Home Work Assignment 3 – Probability & Statistics I 2018

After some research of a specific waiting time X of customers in a system researchers came up with the following model (given by a density function) for the waiting time in minutes, describing the reality adequately:

$$f(x) = \begin{cases} \frac{c}{x^5} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}$$

a. Sow that $c = 4$.
b. Sketch the graph of f and shade the probability $P(X > 1.5)$. Subsequently compute $P(X > 1.5)$.
c. Determine $E(X)$.
d. Determine $E(X^2)$ and $var(X)$.
e. Derive the density function of $Y = X^2$ and verify that this function is a density function.
f. Determine the distribution function (cdf) $F(x)$ of X and use F to determine the median M, the value of X such that $P(X \leq M) = \frac{1}{2}$.
g. Compute $P(X > 2|X > 1.5)$.

Grading:

a	b	c	d	e	f	g	**tot**
2	3	2	3	5	3	2	**20**

Grade = # points /2

Solutions:

a. $\int_{-\infty}^{\infty} f(x)dx = 1$ should hold for any density function (and $f(x) \geq 0$),

so $\int_1^{\infty} \frac{c}{x^5} dx = \int_1^{\infty} cx^{-5} dx = \left[-c \cdot \frac{1}{4} x^{-4}\right]_{x=1}^{x\to\infty} = 0 + \frac{1}{4}c = 1$

Conclusion: $c = 4$.

b. $P(X > 1.5) = \int_{1.5}^{\infty} \frac{4}{x^5} dx = [-x^{-4}]_{x=1.5}^{x\to\infty} = 0 - (-(1.5)^{-4}) = \frac{16}{81}$ ($\approx 19.8\%$)

c. $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_1^{\infty} x \cdot \frac{4}{x^5} dx = \left[-\frac{4}{3} x^{-3}\right]_{x=1}^{x\to\infty} = 0 - \left(-\frac{4}{3}\right) = \frac{4}{3}$

d. $E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx = \int_1^{\infty} x^2 \cdot \frac{4}{x^5} dx = \left[-\frac{4}{2} x^{-2}\right]_{x=1}^{x\to\infty} = 2$

$var(X) = E(X^2) - (EX)^2 = 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9}$.

e. 1. $F_Y(y) = P(X^2 \leq y) = \begin{cases} 0 \text{ if } y < 1 \\ P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \text{ if } y \geq 1 \end{cases}$

2. $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$,

Since $f_X(x) = 0$ if $x < 1$, we have $f_X(-\sqrt{y}) = 0$ and $f_X(x) = \frac{4}{x^5}$ if $x \geq 1$, so:

3. $f_Y(y) = \frac{1}{2\sqrt{y}} \cdot \frac{4}{(\sqrt{y})^5} + \frac{1}{2\sqrt{y}} \cdot 0 = \frac{2}{y^3}$, if $\sqrt{y} \geq 1$ or $y \geq 1$ (and $f_Y(y) = 0$, if $y < 1$)

This is a density function, since $f_y(y) \geq 0$ and

$\int_{-\infty}^{\infty} f_Y(y)dy = \int_1^{\infty} \frac{2}{y^3} dy = [-y^{-2}]_{y=1}^{y\to\infty} = 0 - (-1) = 1$.

f. $F(x) = \int_{-\infty}^{x} f(u)du = \int_1^{x} \frac{4}{u^5} du = [-u^{-4}]_{u=1}^{u=x} = -x^{-4} + 1 = 1 - \frac{1}{x^4}$ if $x \geq 1$

and $F(x) = 0$ if $x < 1$.

$F(M) = \frac{1}{2}$ if $1 - \frac{1}{M^4} = \frac{1}{2}$, so $M^4 = 2$ or $M = 2^{1/4} \approx 1.19$.

g. $P(X > 2|X > 1.5) = \frac{P(X>2 \text{ and } X>1.5)}{P(X>1.5)} = \frac{P(X>2)}{P(X>1.5)} = \frac{1-F(2)}{P(X>1.5)} = \frac{\frac{1}{2^4}}{\frac{16}{81}} = \frac{81}{256}$

$\approx 31.6\%$.