Slides
Probability Theory
Chapter 6

Ch. 6: Continuous distributions

Uniform, exponential and normal distribution

Functions of a r.v., such as $Y = X^2$

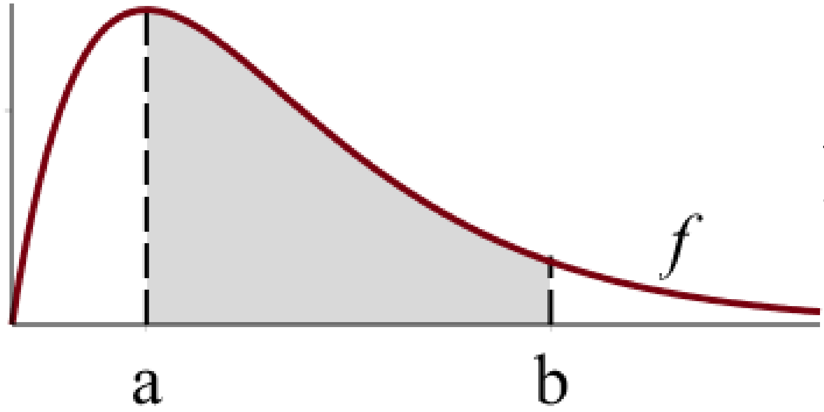
The **density function** $\boldsymbol{f(x)}$ of a continuous r.v. X

For a **continuous** random variable X we have:

$\boldsymbol{S_X}$ **is an interval** and $\boldsymbol{P(X = x) = 0}$

The distribution of X is defined now by the density function $\boldsymbol{f}$**: probabilities are areas** under the graph of f

$$\boldsymbol{P(a < X < b) = \int_a^b f(x)dx}$$



Requirements of $\boldsymbol{f}$:

1. $\boldsymbol{f(x) \geq 0}$
2. $\boldsymbol{\int_{-\infty}^{\infty} f(x)dx = 1}$

The **distribution function** $\boldsymbol{F(x)}$ of a **discrete** r.v. X

<u>Voorbeeld:</u>

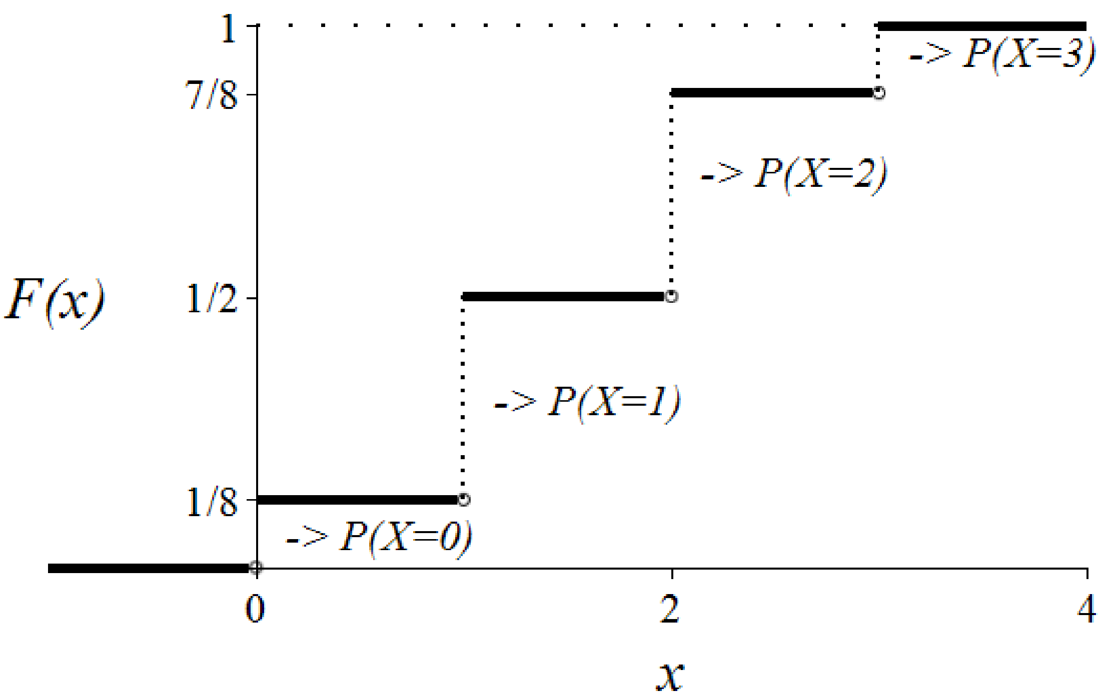
$X \sim$ B(3, ½)

