Solutions Chapter 3 (Confidence Intervals) – Mathematical Statistics

Exercise 1

a. An estimate of the expectation (μ) is the sample mean: $\bar{x} = 28.25$.
An estimate of the variance (σ^2) is the sample variance: $s^2 = 14.37$
(Remark: the notations μ = 28.25 and σ^2 = 14.37 are false is, since μ and σ remain unknown!)

b. Requested: a 95%-CI for μ with unknown σ (*directly from the formula sheet*) has bounds: $\overline{x} \pm c\frac{s}{\sqrt{n}}$,
where $c = 2.131$, such that $P(T_{15} \geq c) = \frac{\alpha}{2} = 0.025$, from the t_{15}-distribution *(→ brief notation for the t-distribution with $df = 15$)*
Substitute into the formula: $95\% - CI(\mu) = \left(28.25 - 2.131 \cdot \frac{\sqrt{14.37}}{\sqrt{16}},\ 28.25 + 2.131 \cdot \frac{\sqrt{14.37}}{\sqrt{16}}\right)$
$= (28.25 - 2.02, 28.25 + 2.02) = (26.23, 30.27)$
(*Note: in the formula we did not use the capitals $\overline{X}$ and S, since $\overline{x}$ and s are the numerical values in a. after observing the sample variables).*

c. Requested is a 95%-CI for σ^2, so we will use (formula sheet):
$\left(\frac{(n-1)S^2}{c_2}, \frac{(n-1)S^2}{c_1}\right)$, with $P(\chi^2_{n-1} \leq c_1) = P(\chi^2_{n-1} \geq c_2) = \frac{\alpha}{2} = 0.025$
We have $s^2 = 14.37$ and $n = 16$ (a.) and in the χ^2_{16-1}-table we find $c_1 = 6.26$ and $c_2 = 27.49$.
So 95%-CI(σ^2) = $\left(\frac{(n-1)S^2}{c_2}, \frac{(n-1)S^2}{c_1}\right) = \left(\frac{15 \cdot 14.37}{27.49}, \frac{15 \cdot 14.37}{6.26}\right) \approx (7.84, 34.4)$.

Exercise 2

a. A 99%-CI for μ has bounds: $\overline{x} \pm c\frac{s}{\sqrt{n}}$, where $c = 2.861$ can be found in the t-table with 19 degrees of freedom, such that $P(T_{19} \geq c) = \frac{\alpha}{2} = 0.005$, so $c\frac{s}{\sqrt{n}} = 2.861 \cdot \frac{39.50}{\sqrt{20}} \approx 25.27$
99%-CI(μ) = $(189.74 - 25.27, 189.74 + 25.27) = (164.47, 215.01)$

b. At a confidence level 99% the mean (expected) number of hours of sunshine in the month of July in De Bilt between 164.47 and 215.01. We are quite sure about this statement: this method will produce an interval containing μ in 99% of the cases that we will apply the method in equivalent situations, that is, if we repeat the random samples over and over again with equally many observations.

c. The value 164.1 of the observation x in the year 1984 should not be compared to confidence interval in a. since the interval concerns μ, the ***mean number*** of hours of sunshine.
(Note that more observations result in a smaller interval and, of course, a relatively smaller proportion of observations within the interval).
If we want to check how exceptional an observation such as $x = 164.1$ is, we can compute its z-score:
$z = \frac{x-\overline{x}}{s} = \frac{164.1-189.74}{39.50} \approx -0.65$.
This value does not at all indicate an exceptionally low observation: it is less than one standard deviation away from the mean.

