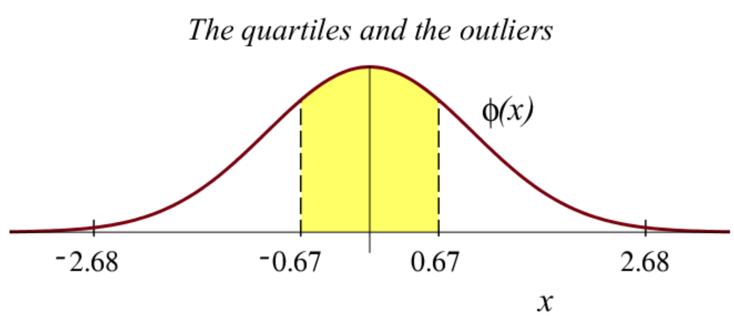
Solutions Chapter 1 (Descriptive Statistics) – Mathematical Statistics

1.

	Sample mean $\overline{x}$	Sample median M	Sample standard deviation s	Sample variance s^2
a.	2.5	3	3.15	9.9
b.	3.08	3	1.19	1.41
c.	49.6	49	8.77	76.93

2. The z-score of 40 is -2, so $\frac{40-\overline{x}}{s} = -2$. And the z-score of 90 is 3, so $\frac{90-\overline{x}}{s} = 3$.
This information leads to two equations with the unknown μ and σ: $40 - \overline{x} = -2s$ and
$90 - \overline{x} = 3s$.
Solving the equations we find: $s = 10$ and $\overline{x} = 60$.
(But you can also argue that the difference between the observations 90 and 40 is 5 standard deviations (2 to the left and 3 to the right of $\overline{x}$*), so* $s = \frac{90-40}{5} = 10$ *and the sample mean is* $40 + 2 \times 10 = 60$*)*

3.



a. $P(Z \leq Q_3) = 0.75$ if $Q_3 \approx 0.67$ (N(0,1)-table)
Because of symmetry we have:
$$Q_1 = -Q_3 \approx -0.67$$

b. $IQR = Q_3 - Q_1 = 1.34$, so
$(Q_1 - 1.5 \times IQR, Q_3 + 1.5 \times IQR)$
$= (-0.67 - 1.5 \times 1.34, 0.67 + 2.01)$
$= (-2.68, +2.68)$

c. Using symmetry: $2 \times P(Z \geq Q_3 + 1.5 \times IKA) = 2 \times P(Z \geq 2.68) = 2[1 - 0.9963] = 0.74\%$

d. The cdf of the exponential distribution is $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$ $(= \int_{-\infty}^{x} f(u)du)$

$F(x) = 0.75$ if $x = -\frac{\ln(0.25)}{\lambda} = \frac{\ln(4)}{\lambda}$. And $F(x) = 0.25$ for $x = -\frac{\ln(0.75)}{\lambda} = \frac{\ln\left(\frac{4}{3}\right)}{\lambda}$,

So $IQR = Q_3 - Q_1 = \frac{\ln(4)}{\lambda} - \frac{\ln\left(\frac{4}{3}\right)}{\lambda} = \frac{\ln(3)}{\lambda}$

$$(Q_1 - 1.5 \times IQR, Q_3 + 1.5 \times IQR) = \left(\frac{\ln\left(\frac{4}{3}\right)}{\lambda} - 1.5 \cdot \frac{\ln(3)}{\lambda}, \frac{\ln(4)}{\lambda} + 1.5 \cdot \frac{\ln(3)}{\lambda}\right)$$
$$= \left(\frac{\ln\left(\frac{4}{3} \cdot \frac{1}{3^{1.5}}\right)}{\lambda}, \frac{\ln(4 \cdot 3^{1.5})}{\lambda}\right) = \left(\frac{\ln\left(\frac{4}{27}\sqrt{30}\right)}{\lambda}, \frac{\ln(12\sqrt{3})}{\lambda}\right)$$

The left bound is negative so the probability of an outlier is
$P\left(X \geq \frac{\ln(12\sqrt{3})}{\lambda}\right) = e^{-\lambda \cdot \frac{\ln(12\sqrt{3})}{\lambda}} = \frac{1}{12\sqrt{3}} \approx 4.8\%$: larger than for a normal distribution (0.74%).