$\boldsymbol{x}$	**0**	**1**	**2**	**3**	***som***
$\boldsymbol{P(X=x)}$	**1/8**	**3/8**	**3/8**	**1/8**	***1***

$\boldsymbol{F(x) = P(X \le x)}$, e.g.: $F(1) = P(X \le 1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

and $F(1.2) = P(X \le 1.2) = P(X \le 1) = \frac{1}{2}$

The distribution function of the B(3,1/2)-distribution



Note that $\boldsymbol{F(x)}$, the *cumulative probabilities* $\boldsymbol{P(X \le x)}$ of the $\boldsymbol{B(n, p)}$- and the Poisson distributions, are given in the tables.

The distribution function is a probability:

$$\boldsymbol{F(x) = P(X \leq x)}$$

For a continuous r.v. X we have:

$$\boldsymbol{F(x) = \int_{-\infty}^{x} f(u)du} \text{ and reverse: } \boldsymbol{f(x) = \frac{d}{dx}F(x)}$$

Properties of $F(x)$, discrete ànd continuous:

1. F is a **non-decreasing** function

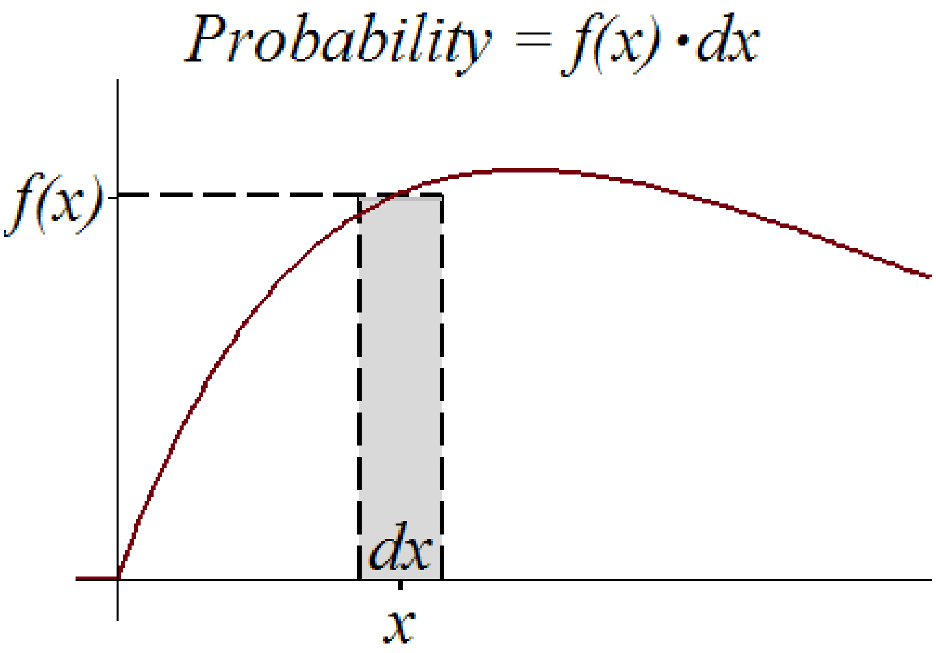
2. $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$

