\frac{1}{2}y}\right) = 1 - F_{X_1}\left(e^{-\frac{1}{2}y}\right) = 1 - e^{-\frac{1}{2}y}$, or $0 \le e^{-\frac{1}{2}y} \le 1$, implying that $y \ge 0$.

Conclusion: $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{2}e^{-\frac{1}{2}y}$ ($y \ge 0$).

This is the exponential density with parameter λ = $\frac{1}{2}$, so $E(Y) = \frac{1}{\lambda} = 2$.

d. $P(Z \le z) = P(\max(X_1, \ldots, X_n) = P(X_1 \le z \text{ and } X_2 \le z \text{ and } \ldots \text{and } X_n \le z\,)$

$\stackrel{\text{ind.}}{=} P(X_1 \le z) \cdot \ldots \cdot P(X_n \le z) = z^n, \quad \text{if } 0 \le z \le 1$

So $f_Z(z) = \frac{d}{dz}[z^n] = nz^{n-1}$, for $0 \le z \le 1$ (and $f_Z(z)$ = 0 elsewhere)

$E(Z) = \int_{-\infty}^\infty zf(z)\,dz = \int_0^1 z \cdot nz^{n-1}dz = \left[\frac{n}{n+1}z^{n+1}\right]_{z=0}^{z=1} = \frac{n}{n+1}$.

$E(Z^2) = \int_0^1 z^2 \cdot nz^{n-1}dz = \left[\frac{n}{n+2}z^{n+2}\right]_{z=0}^{z=1} = \frac{n}{n+2}$, so $(Z) = \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2 = \frac{n}{(n+2)(n+1)^2}$.

5. (We are using the notation X instead of X_1 in a.)

a. $E(X) = \int_{-\infty}^\infty xf(x)dx = \int_0^\infty x \cdot \lambda e^{-\lambda x}dx = ..\,integr.\,by\ parts \ldots = \left[x \cdot -e^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}\,dx$

$= \quad 0 \quad + \quad \frac{1}{\lambda}$

$E(X^2) = \int_0^\infty x^2 \cdot \lambda e^{-\lambda x}dx = \cdots = \frac{2}{\lambda^2}$, so $var(X) = E(X^2) - (EX)^2 = \frac{1}{\lambda^2}$ and $\sigma = \frac{1}{\lambda}$.

$(E(X^3) = \int_0^\infty x^3 \cdot \lambda e^{-\lambda x}dx = \left[x^3 \cdot -e^{-\lambda x}\right]_0^\infty + \int_0^\infty 3x^2e^{-\lambda x}\,dx = 0 + \frac{3}{\lambda} \cdot E(X^2) = \frac{6}{\lambda^3}$.)

b. For $n = 100$ and $\lambda = 2$, S is approximately (CLT) $N(n\mu, n\sigma^2) = N(50, 25)$.
So $P(S > 55)1 - P(S \leq 55) \overset{\text{CLT}}{\approx} 1 - \Phi\left(\frac{55-50}{\sqrt{25}}\right) = 1 - \Phi(1) = 1 - 0.8413 = 15.87\%$.

c. For $n = 50$ and $\lambda = 2$, $\overline{X}$ is approximately $N\left(\mu, \frac{\sigma^2}{n}\right) = N\left(25, \frac{1}{200}\right)$.

$$P(\overline{X} > 0.55) \overset{\text{CLT}}{\approx} 1 - \Phi\left(\frac{0.55-0.50}{\sqrt{\frac{1}{200}}}\right) = 1 - \Phi(0.71) = 0.2389.$$

d. $F_M(m) = P(\min(X_1, \dots, X_n) \leq m) = 1 - P(\min(X_1, \dots, X_n) \geq m)$
$= 1 - P(X_1 \geq m \text{ and} \dots \text{and } X_n \geq m)$
$\overset{\text{ind.}}{=} 1 - P(X_1 \geq m) \cdot \dots \cdot P(X_n \geq m) = 1 - \left[e^{-\lambda m}\right]^n = 1 - e^{-n\lambda m}$ $(m \geq 0)$
$f_M(m) = n\lambda e^{-n\lambda m}$ (≥ 0), so M is exponentially distributed with parameter $n\lambda$.
For $n = 10$ and $= 2$, $E(M) = \frac{1}{10 \cdot 2} = 0.05$.