d. Requested is the 95%-CI for the standard deviation σ, so we use (formula sheet):
$\left(\frac{(n-1)S^2}{c_2}, \frac{(n-1)S^2}{c_1}\right)$, where $P(\chi^2_{n-1} \leq c_1) = P(\chi^2_{n-1} \geq c_2) = \frac{\alpha}{2} = 0.025$
In this case $n = 20$ and $s = 39.50$ are given and in the χ^2_{20-1}-table we find $c_1 = 8.91$ and $c_2 = 32.85$
So a 95%-CI(σ) = $\left(\sqrt{\frac{19s^2}{c_2}}, \sqrt{\frac{19s^2}{c_1}}\right) = \left(\sqrt{\frac{19 \cdot 39.50^2}{32.85}}, \sqrt{\frac{19 \cdot 39.50^2}{8.91}}\right) \approx (30.0,\ 57.7)$

Exercise 3

a. Using the calculator we find: $\bar{x} = 60$ and $s^2 = 51.25$ ($s \approx 7.159$).
($\bar{X}$ and S^2 are unbiased estimators of μ and σ^2)

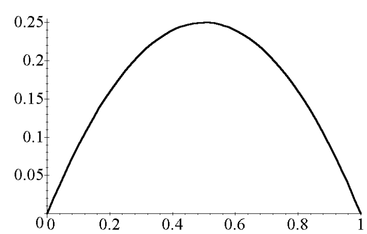
b. We are interested in a confidence interval for μ, where in this case the other parameter σ^2 is unknown. Assuming a normal distribution for the search times we will use: $\overline{X} \pm c\frac{S}{\sqrt{n}}$ with $P(T_{n-1} \geq c) = \frac{1}{2}\alpha$, where $n = 9$, $\bar{x} = 60$, $s \approx 7.159$ and in the t_8-table we find $c = 2.306$, such that $P(T_8 \geq c) = 0.025$.
95%-CI(μ) = $\left(\bar{x} - c\frac{s}{\sqrt{n}}, \bar{x} + c\frac{s}{\sqrt{n}}\right) \approx (54.5, 65.5)$

c. In this part a 95%-BI for σ^2 is requested, so we will use (formula sheet):
$\left(\frac{(n-1)S^2}{c_2}, \frac{(n-1)S^2}{c_1}\right)$, with $P(\chi_8^2 \leq c_1) = P(\chi_8^2 \geq c_2) = \frac{\alpha}{2} = 0.025$, such that $c_1 = 2.18$ and $c_2 = 17.53$
So 95%-CI(σ^2) = $\left(\frac{(n-1)s^2}{c_2}, \frac{(n-1)s^2}{c_1}\right) = \left(\frac{8 \cdot 51.25}{17.53}, \frac{8 \cdot 51.25}{2.18}\right) \approx (23.4, 188.1)$

Exercise 4

a. 95%-CI for p: $\hat{p} \pm c\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ with $\Phi(c) = 1 - \frac{1}{2}\alpha$, *(formula sheet)*
This a large-sample-interval since $n = 100$ is large enough.
Furthermore $\hat{p} = \frac{22}{100} = 0.22$ and $c = 1.96$ from the $N(0,1)$-table such that $\Phi(c) = 0.975$
(***never*** *use the t-distribution if the binomial model applies to the observations!*).
Substitute: $\left(0.22 - 1.96\sqrt{\frac{0.22 \cdot 0.78}{100}}, 0.22 + 1.96\sqrt{\frac{0.22 \cdot 0.78}{100}}\right) \approx (0.22 - 0.08, 0.22 + 0.08) = (0.14, 0.30)$

b. The 95%-CI should not be wider than 0.02, so: $2 \cdot 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq 0.02$.
This implies: $\sqrt{n} \geq \frac{1.96}{0.01}\sqrt{\hat{p}(1-\hat{p})}$, so $n \geq \left(\frac{1.96}{0.01}\right)^2 \cdot \hat{p}(1-\hat{p})$.
Using the observed value in the small sample in a., so $\hat{p} = 0.22$, we find $n \geq 6593$.
(Remark 1: if the indication of $\hat{p}$ in part a. is not given, one could choose a "safe" value ½ for $\hat{p}$ or $\hat{p}(1-\hat{p}) = \frac{1}{4}$: in that case we find: $n \geq 9604$
Remark 2: The inequality $\hat{p}(1-\hat{p}) \leq \frac{1}{4}$ can be shown by considering the function $f(p) = p(1-p) = p - p^2$ *for* $0 \leq p \leq 1$*: de graph of f is a parabola that intersects the X-axis at $p = 0$ and $p = 1$ and the maximum value of f is attained at $p = ½$, for which $f(p) = p(1-p) = ¼$)*