3. F is **continuous from the right**

for any continuous r.v. X

the distribution function $\boldsymbol{F}$ **continuous**

Expectation of a continuous r.v. X



Expectation = "Weighted average"

$$x \cdot f(x)dx$$

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

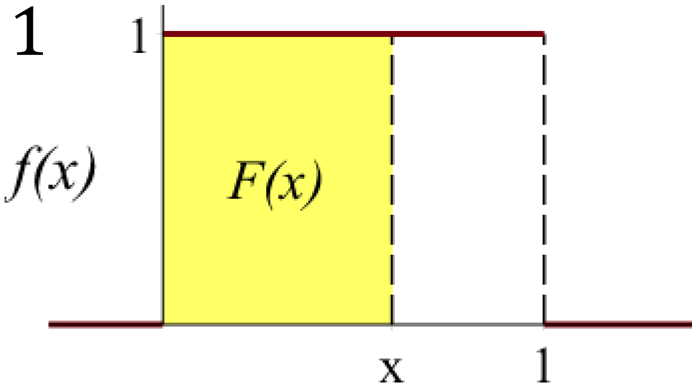
All properties of $E(X)$ remain the same!

If $f(x)$ **is symmetrical** about $x = c$, then $E(X) = c$

Example: *X* = "Random number between 0 en 1"

- $f(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$
- Total area = 1
- $F(x) = P(X \leq x) = x \cdot 1, \text{ if } 0 \leq x \leq 1$

  Dus $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$



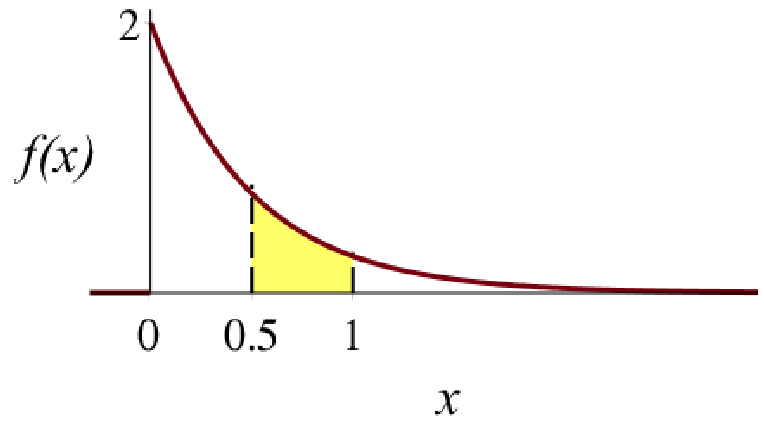
- $E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 x dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$

  Follows immediately from the symmetry of f!

- $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$

  $var(X) = E(X^2) - (EX)^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$. So $\sigma_X = \sqrt{\frac{1}{12}}$

Example: *X* = "Life time of a smartphone screen"

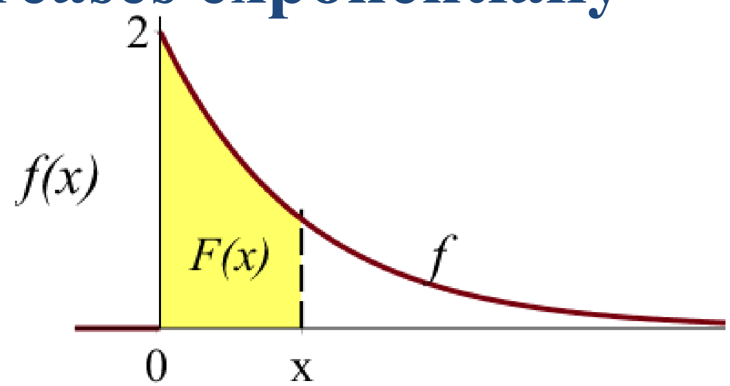


- $f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$
- $\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} 2e^{-2x}dx$
  $= [-e^{-2x}]_0^{\infty} = 0 - (-1) = 1$
- $P(0.5 < X < 1) = \int_{0.5}^{1} 2e^{-2x}dx = [-e^{-2x}]_{0.5}^{1} = e^{-1} - e^{-2}$
- $P(X > x) = \int_x^{\infty} 2e^{-2u}du = [-e^{-2u}]_x^{\infty} = e^{-2x} \quad (x \geq 0)$

"The probability of survival decreases exponentially"

