Slides
Probability Theory
Chapter 2

Combinatorial probability

Computing probabilities combinatorically is possible when all outcomes are **equally likely: count the numbers.**

- The "**art of counting**" is applied often when we have a random sample taken from a finite population at hand: if we arbitrarily choose one element of the population, then every outcome has the **same probability of occurrence**.
- **Example:**
  A car company buys a group of 20 old VW Golf cars, of which the company does not know that 5 suffer from severe hidden defects.
  The company decides to check 4 of the 20 cars at random.

  What is the probability that:

  1. Two of the 4 checked cars suffer from severe defects?

  2. None of the 4 cars has severe defects?

Basic rules of combinatorics

The product rule: if partial experiment 1 has $\boldsymbol{m}$ outcomes and partial experiment 2 $\boldsymbol{n}$ outcomes (no matter what the result of experiment 1 was), then the combination of the two experiments has $\boldsymbol{m} \times \boldsymbol{n}$ outcomes.

"Drawing $\boldsymbol{n}$ **individuals from a population of** $\boldsymbol{N}$ **individuals**" can be executed in different ways:

- Drawing **with** or **without replacement**:
  With replacement → repeated outcomes possible
- Two kinds of outcomes:
  permutations: **ordered outcomes**
  combinations: unordered outcomes,
  subsets of the population

In statistics a **random sample of** $\boldsymbol{n}$ **out of** $\boldsymbol{N}$ usually consists of **combinations <u>without</u> replacement.**

Examples of counting

1. **We roll a dice 3 times:**
   3 partial experiments with each 6 equally likely outcomes:
   Every (ordered) outcome, such as (1, 5, 1) has probability $\frac{1}{216}$,
   since there are $\mathbf{6 \times 6 \times 6 = 216}$ outcomes.
2. **How many lists of the 6 participants of a song contest are possible, if the order of appearance is randomly chosen?**
   Determine the number of permutations - without replacement:
   $\mathbf{6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720}$. **Notation: 6! ("6 factorial")**
3. Generalization**:** the number of lists with $\boldsymbol{n}$ **participants** is $\boldsymbol{n}$!
4. **How many lists of 6 persons are there if we choose them arbitrarily from a group of 15 persons** (without replacement)?
   15 outcomes for the 1st choice, 14 for the 2nd, etc.:

$$15 \times 14 \times 13 \times 12 \times 11 \times 10 = \frac{15!}{9!}$$

Numbers of **permutations and combinations of n out of N**

- The **numbers of permutations** (variations) of $\boldsymbol{n}$ **out of** $\boldsymbol{N}$

$$N \times (N-1) \times \cdots \times (N-n+1) = \frac{N!}{(N-n)!}$$

Calculator: use the button *n*P*r*

- The **numbers of combinations of 6 out of 15**
(= the numbers **subsets of 6** elements chosen out of 15):

For **each** combination of 6 there are 6! permutations.
So: the number of permutations = (# of combinations) × 6!

The number of combinations $= \dfrac{\frac{15!}{9!}}{6!} = \dfrac{15!}{6!9!}$

- In general: the **number of combinations of *n* out of *N*** is

$$\frac{(\#permutations\ of\ n\ out\ of\ N)}{n!} = \frac{N!}{n!(N-n)!}$$ **(say: "*N* choose *n*")**

Notation: *binomial coefficient* $\binom{N}{n}$, calculator button: ***n*C*r***

Combinatorics and probabilities of sample results

If a **combination** of $\boldsymbol{n}$ person is chosen arbitrarily and without replacement from a population of N persons, then the probability of each combination of n persons is the same: $\dfrac{\mathbf{1}}{\binom{\boldsymbol{N}}{\boldsymbol{n}}}$

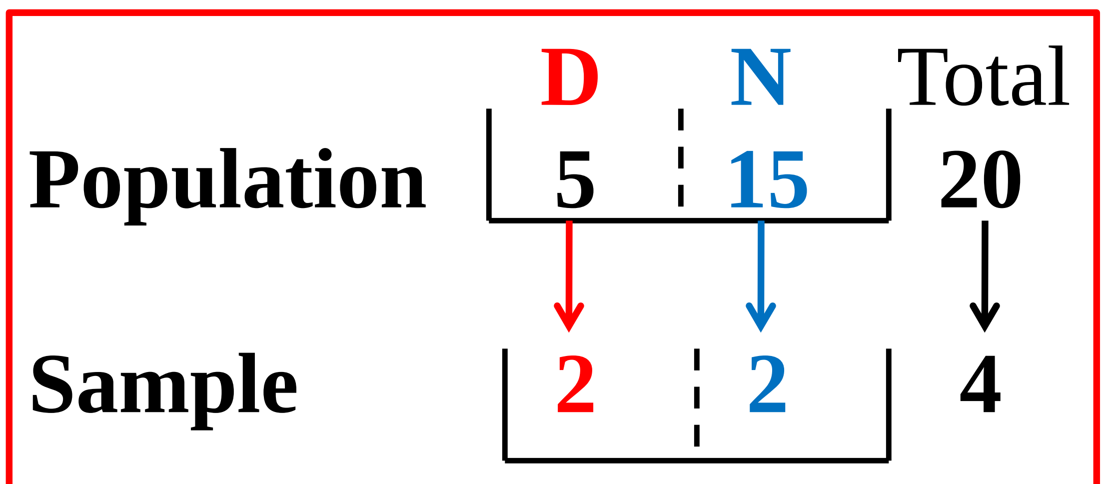
- **Solution of the defect Golf cars problem**

