Slides Probability Theory
Chapter 8

Ch. 8: Distributions of waiting times

Discrete: geometric waiting time

If every trial takes a fixed unit of time, then the total (number of units of) time needed to attain a success has a geometric distribution:

$$\boldsymbol{P(X=k)=(1-p)^{k-1}p} \text{ , met } \boldsymbol{k} = \mathbf{1, 2, \ldots.}$$

Properties:

$$\boldsymbol{E(X)=\frac{1}{p}, var(X)=\frac{1-p}{p^2}} \text{ and } \boldsymbol{P(X>k)=(1-p)^k}$$

The **lack of memory property** of the geometric distribution: $\boldsymbol{P(X>s+t|X>s)=P(X>t)}$

for all t and s in S_X

In words: “the number of trials in the past does not influence the probability of success in future.”

Continuous waiting times: exponential distribution

$\boldsymbol{f_X(x) = \lambda e^{-\lambda x}}$, with $x \geq 0$ $(f_X(x) = 0, x < 0)$

Properties:

$\boldsymbol{E(X) = \frac{1}{\lambda}, var(X) = \frac{1}{\lambda^2}}$ and $\boldsymbol{P(X > x) = e^{-\lambda x}}$

The exponential distribution has **the lack of memory property: "the remaining waiting time does not depend on the elapsed waiting time":**

$\boldsymbol{P(X > s + t | X > s) = P(X > t)}$ for all t and s in S_X

The **geometric** distribution is the **only discrete,** and the **exponential** distribution the **only continuous** distribution that has the lack of memory property.

Sums of independent waiting times $X_1, \dots, X_n$

if the X_i\`s are exponentially distributed with par. λ

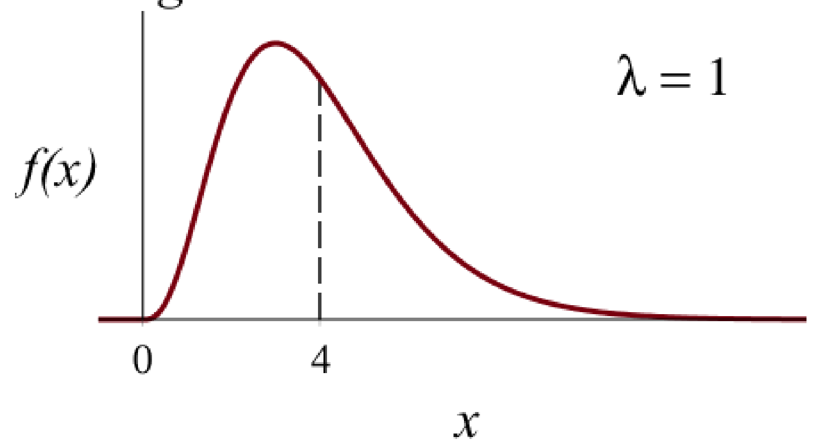
so the total waiting time is $\boldsymbol{S_n = \sum_{i=1}^{n} X_i}$

Properties:

1. $\boldsymbol{E(S_n) = \frac{n}{\lambda}}$ and $\boldsymbol{var(S_n) = \frac{n}{\lambda^2}}$
2. for large n (> 25) the CLT implies: $\boldsymbol{S_n \sim N\left(\frac{n}{\lambda}, \frac{n}{\lambda^2}\right)}$
3. $$\boldsymbol{f_{S_n}(s) = \frac{\lambda(\lambda s)^{n-1} e^{-\lambda s}}{(n-1)!}}$$
   $$\boldsymbol{(s \geq 0)}$$

The Erlang distribution with parameters *n* and λ.

Erlang distribution with n = 4 and



Poisson process

for service times / waiting times / interarrival times

If $X_1, \ldots, X_n$ are **independent** and **exponentially distributed** with parameter λ, then we have:

- The total $S_n = \sum X_i$ has an **Erlang distribution** with **parameters $\boldsymbol{n}$** and **λ.**
- The number of arrivals (serviced customers) in the period $[0, s]$ has **Poisson** distribution with $\mu = \boldsymbol{\lambda s}$.

The last property can be proven, by deriving:

$$\boldsymbol{F_{S_n}(s) = 1 - \sum_{k=0}^{n-1} \frac{(\lambda s)^k e^{-\lambda s}}{k!} \quad (s \geq 0)}$$