- $F(x) = P(X \leq x)$
  $= 1 - P(X > x)$
  $= 1 - e^{-2x}$, als $x \geq 0$

  En $F(x) = 0$, als $x < 0$



Continuation Ex.: X = "Life time of a smartphone screen"

- Using the definition we find:

  $$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{\infty} x \cdot 2e^{-2x}\,dx$$

- *This integral can be determined using* ***partial integration****:*

  $$\int_a^b \boldsymbol{f(x)g'(x)dx} = [\boldsymbol{f(x)g(x)}]_{x=a}^{x=b} - \int_a^b \boldsymbol{f'(x)g(x)dx}$$

  (*reverse of the product rule for derivatives:* $(\boldsymbol{fg})' = \boldsymbol{f'g} + \boldsymbol{fg'}$)

- In the example we choose: $f(x) = x$ and $g'(x) = 2e^{-2x}$, for which the anti-derivative is $g(x) = -e^{-2x}$

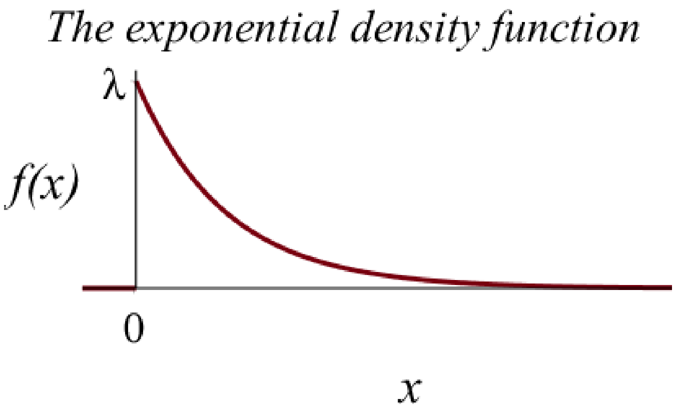
- $$\int_0^{\infty} x \cdot 2e^{-2x}\,dx = x \cdot -e^{-2x}|_0^{\infty} + \int_0^{\infty} e^{-2x}\,dx$$

  $$= (0 - 0) \qquad + \left[-\frac{1}{2}e^{-2x}\right]_{x=0}^{x=\to\infty} = +\frac{1}{2}$$

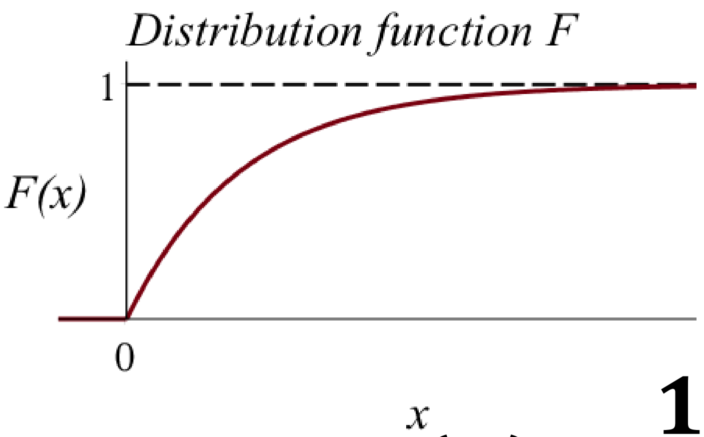
- $E(X^2) = \int_0^{\infty} x^2 \cdot 2e^{-2x}\,dx$: *(twice) partial integration*