Exercise 5

a. This is another binomial situation: Use $\hat{p} \pm c\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ with $\Phi(c) = 1 - \frac{1}{2}\alpha$, *(formula sheet)*
In this case $1 - \alpha = 0.95$ and $\Phi(c) = 0.975$, so $c = 1.96$, $\hat{p} = \frac{73}{400} = 0.1825$ and $n = 400$.
Substitution in the formula results in:
$$95\%\text{-CI}(p) = \left(0.1825 - 1.96 \cdot \sqrt{\frac{0.1825 \cdot 0.8175}{400}}, 0.1825 + 1.96 \cdot \sqrt{\frac{0.1825 \cdot 0.8175}{400}}\right)$$
$$= (0.1825 - 0.0379, 0.1825 + 0.0379) \approx (0.145, 0.220).$$
Interpretation: "We are 95% confident that the real population proportion of cars that do not meet the conditions is between 14.5 and 22%."
The more precise (frequency-)interpretation: "If we would repeat the sample many times (independently), 95% of all computed confidence intervals will contain the population proportion p of substandard cars."

b. We want only an upper bound for the proportion p.

So we should use: 95%-CI(p) $= \left(0, \hat{p} + c\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$, where $n = 400$ and $\hat{p} = \frac{73}{400}$ are the same as in a.,

but now $\Phi(c) = 0.95$ (one sided implies one tail probability with area α): $c = 1.645$.

95%-CI(p) $= \left(0, 0.1825 + 1.645 \cdot \sqrt{\frac{0.1825 \cdot 0.8175}{400}}\right) \approx (0, 0.214)$ (or, including $p = 0$: $[0, 0.214)$)

Interpretation: "We are 95% confident that the proportion of cars with deficiencies is at most 21.4%."

c. **Estimation error = half of the width**: $c\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq 0.02$, it follows that $n \geq \left(\frac{c}{0.02}\right)^2 \hat{p}(1-\hat{p})$.

For a confidence level 99% is $\Phi(c) = 0.995$, so $c = 2.575$ en in and from a. we can use $\hat{p} = \frac{73}{400}$.

We find: $n \geq 2473.1$, or: $\boldsymbol{n = 2474}$.

Exercise 6

Consider the sample variance S^2 of a random sample $X_1, \ldots, X_n$ drawn from a normal distribution with unknown μ and σ^2. The sample size is large: $n > 25$ (sufficiently large to apply the CLT)

a. According to the CLT $\frac{(n-1)S^2}{\sigma^2}$ is approximately normal distributed with expectation $n-1$ and variance $2(n-1)$: $\frac{(n-1)S^2}{\sigma^2} \sim N\big(n-1, 2(n-1)\big)$

Hence we have approximately

$$\frac{\frac{(n-1)S^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} \sim N(0,1) \text{ and}$$

$$P\left(-c < \frac{\frac{(n-1)S^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} < c\right) \overset{\text{CLT}}{\approx} 1-\alpha, \quad \text{where } \Phi(\text{c}) = 1 - \frac{1}{2}\alpha.$$

This equivalent to $P\left((n-1) - c\sqrt{2(n-1)} < \frac{(n-1)S^2}{\sigma^2} < (n-1) + c\sqrt{2(n-1)}\right) \approx 1-\alpha$

or: $P\left(\frac{(n-1)S^2}{(n-1)+c\sqrt{2(n-1)}} < \sigma^2 < \frac{(n-1)S^2}{(n-1)-c\sqrt{2(n-1)}}\right) \approx 1-\alpha$ (assuming that $(n-1) - c\sqrt{2(n-1)} > 0$)

b. Substituting $n = 101$, $s^2 = 50$ and $c = 1.96$ ($\Phi(\text{c}) = 0.975$) in the formula of a. we find:

95%-$CI(\sigma^2) \approx \left(\frac{100 \cdot 50}{100+1.96\sqrt{200}}, \frac{100 \cdot 50}{100-1.96\sqrt{200}}\right) \approx \mathbf{(39.15, 69.17)}$.

c. 95%-$CI(\sigma^2) = \left(\frac{(n-1)S^2}{c_2}, \frac{(n-1)S^2}{c_1}\right)$, where $n = 101, s^2 = 50$ and

$P(\chi^2_{100} \leq c_1) = P(\chi^2_{100} \geq c_2) = \frac{\alpha}{2} = 0.025$, so that $c_1 = 74.22$ and $c_2 = 129.56$:

95%-$CI(\sigma^2) = \left(\frac{100 \cdot 50}{129.56}, \frac{100 \cdot 50}{74.22}\right) \approx (\mathbf{38.59}, \mathbf{67.37})$

(Note: considering the differences of the results in b. and c. it seems to be better to use the χ^2-tables, and linear interpolation, if $df \leq 100$)

Exercise 7

a. $\overline{x} = 12.6$ is an estimate of $E(X) = \frac{1}{\lambda}$, so $\frac{1}{\overline{x}} = \frac{1}{12.6} \approx 0.0794$ is an estimate of λ.

Note: In chapter2 we showed that $1/\overline{X}$ *is the maximum likelihood estimator* (mle).

b. Construction of a confidence interval of λ:

$\overline{X}$ is according to the CLT approximately $N\left(\mu, \frac{\sigma^2}{50}\right)$ with $E(X) = \frac{1}{\lambda}$, then approximately $\frac{\overline{X} - \frac{1}{\lambda}}{\sqrt{\frac{\sigma^2}{50}}} \sim N(0,1)$.

σ^2 is unknown, but we can approximate σ^2 with S^2. (In this case we have $\sigma^2 = \frac{1}{\lambda^2}$, therefore an alternative estimator would be $\overline{X}^2$! Check that $\overline{x}^2 \approx s^2$.)

We can use the $N(0,1)$-distribution to choose the value of c such that $P(-c < Z < c) = 1 - \alpha$

$$P\left(-c < \frac{\overline{X} - \frac{1}{\lambda}}{\sqrt{\frac{S^2}{50}}} < c\right) \approx 1 - \alpha$$

$$P\left(-c\sqrt{\frac{S^2}{50}} < \overline{X} - \frac{1}{\lambda} < c\sqrt{\frac{S^2}{50}}\right) \approx 1 - \alpha$$

$$P\left(\overline{X} - c\sqrt{\frac{S^2}{50}} < \frac{1}{\lambda} < \overline{X} + c\sqrt{\frac{S^2}{50}}\right) \approx 1 - \alpha$$

$$P\left(\overline{X} - c\sqrt{\frac{S^2}{50}} < \frac{1}{\lambda} < \overline{X} + c\sqrt{\frac{S^2}{50}}\right) \approx 1 - \alpha$$

$$P\left(\frac{1}{\overline{X} + c\sqrt{\frac{S^2}{50}}} < \lambda < \frac{1}{\overline{X} - c\sqrt{\frac{S^2}{50}}}\right) \approx 1 - \alpha$$

(*If* $\overline{X} - c\sqrt{\frac{S^2}{50}} < 0$, *the upper bound should be* ∞.)

An alternative (better?) approach is to use that $\sigma^2 = \frac{1}{\lambda^2}$ so that $\overline{X}$ is approximately $N\left(\frac{1}{\lambda}, \frac{1}{\lambda^2 n}\right)$.

