Slides
Probability Theory
Chapter 1

Ch1: Introduction to the probability concept

Model: sample space *S* and probability *P*

- **<u>Example 1</u>: one roll with a dice:**
  Outcomes are 1, 2, 3, 4 , 5 or 6 → S = {1, 2, …, 6}
  Event A = "even number" = {2, 4, 6}
  A occurs in 3 out of 6 outcomes: probability $P(A) = \frac{3}{6}$

  Condition for the correctness of this calculation:
  The dice is "fair": every outcome is **equally likely**.

- **<u>Example 2</u>:** We observe the times (in *min*) between the consecutively arriving customers: 24 interarrival times.

$$S = \{(x_1, \dots, x_{24}) \mid x_i \geq 0, i = 1, 2, \dots, 24\}$$

A = "the total interval of time is greater than 50 *min*."

$$S = \{(x_1, \dots, x_{24}) \text{ in } S \mid \sum_{i=1}^{24} x_i \geq 50\}$$

Concepts (1)

- **Stochastic experiment:** outcome depends on chance
- **Sample space *S*** = {all outcomes}: an outcome $\boldsymbol{s} \in \boldsymbol{S}$
- **Event $\boldsymbol{A} \subset \boldsymbol{S}$:** A is a subset of S

  Special events:

  The largest is ***S*** itself: the **certain** event and

  the smallest is ø, the **impossible** event

  Note that in this course $\boldsymbol{A} \subset \boldsymbol{S}$ *means the same as* $\boldsymbol{A} \subseteq \boldsymbol{S}$

  (math A)

- Properties: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

  $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

  $A \cup B = A \cup (\overline{A} \cap B)$ and $A = (A \cap B) \cup (A \cap \overline{B})$

  De Morgan`s rules:

  $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Concepts (2)

- The ***P***robability of an event A: $\boldsymbol{P(A)}$
- Evident properties that should be true:

$$\mathbf{0} \leq \boldsymbol{P(A)} \leq \mathbf{1}, \qquad \boldsymbol{P(\emptyset)} = \mathbf{0} \text{ and } \boldsymbol{P(S)} = \mathbf{1}$$

- **the frequency-interpretation** of e. g. $P(A) = 0.80$:

  "If the stochastic experiment is repeated many times, *A* will occur in about in 80% of the repetitions".

- This notion reflects an "experimentally determined probability": the **relative frequency of an event *A*** is the proportion of occurrences of A in n repetitions of the experiment:

$$\boldsymbol{f_n(A) = \frac{n(A)}{n}}$$

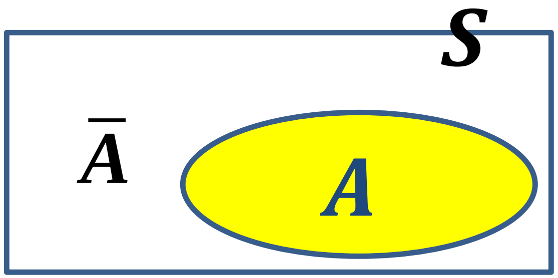
- **Experimental law of large numbers**

  for large n we have $\boldsymbol{f_n(A) \approx P(A)}$

Basic rules of probability (1)

- The **complement** of *A*: $\overline{A} = \{s \in S \mid s \notin A\}$

  say "not-A"

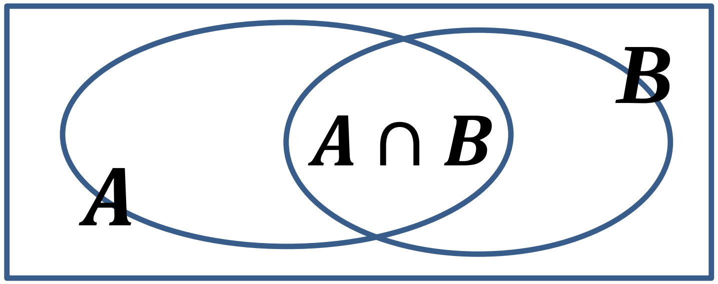


- The **complement rule:**

  $$P(\overline{A}) = 1 - P(A)$$

- The **intersection** $A \cap B = AB$: set of outcomes in *A* and *B*.

  say "A ànd B"



- The **union** of *A* and *B*: $A \cup B$

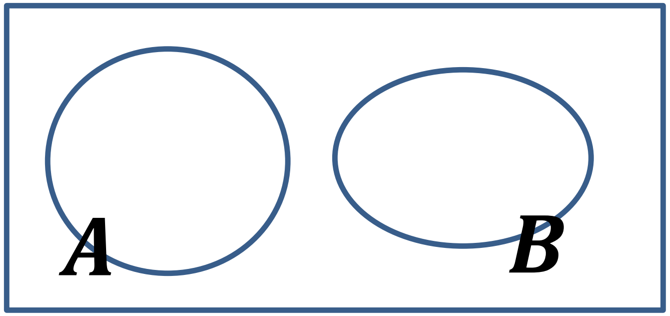
  say "A or B (or both)"