Exponential distribution with parameter λ

Applications: waiting / life / interarrival times



$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



$$P(X > x) = e^{-\lambda x}, \quad x \geq 0$$

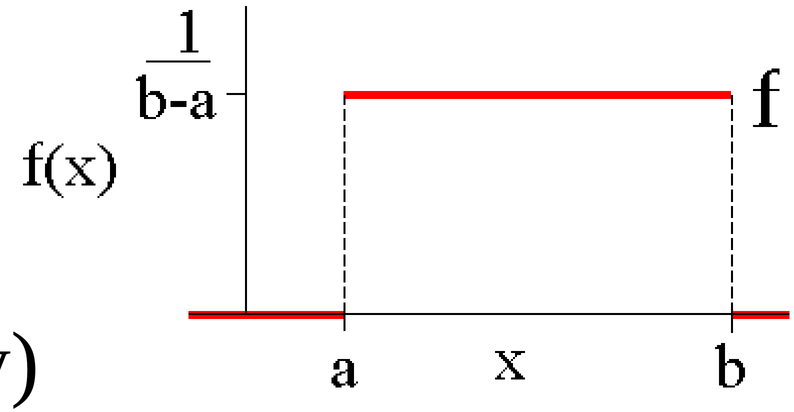
$$F(x) = P(X \leq x)$$
$$= 1 - e^{-\lambda x}, \quad x \geq 0$$

$$E(X) = \frac{1}{\lambda} \; and \; var(X) = \frac{1}{\lambda^2}, so \; \sigma_X = \frac{1}{\lambda}$$

Intensity **large ⇒ expected time small**

$U(a,b)$: **Uniform distribution** on the interval [a, b]

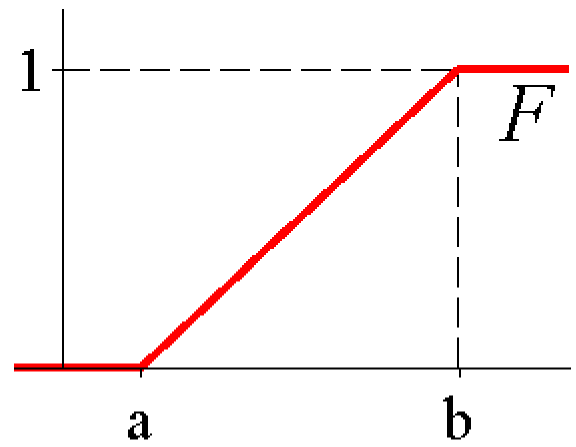
$$f(x) = \begin{cases} \frac{1}{b-a}, \text{if } a \le x \le b \\ 0, \quad \text{elsewhere} \end{cases}$$



$$E(X) = \frac{a+b}{2} \text{ (line of symmetry)}$$

$$var(X) = \frac{(b-a)^2}{12}$$

$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1, & x > a \end{cases}$$



Application: random numbers, drawn from an interval, usually [0, 1], often used in simulations

$X \sim U(0,1)$: what is the distribution of $Y = 3X + 4$?

1. $F_Y(y) = P(Y \leq y) = P(3X + 4 \leq y)$

$$= P\left(X \leq \frac{y-4}{3}\right) = F_X\left(\frac{y-4}{3}\right)$$

F_Y expressed in F_X

2. $f_Y(y) = \frac{d}{dy} F_Y(y)$, **apply the chain rule (!):**

$$f_Y(y) = f_X\left(\frac{y-4}{3}\right) \cdot \frac{1}{3}$$

f_Y expressed in f_X

3. Since $f_X(x) = 1$, for $0 \leq x \leq 1$, we have:

$$f_Y(y) = 1 \cdot \frac{1}{3}, \text{ if } 0 \leq \frac{y-4}{3} \leq 1, \text{so if } 4 \leq y \leq 7$$

Conclusion: $Y = 3X + 4 \sim U(4,7)$

Functions of a continuous r.v. $X : Y = g(X)$

E.g. $Y = \ln(X)$, $Y = eX$, $Y = |X|$ or $Y = X^2$,
for a given distribution ($f_X(x)$) of X

We derive $f_Y(y)$ in **three steps**:

1. Express the distribution function $\boldsymbol{F_Y(y)}$ **in** $\boldsymbol{F_X}$:
use $\boldsymbol{F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \ldots}$

2. Express $\boldsymbol{f_Y(y)}$in $\boldsymbol{f_X}$, using the derivative of $F_Y(y)$

3. Apply the **given** $\boldsymbol{f_X(x)}$ to determine $f_Y(y)$.

If g is a **linear function:** $\boldsymbol{Y = g(X) = aX + b}$

then $$\boldsymbol{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

What is the distribution of $\boldsymbol{Y = -\frac{ln(X)}{2}}$, if $\boldsymbol{X \sim U(0, 1)}$?