1. For a combination of 2 **D**efective and 2 **N**on-defective cars we have to choose (first) 2 out of 5 defect Golfs and then 2 out of 15 non-defective Golfs. # of these combinations is $\binom{5}{2} \times \binom{15}{2}$ on a total $\binom{20}{4}$.

$$P(2\boldsymbol{D} \text{ and } 2\ \boldsymbol{N}) = \frac{\binom{5}{2} \times \binom{15}{2}}{\binom{20}{4}}$$

2. $P(\text{no Defective Golfs})$

$$= \frac{\binom{5}{2} \times \binom{15}{2}}{\binom{20}{4}} \approx 0.217$$



Combinatorial Probability

Applying definition by Laplace: $\boldsymbol{P(A) = \frac{N(A)}{N(S)}}$

For $\boldsymbol{k}$ **draws out of** $\boldsymbol{n}$ (different) "**things**", we distinguish:

<table>
<tr><td colspan="2"></td><td colspan="2">Way of drawing</td></tr>
<tr><td colspan="2"></td><td>Without replacement</td><td>With replacement</td></tr>
<tr><td rowspan="2">Type of out-come</td><td>Ordered</td><td>$\frac{n!}{(n-k)!}$ permutations</td><td>n^k permutations with repetition</td></tr>
<tr><td>Un-ordered</td><td>$\binom{n}{k}$ combinations</td><td>Combinations with repetition: non-symmetric!</td></tr>
</table>

Generalizing the example: **hypergeometric formula**

Applicable for situation **of draws without replacement and unordered outcomes, e.g. choose n out of N** balls:
R red and $N - R$ **white.**

The diagram shows the outcome of k red and $n - k$ white.

	red	white	total
population:	R	$N - R$	N
	↓	↓	↓
sample	k	$n - k$	n

Random variable
X = "the number of red balls in n draws"

$$P(X = k) = \frac{\binom{R}{k}\binom{N-R}{n-k}}{\binom{N}{n}}, \quad k = 0, \ldots, n$$

X has a **hypergeometric distribution.**

Instruments of Combinatorial Probability

- **Product rule**: the numbers of outcomes of partial experiments can be multiplied (if "independent").
- Use in case of "arbitrary draws from populations" the so called **vase model** to visualize the problem.
- Are the draws **with** or **without replacement**?
  Can outcomes repeatedly occur?
- Is the order of the outcomes of interest for the problem?
  Use **Permutations** (orders, variations) or
  Combinations (unordered, subsets).
- Is **the probability of the complement** easier to compute?
- Avoid "**double counting**" of outcomes.

Overview Basic Probability in Ch. 1+2

Axioms of Kolmogorov: 1. $P(A) \geq 0$

2. $P(S) = 1$

3. $P(\cup_i A_i) = \sum_i P(A_i)$,

if the A_i`s are mutually exclusive

Complement rule: $P(\overline{A}) = 1 - P(A)$

Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

For **disjoint** events we have: $P(A \cap B) = 0$

Symmetric probability space (Laplace): $P(A) = \frac{N(A)}{N(S)}$

number of permutations of n out of N is $\frac{N!}{(N-n)!}$

number of combinations of n out of N is $\binom{N}{n} = \frac{N!}{n!(N-n)!}$

Examples of draws without replacement

How many matches are there in a competition of 6 teams?

- A full competition (home and away games a-b and b-a)?

  Solution: $6 \cdot 5$ games: $\frac{6!}{4!}$ permutations of 2 out of 6.

- A tournament (unordered pairs $\{a, b\}$ of 2 teams a and b).

  Solution: $\frac{6 \cdot 5}{2}$ games: $\frac{6!}{4!2!} = \binom{6}{2}$ combinations of 2 out of 6.

How many (different) compositions are possible, if you have 10 persons to compose

- A board with the chair, secretary and treasurer?

  Solution: $10 \cdot 9 \cdot 8 = \frac{10!}{(10-3)!}$ **permutations of 3 out of 10**

- A **committee** of 3 persons?

  $\frac{10 \cdot 9 \cdot 8}{3!} = \frac{10!}{3!7!} = \binom{10}{3}$ **combinations of 3 out of 10**