Then we can solve λ from the inequality in the following (approximate) probability statement:

$$P\left(-c < \frac{\overline{X} - \frac{1}{\lambda}}{\sqrt{\frac{1}{\lambda^2 n}}} < c\right) \approx 1 - \alpha, \qquad \text{where } \Phi(c) = 1 - \frac{1}{2}\alpha$$

$$\Leftrightarrow P\left(-c < \left(\lambda\overline{X} - 1\right)\sqrt{n} < c\right) \approx 1 - \alpha$$

$$\Leftrightarrow P\left(-c/\sqrt{n} < (\lambda\overline{X} - 1) < c/\sqrt{n}\right) \approx 1 - \alpha,$$

$$\Leftrightarrow P\left(1 - \frac{c}{\sqrt{n}} < \lambda\overline{X} < 1 + \frac{c}{\sqrt{n}}\right) \approx 1 - \alpha,$$

$$\Leftrightarrow P\left(1/\overline{X}\left(1 - \frac{c}{\sqrt{n}}\right) < \lambda < 1/\overline{X}\left(1 + \frac{c}{\sqrt{n}}\right)\right) \approx 1 - \alpha$$

c. $n = 50, \overline{x} = 12.6, s^2 = 150.3$ and $c = 1.96$ if $P(-c < Z < c) = 0.95$

$$95\% - BI(\lambda) = \left(\frac{1}{\overline{X} + c\sqrt{S^2/50}}, \frac{1}{\overline{X} - c\sqrt{S^2/50}}\right) \approx (0.0625, 0.1087)$$

(*the alternative formula results in the (slightly wider) interval* $(0.057, 1.014)$)

Exercise 8: *Analysis of the problem to find a correct approach:*

a. *We want to know what proportion (p) of the 2250 visitors are buyers. Firstly, on the basis of the sample of* $\boldsymbol{n = 75}$ *visitors, we have to determine an interval estimate of p (do not use* $n = 2250$*), and then we can estimate* $2250 \times p$*. Note that from the construction of the* $CI(p)$ *it follows:*

$$P(L < p < U) = 1 - \alpha \quad \Leftrightarrow \quad P(a \cdot L < a \cdot p < a \cdot U) = 1 - \alpha \quad (a = 2250)$$

b. *The turnover = the number of buyers (1350) times μ: μ = "the expected turnover per buyer".*
So first we determine a CI of μ on the basis of a sample of $\boldsymbol{n = 45}$ *buyers (60% of the 75 visitors).*

Solutions:

a. Binomial Model: X = "the number of buyers in a random sample of $n = 75$ visitors."
X is $B(75, p)$-distributed with unknown p = "proportion of buyers". *(Formula sheet:)*

$$95\%\text{-}CI(p) = \left(\hat{p} - c\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + c\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = \left(0.60 - 1.96\sqrt{\frac{0.6\times0.4}{75}}, 0.60 - 1.96\sqrt{\frac{0.6\times0.4}{75}}\right)$$

$\approx (0.4891 , 0.7109)$, where $n = 75$ (visitors), $\hat{p} = 0.60$ (so $x = 45$) and $c = 1.96$.

Then for the expected number of buyers on a day with 2250 visitors we have:

$95\%\text{-}CI(2250p) \approx (2250\times0.4891 , 2250\times0.7109) \approx (1100, 1600)$

(*"We are 95% confident that the number of buyers on a day with 2250 visitors is between 1100 and 1600"*)

b. As stated in a, we assume that $X_1, \ldots, X_{45}$ are the turnovers of the 45 buyers in the sample:
$X_1, \ldots, X_{45}$ are independent and (approximately) normally distributed with unknown μ and σ^2.
μ = the "mean" (expected) turnover of all (potential) buyers.