4.

a. First we will determine the 25th and the 75th percentile: $Q_1 = x_{(10)} = 141$ and $Q_3 = x_{(29)} = 154$,
since 25% (75%) of $n = 38$ is 9.5 (28.5).
The Inter Quartile Range $IQR = 154 - 141 = 13$
Possible outliers are outside $(Q_1 - 1.5 \times IQR, Q_3 + 1.5 \times IQR) = (121.5, 173.5)$:
we have one (potential) **outlier: 121**

b. 1. The numerical summary: sample skewness coefficient = -0.41 deviates somewhat from 0, the reference value of the normal distribution. The kurtosis = 3.09 is close to the reference value 3.
2. The histogram reveals a distribution which follows the normal distribution roughly, though there

seem to be two peaks. But the histogram is roughly symmetric and mound shaped.

3. The **normal** Q-Q plot shows points around the straight line $y = x$: normality seems reasonable.

The total "verdict" based on these three aspects, is that the normal distribution can be assumed a model of these observations.

5.

a. $E(X_i) = \frac{1}{2}$ (symmetry), $E(X_i^2) = \int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_{x=0}^{x=1} = \frac{1}{3}$, so $var(X) = E(X^2) - (EX)^2 = \frac{1}{12}$.

$\gamma_2 = \frac{E(X-\mu)^4}{\sigma^4}$, where $\sigma^4 = \left(var(X)\right)^2 = \frac{1}{144}$ and

$$E(X-\mu)^4 = \int_0^1 \left(x - \frac{1}{2}\right)^4 \cdot 1dx = \left[\frac{1}{5}\left(x-\frac{1}{2}\right)^5\right]_{x=0}^{x=1} = \frac{1}{5}\left[\frac{1}{2^5} - \left(-\frac{1}{2^5}\right)\right] = \frac{1}{80}$$

Hence $\gamma_2 = \frac{E(X-\mu)^4}{\sigma^4} = \frac{144}{80} = 1.8$.

b. $F_{X_1}(x) = \int_0^x 1du = [u]_{u=0}^{u=x} = x$ (if $0 \le x \le 1$). $F_{X_1}(x) = 0$ for $x < 0$ and $F_{X_1}(x) = 1$ for $x > 1$.

c. $F_Y(y) = P(Y \le y) = P(-2\ln(X_1) \le y) = P\left(\ln(X_1) \ge -\frac{1}{2}y\right)$

$= P\left(X_1 \ge e^{-\frac{1}{2}y}\right) = 1 - F_{X_1}\left(e^{-\frac{1}{2}y}\right) = 1 - e^{-\frac{1}{2}y}$, or $0 \le e^{-\frac{1}{2}y} \le 1$, implying that $y \ge 0$.

Conclusion: $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{2}e^{-\frac{1}{2}y}$ ($y \ge 0$).

This is the exponential density with parameter $\lambda = \frac{1}{2}$, so $E(Y) = \frac{1}{\lambda} = 2$.

d. $P(Z \le z) = P(\max(X_1, \dots, X_n) = P(X_1 \le z \text{ and } X_2 \le z \text{ and } \dots \text{and } X_n \le z)$

$\overset{\text{ind.}}{=} P(X_1 \le z) \cdot \dots \cdot P(X_n \le z) = z^n, \quad \text{if } 0 \le z \le 1$

So $f_Z(z) = \frac{d}{dz}[z^n] = nz^{n-1}$, for $0 \le z \le 1$ (and $f_Z(z) = 0$ elsewhere)

$E(Z) = \int_{-\infty}^{\infty} zf(z)\,dz = \int_0^1 z \cdot nz^{n-1}dz = \left[\frac{n}{n+1}z^{n+1}\right]_{z=0}^{z=1} = \frac{n}{n+1}$.