6. $Z_1, Z_2, \dots, Z_n$ are independent and all standard normal.

a. $E(Z_1^2) = var(Z_1) + (EZ_1)^2 = 1 + 0 = 1$,

$$E(Z_1^4) = \int_{-\infty}^{\infty} z^4 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \left[z^3 \cdot -\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\right]_{-\infty}^{-\infty} + \int_{-\infty}^{\infty} 3z^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 0 + 3E(Z_1^2) = 3$$

$var(Z_1^2) = E(Z_1^4) - E(Z_1^2)^2 = 3 - 1^2 = 2$,
$E(Z_1^2 + \dots + Z_n^2) = n \cdot E(Z_1^2) = n$ and $var(Z_1^2 + \dots + Z_n^2) = n \cdot var(Z_1^2) = 2n$

b. (Below $Z = Z_1$ and $Y = Z_1^2$)

1. $F_Y(y) = P(Y \leq y) = P(Z^2 \leq y) = \begin{cases} 0 & \text{if } y \leq 0 \\ P(-\sqrt{y} \leq Z \leq \sqrt{y}) & \text{if } y > 0 \end{cases}$

   So, if $y > 0$: $F_Y(y) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$

2. If $y \leq 0$, then $f_Y(y) = \frac{d}{dy} F_Y(y) = 0$

   If $y > 0$, then $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[\Phi(\sqrt{y}) - \Phi(-\sqrt{y})\right]$
   $= \frac{1}{2\sqrt{y}}\left[\varphi(\sqrt{y}) + \varphi(-\sqrt{y})\right]$

3. $f_Y(y) = 0$, for $y \leq 0$ and

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{y})^2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{y})^2}\right] = \frac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2}y}, \qquad \text{for } y > 0.$$

c. $f_{X_1+X_2}(z) = \int_{-\infty}^{\infty} f_{X_1}(x) f_{X_2}(z-x) dx = \int_0^z \frac{1}{\sqrt{2\pi x}} e^{-\frac{1}{2}x} \cdot \frac{1}{\sqrt{2\pi x}} e^{-\frac{1}{2}(z-x)} \mathrm{d}x = \frac{e^{-\frac{1}{2}z}}{2\pi} \int_0^z \frac{1}{\sqrt{x(z-x)}} dx$

The last integral equals π (given): $f_{X_1+X_2}(z) = \frac{1}{2} e^{-\frac{1}{2}z}$, *for* z > 0.

A Chi-square distribution with 2 degrees of freedom is apparently the same as an Exp $\left(\frac{1}{2}\right)$*-distribution*

d. $f(z) = \frac{z^{\alpha-1} e^{-\frac{z}{\beta}}}{\Gamma(\alpha)\beta^\alpha} = \frac{1 \cdot e^{-z/2}}{1 \cdot 2^1} = \frac{1}{2} e^{-\frac{1}{2}z}$.

7. a. $M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2tx)} dx$
$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2} dx = e^{+\frac{1}{2}t^2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx = e^{\frac{1}{2}t^2}$

$M`(t) = te^{\frac{1}{2}t^2}$. So ``$(t) = M`(0) = 0 = E(X)$.

$M``(t) = \frac{d}{dt}\left[te^{\frac{1}{2}t^2}\right] = e^{\frac{1}{2}t^2} + 1 \cdot t \cdot te^{\frac{1}{2}t^2} = e^{\frac{1}{2}t^2}(1+t^2)$. Hence $M``(0) = 1 = E(X^2) = var(X)$

b. $M(t) = E(e^{tX}) = \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dx = \int_0^\infty \lambda e^{(t-\lambda)x} dx = \left[\frac{\lambda}{t-\lambda} e^{(t-\lambda)x}\right]_{x=0}^\infty = 0 - \frac{\lambda}{t-\lambda} = \frac{\lambda}{\lambda - t}$

if $t < \lambda$ (otherwise $M(t)$ is not defined!)

$M`(t) = -1 \cdot -1 \cdot \frac{\lambda}{(\lambda-t)^2}$, so $M`(0) = \frac{1}{\lambda} = E(X)$

$M``(t) = \frac{d}{dt}\left[\frac{\lambda}{(\lambda-t)^2}\right] = \frac{2\lambda}{(\lambda-t)^3}$, so $M``(0) = \frac{2}{\lambda^2} = E(X^2)$ and $var(X) = E(X^2) - (EX)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

c. $M(t) = E(e^{tX}) = \sum_{x=0}^\infty e^{tx} \cdot \frac{\mu^x e^{-\mu}}{x!} = \sum_{x=0}^\infty \frac{(e^t\mu)^x e^{-\mu}}{x!} = \sum_{x=0}^\infty \frac{(e^t\mu)^x e^{-e^t\mu}}{x!} \cdot e^{\mu(e^t-1)} = e^{\mu(e^t-1)}$

$M`(t) = \mu e^t e^{\mu(e^t-1)} = \mu e^{\mu(e^t-1)-t}$. Hence $M`(0) = \mu$

$M``(t) = \frac{d}{dt}\left[\mu e^{\mu(e^t-1)-t}\right] = \mu(\mu e^t - 1)e^{\mu(e^t-1)-t}$, so $M``(0) = \mu^2 - \mu = E(X^2)$.