Formula sheet: $95\%\text{-}CI(\mu) = \left(\overline{X} - c \cdot \frac{S}{\sqrt{n}}, \overline{X} + c \cdot \frac{S}{\sqrt{n}}\right)$

Where $n = 45$, $\overline{X} = 40$, $s = 10$ and $c = 2.021$ from the t_{44}-tabel (*use the t_{40}-table*): $P(T_{40} \geq c) = 0.025$

$95\%\text{-}CI(\mu) \approx (36.99, 43.01)$

An interval estimate of the total turnover 1350μ is:

$95\%\text{-}CI(1350\mu) = (1350\times36.99, 1350\times43.01) = (49937, 58064)$

(*"At a 95%level of confidence the total turnover of the shop on a day with 1350 buyers is between 50 and 58 thousand Euro"*).

Exercise 9

a. For a (1-α)100%-confidence interval of p the probability statement is (for sufficiently large n):

$$\boldsymbol{P\left(-c < \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} < c\right) \overset{\text{CLT}}{\approx} 1 - \alpha} \qquad (c \text{ such that } \Phi(c) = 1 - \frac{1}{2}\alpha.)$$

Solving $(\hat{p} - p)^2 < c^2 \dfrac{p(1-p)}{n}$ with respect to p (e.g. first solving the equality):

Equality: $\hat{p}^2 - 2\hat{p} \cdot p + p^2 = \frac{c^2}{n} \cdot p - \frac{c^2}{n} \cdot p^2$

$$\Leftrightarrow \left[1 + \frac{c^2}{n}\right] \cdot p^2 - \left(2\hat{p} + \frac{c^2}{n}\right) \cdot p + \hat{p}^2 = 0$$

(*the left hand side is parabola that opens upward: solutions of p for the inequality lie between the two zero's*)

$$\Leftrightarrow p_{1,2} = \frac{\left(2\hat{p}+\frac{c^2}{n}\right) \pm \sqrt{\left(2\hat{p}+\frac{c^2}{n}\right)^2 - 4\left[1+\frac{c^2}{n}\right]\cdot \hat{p}^2}}{2\left[1+\frac{c^2}{n}\right]} = \frac{\left(2\hat{p}+\frac{c^2}{n}\right) \pm \sqrt{\left(2\hat{p}+\frac{c^2}{n}\right)^2 - 4\left[1+\frac{c^2}{n}\right]\cdot \hat{p}^2}}{2\left[1+\frac{c^2}{n}\right]}$$

$$\Leftrightarrow p = \frac{\left(2\hat{p}+\frac{c^2}{n}\right) \pm \sqrt{\left(2\hat{p}+\frac{c^2}{n}\right)^2 - 4\left[1+\frac{c^2}{n}\right]\cdot \hat{p}^2}}{2\left[1+\frac{c^2}{n}\right]} = \frac{\left(2\hat{p}+\frac{c^2}{n}\right) \pm \sqrt{\left(2\hat{p}+\frac{c^2}{n}\right)^2 - 4\left[1+\frac{c^2}{n}\right]\cdot \hat{p}^2}}{2\left[1+\frac{c^2}{n}\right]}$$

$$\Leftrightarrow p = \frac{\left(2\hat{p}+\frac{c^2}{n}\right) \pm \sqrt{4\hat{p}^2 + \frac{4c^2}{n}\hat{p} + \frac{c^4}{n^2} - 4\hat{p}^2 - \frac{4c^2}{n}\hat{p}^2}}{2\left[1+\frac{c^2}{n}\right]} = \frac{\left(2\hat{p}+\frac{c^2}{n}\right) \pm \sqrt{\frac{4c^2}{n}\cdot\left[\hat{p}(1-\hat{p})+\frac{c^2}{4n}\right]}}{2\left[1+\frac{c^2}{n}\right]}$$

$$\Leftrightarrow p = \frac{n}{n+c^2}\hat{p} + \frac{c^2}{2(n+c^2)} \quad \pm \frac{n}{n+c^2} c\sqrt{\frac{\hat{p}(1-\hat{p})+\frac{c^2}{4n}}{n}}$$
$$= \frac{n}{n+c^2}\left[\left(\hat{p}+\frac{c^2}{2n}\right) \pm c\sqrt{\frac{\hat{p}(1-\hat{p})+\frac{c^2}{4n}}{n}}\right]$$