1. $\boldsymbol{F_Y(y)} = P\left(-\frac{ln(X)}{2} \leq y\right) = P(ln(X) \geq -2y)$

$= P(X \geq e^{-2y}) = \boldsymbol{1 - F_X(e^{-2y})}$

2. $\boldsymbol{f_Y(y)} = \frac{d}{dy} F_Y(y) = +2e^{-2y} \cdot \boldsymbol{f_X\left(e^{-2y}\right)}$

3. Since $f_X(x) = 1$ for $0 \leq x \leq 1$, we have:

$f_Y(y) = 2e^{-2y}$, if $0 \leq e^{-2y} \leq 1$, so if $y \geq 0$.

Conclusion: $\boldsymbol{Y = -\frac{ln(X)}{2} \sim Exp(\lambda = 2)}$

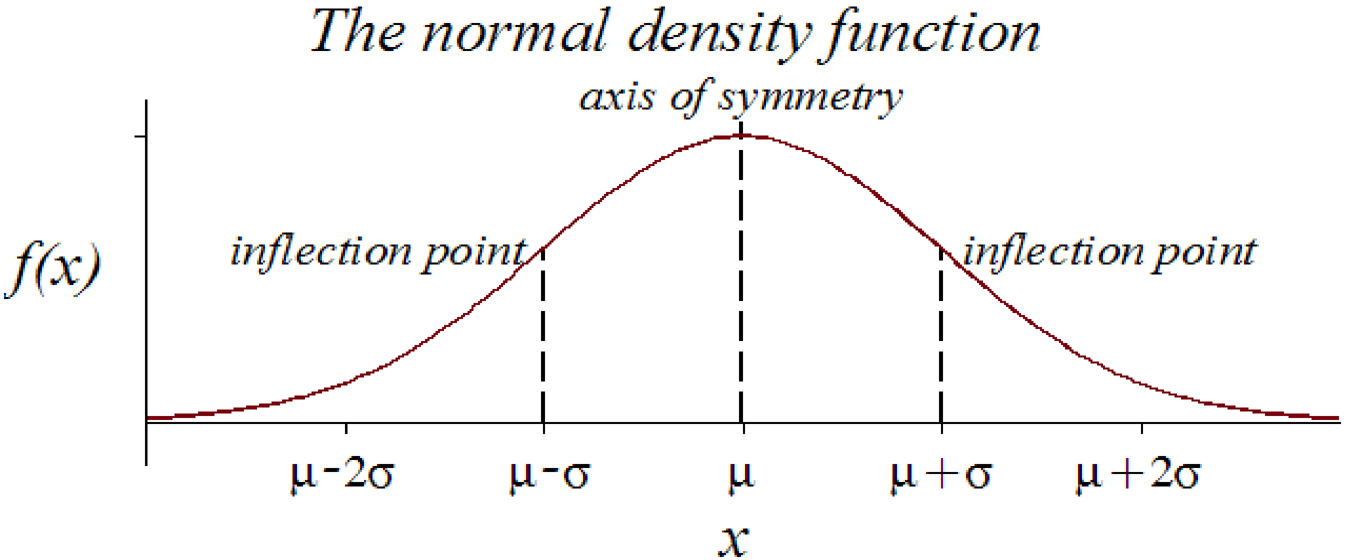
Generalization: $\boldsymbol{Y = -\frac{ln(X)}{\lambda} \sim Exp(\lambda)}$

(Application: simulation of $\boldsymbol{Exp(\lambda)}$-distributed times).