$E(Z^2) = \int_0^1 z^2 \cdot nz^{n-1}dz = \left[\frac{n}{n+2}z^{n+2}\right]_{z=0}^{z=1} = \frac{n}{n+2}$, so $(Z) = \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2 = \frac{n}{(n+2)(n+1)^2}$.

6. (We are using the notation X instead of X_1 in a.)

a. $E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} x \cdot \lambda e^{-\lambda x}dx = ..integr.by\ parts \dots = \left[x \cdot -e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx$

$= \quad 0 \quad + \quad \frac{1}{\lambda}$

$E(X^2) = \int_0^{\infty} x^2 \cdot \lambda e^{-\lambda x}dx = \dots = \frac{2}{\lambda^2}$, so $var(X) = E(X^2) - (EX)^2 = \frac{1}{\lambda^2}$ and $\sigma = \frac{1}{\lambda}$.

$E(X^3) = \int_0^{\infty} x^3 \cdot \lambda e^{-\lambda x}dx = \left[x^3 \cdot -e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} 3x^2 e^{-\lambda x}\,dx = 0 + \frac{3}{\lambda} \cdot E(X^2) = \frac{6}{\lambda^3}$.

$E(X-\mu)^3 = E(X^3 - 3\mu \cdot X^2 + 3\mu^2 \cdot X - \mu^3) = E(X^3) - 3\mu \cdot E(X^2) + 2\mu^3 = \frac{6-6+2}{\lambda^3} = \frac{2}{\lambda^3}$.

So $\gamma_1 = \frac{E(X-\mu)^3}{\sigma^3} = \frac{2/\lambda^3}{\left(\frac{1}{\lambda}\right)^3} = 2$.

b. For $n = 100$ and $\lambda = 2$, S is approximately (CLT) $N(n\mu, n\sigma^2) = N(50, 25)$.

So $P(S > 55)1 - P(S \le 55) \overset{\text{CLS}}{\approx} 1 - \Phi\left(\frac{55-50}{\sqrt{25}}\right) = 1 - \Phi(1) = 1 - 0.8413 = 15.87\%$.

c. For $n = 50$ and $\lambda = 2$, $\overline{X}$ is approximately $N\left(\mu, \frac{\sigma^2}{n}\right) = N\left(25, \frac{1}{200}\right)$.

$$P(\overline{X} > 0.55) \overset{\text{CLT}}{\approx} 1 - \Phi\left(\frac{0.55-0.50}{\sqrt{\frac{1}{200}}}\right) = 1 - \Phi(0.71) = 0.2389.$$

d. $F_M(m) = P(\min(X_1, \dots, X_n) \le m) = 1 - P(\min(X_1, \dots, X_n) \ge m)$

$= 1 - P(X_1 \ge m \text{ and } \dots \text{ and } X_n \ge m)$

$$\overset{\text{ind.}}{=} 1 - P(X_1 \geq m) \cdot \ldots \cdot P(X_n \geq m) = 1 - \left[e^{-\lambda m}\right]^n = 1 - e^{-n\lambda m} \ (m \geq 0)$$

$f_M(m) = n\lambda e^{-n\lambda m}$ (≥ 0), so M is exponentially distributed with parameter $n\lambda$.

For $n = 10$ and $\lambda = 2$, $E(M) = \frac{1}{10 \cdot 2} = 0.05$.

7. a. $X + Y \sim Poisson(\mu = 3 + 4)$.

You can state this if you know this property: if X and Y are independent and both Poisson-distributed with parameters μ_X and μ_Y, resp., then $Z = X + Y$ has a Poisson distribution with parameter $\mu_X + \mu_Y$. This is proven as follows (using the **convolution sum,** here with $\mu_X = 3$ and $\mu_Y = 4$.):

$$P(Z = n) \overset{ind.}{=} \sum_{k=0}^{n} P(X = k) \cdot P(Y = n - k) \text{ (using the assumed independence of } X \text{ and } Y).$$

$$= \sum_{k=0}^{n} \frac{3^k\, e^{-3}}{k!} \cdot \frac{4^{n-k}\, e^{-4}}{(n-k)!} = e^{-7} \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} 3^k 4^{n-k} = \frac{e^{-7}}{n!} \sum_{k=0}^{n} \binom{n}{k} 3^k 4^{n-k}$$