For $n = 1200$, $c = 1.96$ (95% confidence level) and the observed $\hat{p} = \frac{300}{1200} = 0.25$ we find:
0.25080 ± 0.02447, so 95%-BI(p) ≈ **(0.2263, 0.2753)**

Comparing this interval to the "standard" interval on the formula sheet with bounds $\hat{p} \pm c\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ we find:
95%-$CI(p)$ ≈ (0.25 – 0.0245, 0.25 + 0.0245) = $(\mathbf{0.2255}, \mathbf{0.2745})$ *(rounded at 3 decimals the same result).*

Exercise 10

a. Requested: a 90%-PI for a new observation X_{17} (assuming now that $X_1, \ldots X_{16}, X_{17}$ is a random sample from a normal distribution with unknown μ and σ^2 (*directly from the formula sheet*) has bounds: $\overline{x} \pm cs\sqrt{1+\frac{1}{n}}$, where $n = 16$, $\overline{x} = 189.74$, $s^2 = 14.37$ and $c = 1.753$, such that $P(T_{15} \geq c) = \frac{\alpha}{2} = 0.05$, from the table of the t_{15}-distribution.
Substitution into the formula:

$$90\%\text{-}PI(X_{17}) = \left(28.25 - 1.753\cdot\sqrt{14.37}\sqrt{1+\tfrac{1}{16}},\ 26.25 + 1.753\cdot\sqrt{14.37}\sqrt{1+\tfrac{1}{16}}\right)$$
$$= (28.25 - 6.85, 28.25 + 6.85) = (21.40, 35.10)$$

b. 90% of the 16 observations are expected to be in the interval: 10% (1.6 observations) outside.
In this case none of the observations is outside the interval, which is not strange: if we have 16 trials and the success rate is 10%, the probability of no success is $0.9^{16} = 18.5\%$.

c. Requested: a 95%-CI with only a lower for μ with unknown σ (*using the formula sheet*) $\left(\overline{x} - c\frac{s}{\sqrt{n}}, \infty\right)$, where $c = 1.753$, such that $P(T_{15} \geq c) = \alpha = 0.05$, from the t_{15}-distribution.
Substitute into the formula: $95\% - CI(\mu) = \left(28.25 - 1.753\cdot\frac{\sqrt{14.37}}{\sqrt{16}},\ \infty\right)$
$\approx (26.6, \infty)$

("*The expected pressure resistance is at least 26.6, at a 95%-confidence level*".)

Exercise 11

a. Requested: a 90%-PI for a new observation X_{201} (assuming that $X_1, \ldots X_{200}, X_{201}$ is a random sample from a normal distribution with unknown μ and σ^2) has bounds: $\bar{x} \pm cs\sqrt{1+\frac{1}{n}}$,
where $n = 200$, $\bar{x} = 47.3$, $s = 4.0$ and $c = 1.645$, such that $P(T_{199} \geq c) = \frac{\alpha}{2} = 0.05$, from the table of the t_∞-distribution, or use the approximate standard normal distribution: $\Phi(c) = 0.95$
Then the bounds are $47.3 \pm 1.645 \cdot 4.0\sqrt{1+\frac{1}{200}}$, so 90%-$PI(X_{17}) \approx (40.7, 53.9)$
(*We are 90% confident that John works between 40.7 and 53.9 hours per week*).

b. Using the result of exercise 6a: $CI(\sigma^2) \overset{\text{CLT}}{\approx} \left(\frac{(n-1)S^2}{(n-1)+c\sqrt{2(n-1)}}, \frac{(n-1)S^2}{(n-1)-c\sqrt{2(n-1)}}\right)$, where $\Phi(c) = 1 - \frac{1}{2}\alpha$.
We have $n = 200$, $s = 4.0$ and $c = 1.645$, since $\Phi(c) = 1 - \frac{1}{2}\alpha = 0.95$.