The density function of the **normal distribution** with **expectation μ** and **variance σ²**

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, x \in \mathbb{R}$$

Brief notation:
$\boldsymbol{X \sim N(\mu, \sigma^2)}$



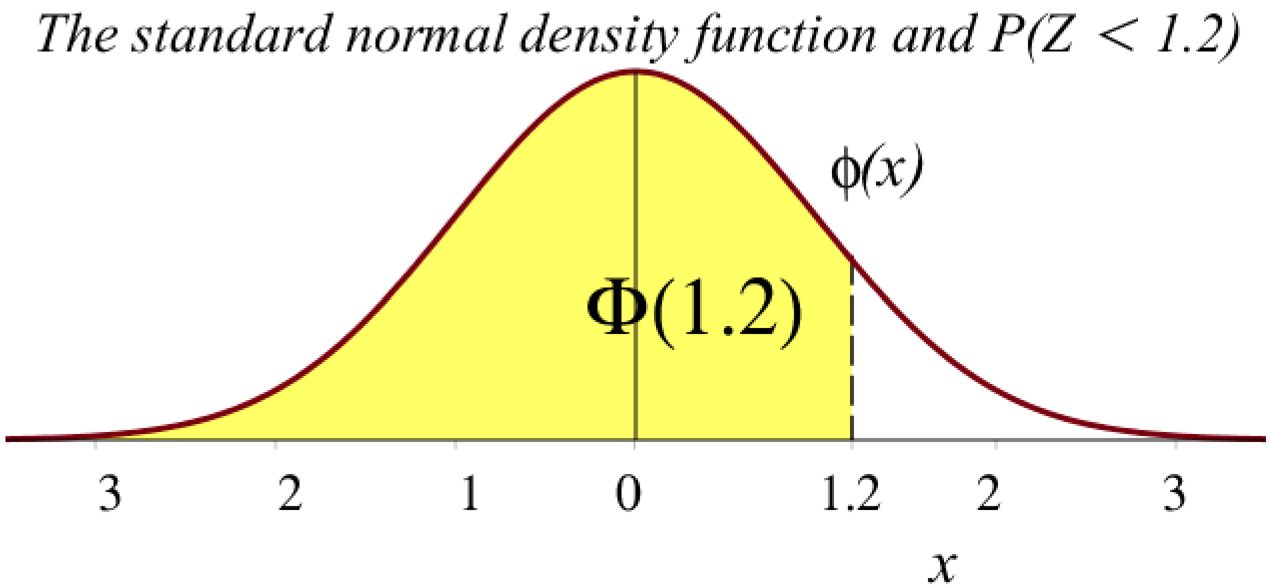
Special case**:** the **standard normal** distribution of $\boldsymbol{Z \sim N(0, 1)}$

<u>Applications:</u> "natural" quantities in nature, physics, economy, etc., such as: weights, lengths, times, IQ`s, profits.

Standard normal distribution: $Z \sim N(0, 1)$

$N(0,1)$**-density function** $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, **for all** z

So $\mu = 0$
(symmetry
about $z = 0$)
And $\sigma^2 = 1$



Distribution function $\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \varphi(z)dz$

The probabilities $\Phi(z) = P(Z \leq z)$ are numerically approximated and presented in the **standard normal table**, for positive values of z.

Numerical approximation of the $N(0,1)$-distr.

probabilities $\boldsymbol{P(Z \leq z) = \Phi(z)}$: in the $N(0,1)$-table for $z \geq 0$.

Probabilities for $X \sim N(\mu, \sigma^2)$ can be computed by **standardization:** $\boldsymbol{Z = \frac{X-\mu}{\sigma} \sim N(0,1)}$

<u>Example</u> for $X \sim N(20,4)$, so $\mu = 20$ and $\sigma = 2$.

$$\boldsymbol{P(X \leq 22.3) = P\left(\frac{X-20}{2} \leq \frac{22.3-20}{2}\right) = P(Z \leq 1.15)}$$
$$\boldsymbol{= \Phi(1.15)}$$
$$\boldsymbol{= 0.8749}$$

Linear transformation of $\boldsymbol{X}$:

$$\boldsymbol{X \sim N(\mu, \sigma^2) \implies aX + b \sim N(a\mu + b, a^2\sigma^2)}$$

Overview common continuous distributions

Name + parameters	Density function	$E(X)$	var(X)
Uniform $U(a,b)$	$f(x)=\frac{1}{b-a}, x\in[a,b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential $\mathrm{Exp}(\lambda)$	$f(x)=\lambda e^{-\lambda x}, x\geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Standard normal $N(0,1)$	$\varphi(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$	**0**	**1**
Normal $N(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2