In the last summation we recognize Newton`s Binomial Theorem: $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$:

$$P(Z = n) = \frac{e^{-7}}{n!} \cdot (3+4)^n = \frac{7^n e^{-7}}{n!}, \quad \text{for } n = 0, 1, 2, \ldots.$$

Alternatively one can use moment generating functions: for the Poisson distribution of N we have:

$$M_N(t) = E(e^{tN}) = \sum_{n=0}^{\infty} e^{tn} \cdot \frac{\mu^n}{n!} e^{-\mu} = \sum_{n=0}^{\infty} \frac{(e^t\mu)^n}{n!} e^{-\mu} = e^{-\mu} e^{e^t\mu} \sum_{n=0}^{\infty} \frac{(e^t\mu)^n}{n!} e^{-e^t\mu} = e^{-\mu(1-e^t)}$$

For independent $X \sim Poisson(\mu_1)$ and $Y \sim Poisson(\mu_2)$, we have

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = e^{-\mu_1(1-e^t)} \cdot e^{-\mu_2(1-e^t)} = e^{-(\mu_1+\mu_2)(1-e^t)}$$

Conclusion: $X + Y \sim Poisson(\mu_1 + \mu_2)$ (*proof is easily extended to n independent* $X_i \sim Poisson(\mu_i)$)

b. Since X and Y are independent we can apply the convolution integral, but we have to take into account that $f_X(x)$ and $f_Y(y)$ are both 1 if both x and y are between 0 and 1, otherwise $f_X(x) \cdot f_Y(y) = 0$.
Distinguishing $z = x + y$ between 0 and 1, between 1 and 2 and all other values we find:

$0 \leq z \leq 1$: $f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx = \int_0^z 1 \cdot 1dx = [x]_{x=0}^{z} = z$

$1 < z \leq 2$: $f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx = \int_{z-1}^{1} 1 \cdot 1dx = [x]_{x=z-1}^{1} = 1 - (z-1) = 2 - z$

$$So: f(x) = \begin{cases} z & \text{if } 0 \leq z \leq 1 \\ 2 - z & \text{if } 1 \leq z \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

c. $X + Y \sim B(m + n, p)$.

This a property of the binomial distribution which can be proven as follows:
Suppose that $X_1, \ldots X_m, X_{m+1}, \ldots, X_{m+n}$ are independent alternatives with success probability p, then $X = \sum_{i=1}^{m} X_i \sim B(m,p)$, $Y = \sum_{i=m+1}^{m+n} X_i \sim B(n,p)$ and $X + Y = \sum_{i=1}^{m+n} X_i \sim B(m+n,p)$.

d. $X + Y \sim N(20 + 30, 81 + 144)$.

According a well known property of the normal distribution.
(Using the moment generating of a $N(\mu, \sigma^2)$-distribution, $M(t) = e^{\mu t + \sigma^2 t^2/2}$, it easily shown that the sum $X + Y$ of two independent $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ has a $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$-distribution)

e. $X \sim Exp(\lambda = 2)$ and $Y \sim Exp(\lambda = 3)$ are independent, so:

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx \overset{z \geq 0}{=} \int_0^z 2e^{-2x} \cdot 3e^{-3(z-x)}dx$$

$$= 6e^{-3z} \int_0^z e^x dx = 6e^{-3z} \cdot [e^x]_{x=0}^{z} = 6e^{-3z} \cdot (e^z - 1), \qquad \text{for } z > 0$$

And $f_{X+Y}(z) = 0$ if $z < 0$.

This is a density function since $f_{X+Y}(z) \geq 0$ and
$\int_{-\infty}^{\infty} f_{X+Y}(z)dz = \int_0^{\infty} (6e^{-2z} - 6e^{-3z})dz = [-3e^{-2z} + 2e^{-3z}]_{x=0}^{\infty} = 0 - (-3 + 2) = 1$.