$$90\% - CI(\sigma) = \left(\sqrt{\frac{199 \cdot 4.0^2}{199 + 1.645\sqrt{398}}}, \sqrt{\frac{199 \cdot 4.0^2}{199 - 1.645\sqrt{398}}}\right) \approx (3.71, 4.38)$$

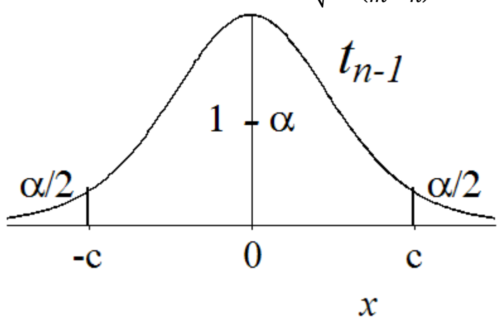
Exercise 12

Copying the approach in section 3.5:

- We assume that all observations are independent and all $N(\mu, \sigma^2)$. In particular $\bar{X}$ and $\bar{Y}$ are independent.
- $\bar{X}$ serves as a prediction of $\bar{Y}$: $\bar{Y} - \bar{X}$ is the prediction error, with
  Expectation $E(\bar{Y} - \bar{X}) = E(\bar{Y}) - E(\bar{X}) = \mu - \mu = 0$ and
  and variance $var(\bar{Y} - \bar{X}) \overset{\text{ind.}}{=} var(\bar{Y}) + var(\bar{X}) = \frac{\sigma^2}{m} + \frac{\sigma^2}{n} = \sigma^2\left(\frac{1}{m} + \frac{1}{n}\right)$

$$So\text{: } X_{n+1} - \bar{X} \sim N\left(0, \sigma^2\left(\frac{1}{m} + \frac{1}{n}\right)\right) \quad \text{and} \quad \frac{\bar{Y} - \bar{X}}{\sqrt{\sigma^2\left(\frac{1}{m} + \frac{1}{n}\right)}} \sim N(0,1)$$

Substituting S^2 for σ^2 results in a t-distributed "pivot" with $n - 1$ degrees of freedom.
(Verify: $T = \frac{Z}{\sqrt{W/n}} = \frac{\bar{Y}-\bar{X}}{\sqrt{S^2\left(\frac{1}{m}+\frac{1}{n}\right)}} \sim t_{n-1}$ if we choose $Z = \frac{\bar{Y}-\bar{X}}{\sqrt{\sigma^2\left(\frac{1}{m}+\frac{1}{n}\right)}} \sim N(0,1)$ and $W = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$)



$$P\left(-c < \frac{\bar{Y} - \bar{X}}{\sqrt{S^2\left(\frac{1}{m} + \frac{1}{n}\right)}} < c\right) = 1 - \alpha$$

$$\Leftrightarrow P\left(-c\sqrt{S^2\left(\frac{1}{m} + \frac{1}{n}\right)} < \bar{Y} - \bar{X} < c\sqrt{S^2\left(\frac{1}{m} + \frac{1}{n}\right)}\right) = 1 - \alpha$$

$$\Leftrightarrow P\left(\bar{X} - c\sqrt{S^2\left(\frac{1}{m} + \frac{1}{n}\right)} < \bar{Y} < \bar{X} + c\sqrt{S^2\left(\frac{1}{m} + \frac{1}{n}\right)}\right) = 1 - \alpha$$

$$(1-\alpha)100-PI(\overline{Y})=\left(\overline{X}-c\sqrt{S^2\left(\frac{1}{m}+\frac{1}{n}\right)}\,,\ \overline{X}+c\sqrt{S^2\left(\frac{1}{m}+\frac{1}{n}\right)}\right)$$

Note that for $m=1$ this formula gives the prediction interval for one observation and for $m\rightarrow\infty$ we find the confidence interval of μ ($\overline{Y}$ converges to μ).