So $var(X) = E(X^2) - (EX)^2 = (\mu^2 - \mu) - \mu^2 = \mu$.

8. a. From $Y = \ln\left(\frac{1}{X}\right)$ it follows that for $y > 0$:

$$F_Y(y) = P\left(ln\left(\frac{1}{X}\right) \le y\right) = P\left(\frac{1}{X} \le e^y\right) = P\left(X \ge \frac{1}{e^y}\right) = 1 - F_X(e^{-y})$$

So $f_Y(y) = \frac{d}{dy}F_Y(y) = -f_X(e^{-y}) \cdot -e^{-y} = +e^{-y}f_X(e^{-y})$.

Since $f_X(x) = 1$ for $0 \le x \le 1$, we have: $f_Y(y) = e^{-y} \cdot 1$ if $0 \le e^{-y} \le 1$, so if $y \ge 0$.

Conclusion: $Y = \ln\left(\frac{1}{X}\right)$ is exponentially distributed with parameter λ = 1.

b. $E(Y) = \frac{1}{\lambda} = 1$ and $E(X) = \frac{1}{2}$, so $1 = E(Y) \ne \ln\left(\frac{1}{EX}\right) = \ln(2)$

9. a. Given is that every $X_i \sim Exp(\lambda)$, where $E(X_i) = 12 = \frac{1}{\lambda}$, so $\lambda = \frac{1}{12}$.

$var(X_i) = \frac{1}{\lambda^2} = 144$ and $E(\sum_{i=1}^6 X_i) = 6 \cdot E(X_i) = 6 \cdot 12 = 72$.

b. Because of the lack of memory property: $P(X_1 > 15 | X_1 > 3) = P(X_1 > 12) = e^{-\frac{1}{12} \cdot 12} = e^{-1}$.

10. a. (According results in probability theory the sum of independent, $Exp\left(\frac{1}{4}\right)$-distributed is Erlang distributed: S_n has an Erlang distribution with parameters n and λ = ¼.)

$E(S_n) = n \cdot \frac{1}{\lambda} = 4n$ and $var(S_n) = n \cdot \frac{1}{\lambda^2} = 16n$.

b. (applying the convolution-integral:)

$f_{X_1+X_2}(z) = \int_{-\infty}^\infty f_{X_1}(x) f_{X_2}(z-x)dx$, *where* $f_{X_1}(x) = 0$, *if* $x < 0$ *and* $f_{X_2}(z-x) = 0$, *if* $x > z$

$= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx$ $\left(\text{met } \lambda = \frac{1}{4}\right)$

$= \int_0^z \lambda^2 e^{-\lambda z} dx$ *Notice that* $\lambda^2 e^{-\lambda z}$ *is a constant in the integration w.r.t.* x.

$= \lambda^2 e^{-\lambda z} \cdot x|_{x=0}^{x=z}$

$= \lambda^2 z e^{-\lambda z}$, for $z \ge 0$ (*where* $\lambda = \frac{1}{4}$)

And $f_{X_1+X_2}(z) = 0$, if $z < 0$.

c. $P(\overline{X}_2 > 5) = P\left(\frac{S_2}{2} > 5\right) = P(S_2 > 10) = \int_{10}^\infty \lambda^2 z e^{-\lambda z} dz = \cdots$ *integration by parts ….*

$= \left[\lambda z \cdot -e^{-\lambda z}\right]_{10}^\infty + \int_{10}^\infty \lambda e^{-\lambda z} dz$

$= \left(0 + 10\lambda e^{-10\lambda}\right) + \left(0 + e^{-10\lambda}\right) \overset{\lambda=1/4}{=} \frac{7}{2} e^{-\frac{5}{2}}$

d. For large n we can apply the CLT: $\overline{X}_n$ is approximately $N\left(\mu, \frac{\sigma^2}{n}\right)$, where in this case $n = 100$, $\mu = \frac{1}{\lambda} = 4$ and $\sigma^2 = \frac{1}{\lambda^2} = 16$, so: $P\left(\overline{X}_{100} > 5\right) \overset{\text{CLT}}{\approx} P\left(Z > \frac{5-4}{\sqrt{\frac{16}{100}}}\right) = P(Z > 2.5) = 1 - \Phi(2.5) = 0.62\%$