- **General addition rule:**

  $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Basic rules of probability (2)

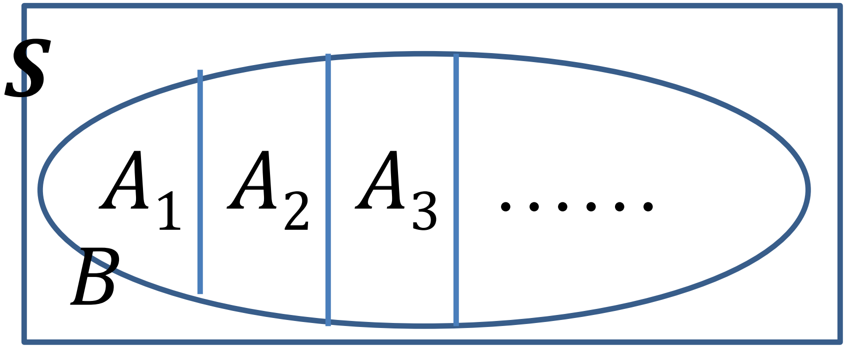
- **Mutually exclusive** or **disjoint** events.
  $\boldsymbol{A \cap B = \emptyset}$, so $\boldsymbol{P(A \cap B) = 0}$



- (Special) **addition rule for mutually exclusive events:**
- A series of events $\boldsymbol{A_1, A_2}$, … **are mutually exclusive (disjoint),** if $\boldsymbol{A_i \cap Aj = \emptyset}$**,** for each pair (i, j)
- A **partition** of B:

$$B = \bigcup_i A_i$$

such that A_1, A_2, …
are mutually exclusive.



Computing probabilities if all **outcomes are equally likely**: **count** the favourable and the total numbers.

- This principle can be applied often when random samples are conducted: an arbitrary choice from a finite population: every outcome of the population is equally likely.
- **Probability definition of Laplace**: compute the probability of A by counting the proportion of favourable outcomes.

$$\boldsymbol{P(A) = \frac{N(A)}{N(S)} = \frac{\text{Number of favourable outcomes of A}}{\text{Total number of outcomes in S}}}$$

- A pair $(\boldsymbol{S}, \boldsymbol{P})$ of a finite sample space S and its Laplace probability P is called a **symmetric probability space.**

What conditions should we impose on P for an arbitrary probability space (S, P) , e.g. if $N(S)$ is finite?

Foundation Probability Theory: **Kolmogorov`s axioms**

A function $\boldsymbol{P}$, that assigns a real number $\boldsymbol{P}(\boldsymbol{A})$ to each event $\boldsymbol{A}$ **Probability (measure)** if:

1. $\boldsymbol{P}(\boldsymbol{A}) \geq \mathbf{0}$
2. $\boldsymbol{P}(\boldsymbol{S}) = \mathbf{1}$
3. If $A_1, \dots, A_n$ or $A_1, A_2, \dots$ are **mutually exclusive**, then

$$\boldsymbol{P}\left(\bigcup_i \boldsymbol{A}_i\right) = \sum_i \boldsymbol{P}(\boldsymbol{A}_i)$$

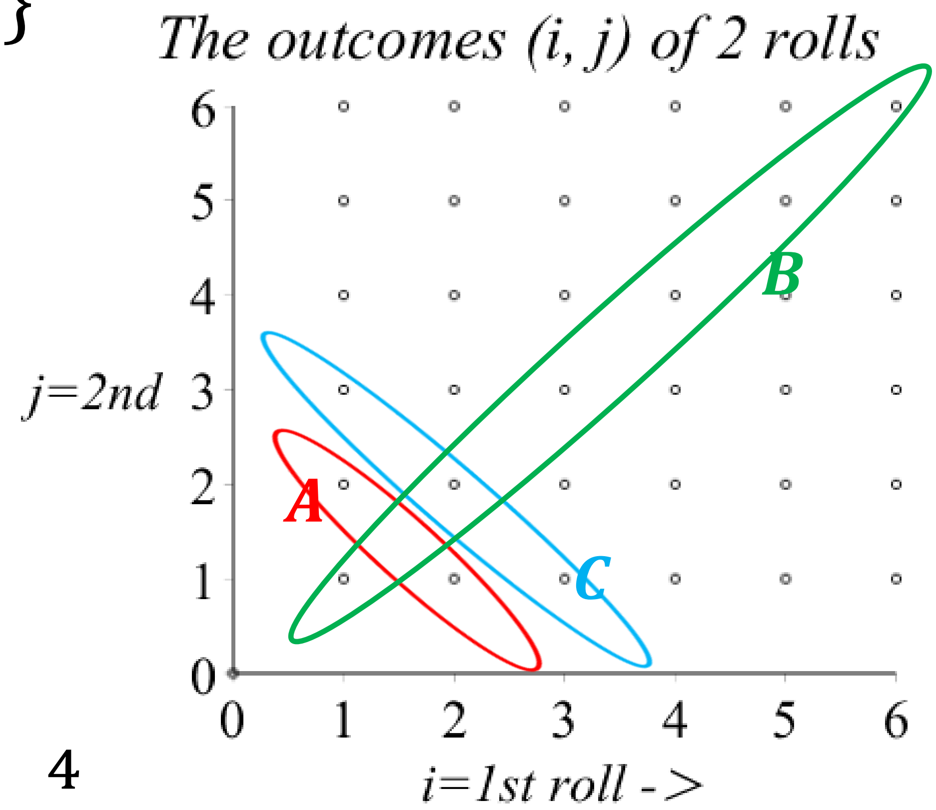
All other desirable properties can be derived from the axioms:

- $\boldsymbol{P}(\emptyset) = \mathbf{0}$ and $\mathbf{0} \leq \boldsymbol{P}(\boldsymbol{A}) \leq \mathbf{1}$
- $\boldsymbol{P}(\overline{\boldsymbol{A}}) = \mathbf{1} - \boldsymbol{P}(\boldsymbol{A})$
- $\boldsymbol{P}(\boldsymbol{A} \cup \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A}) + \boldsymbol{P}(\boldsymbol{B}) - \boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B})$
- **If** $\boldsymbol{A} \subset \boldsymbol{B}$, **then** $\boldsymbol{P}(\boldsymbol{A}) \leq \boldsymbol{P}(\boldsymbol{B})$

Examples (1)

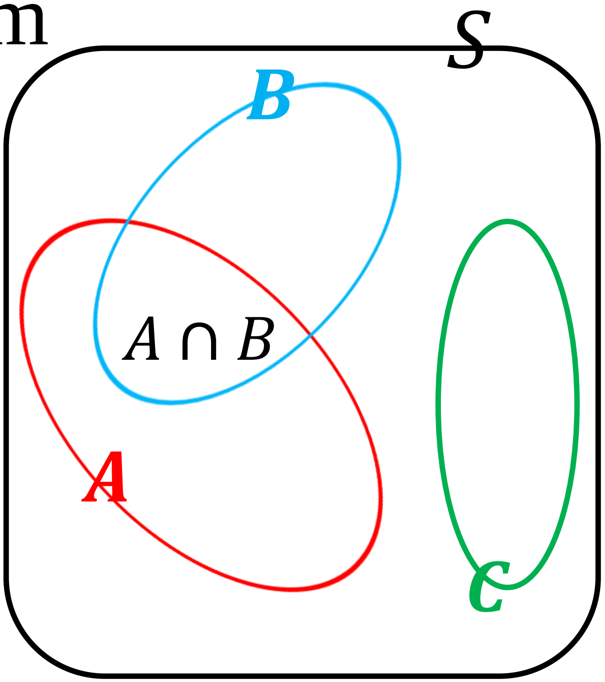
Example: roll a dice twice:

- $S = \{(i,j)|i,j = 1,2,\ldots,6\}$
- $A =$ "total $= 4$"
  $= \{(1,3), 2,2), (3,1)\}$
  $P(A) = \frac{N(A)}{N(S)} = \frac{3}{36}$
- $B =$ "$1^{st} = 2^{nd}\ roll$"
  $P(B) = \frac{6}{36}$
- $C =$ "total $= 5$": $P(C) = \frac{4}{36}$
- $P(\overline{A}) = \frac{33}{36} = 1 - P(A)$



Examples (2)

Example (continues): Venn-diagram



- $P(A \cap B) = P\big((2,2)\big) = \frac{1}{36}$
- $P(A \cup B) = P(A) + P(B) - P(AB)$
  $\frac{8}{36} = \frac{3}{36} + \frac{6}{36} - \frac{1}{36}$
- A and C are mutually exclusive,
  B and C as well,
  but A and B are not: $P(AB) \neq 0$
- Since $A = AB \cup A\overline{B}$, we have: $P(A) = P(AB) + P(A\overline{B})$,
  so $P\big(A\overline{B}\big) = P(A) - P(AB) = \frac{3}{36} - \frac{1}{36} = \frac{2}{36}$
  (correct, since $A\overline{B} = \{(1,3), (3,1)\}$)

Examples (3)

Example: The probability of a prize in a lottery is 10% at each draw,

a. What is the probability of the first win at the tenth draw?

b. What is the probability that the first win after the 20^{th} draw?

Solutions: each outcome is the number of draws, needed for the first win: $S = \{1, 2, 3, \ldots\ldots\}$

a. $P(10) = 0.9 \cdot \cdots \cdot 0.9 \cdot 0.1 = 0.9^9 \cdot 0.1 \approx 3.9\%$

Reasoning: at each draw you will have a success probability of 0.1 and a 0.9 probability of no prize. "10" will occur if in the first 9 draws no prize is won and at the tenth draw a prize is won.

b. $A = \{21, 22, \ldots\}$, so $P(A) = 0.9^{20} \approx 12.2\%$

A occurs if in all first 20 draws you win no prize.