HWA 3 - Calculus I – 2018 hand in on Monday May 14, 14.30

1. Find the derivative of the following functions $h(x) = \sqrt{3 - 2x^2}$
2. Find the equations of all horizontal and vertical asymptotes of the function
$$f(x) = \frac{12 - 10x + 2x^2}{x^2 - 3x}$$
   Motivate your answers by showing the appropriate computation/limit.
3. Consider the function $g(x) = 20x^3 - 3x^5$ on the domain [-2, 3]
   a. Find the absolute maximum and the absolute minimum of $g(x)$ (on its domain [-2, 3])
   b. Find the inflection points of g.
   c. Is the function g even or odd? (Consider the domain = $\mathbb{R}$ for this question)
4. The function f is defined as $f(x) = x^4 e^x$.
   a. Find all critical values of f.
   b. Give all local maxima and minima. Motivate each extreme value with the first derivative test.
   c. Give the second derivative of f and the interval(s), where f is concave downward.

1	2	3a	3b	3c	4a	4b	4c	Total
1	1.5	2	1.5	1	1	1	1	10

Solutions:

1. $h'(x) = \frac{1}{2\sqrt{3-2x^2}} \cdot -4x = -\frac{2x}{\sqrt{3-2x^2}}$

2. The equations of all horizontal and vertical asymptotes of $f(x) = \frac{12-10x+2x^2}{x^2-3x} = \frac{2(x^2-5x+6)}{x(x-3)} = \frac{2(x-3)(x-2)}{x(x-3)}$ Vertical asymptotes $x = a$ if $f(a)$ has the form $\frac{c}{0}$, where $c \neq 0$:
At $x = 0$ $f(x)$ has the form $\frac{12}{0} \Rightarrow$ **VA: $x = 0$**, but at $x = 3$ the form $\frac{0}{0} \Longrightarrow$ no V.A. at $x = 3$.
Note that f has at $x = 3$ a removable discontinuity: $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{2(x-3)(x-2)}{x(x-3)} = \lim_{x\to 3} \frac{2(x-2)}{x} = \frac{2}{3}$

Horizontal asymptotes: $\lim_{x\to\infty} f(x) = \lim_{x\to\pm\infty} \frac{12-10x+2x^2}{x^2-3x} = \lim_{x\to\infty} \frac{\frac{12}{x^2} - \frac{10}{x} + 2}{1 - \frac{3}{x}} = \frac{0-0+2}{1-0} = 2$

$\lim_{x\to-\infty} f(x) = 2$ (similarly) $\Rightarrow$ **HA: $y = 2$**

3. **a**. $g(x) = 20x^3 - 3x^5$ has derivative

$$g'(x) = 60x^2 - 15x^4 = 15x^2(4-x^2) = 15x^2(x-2)(x+2)$$

Using the closed interval method for this continuous function on [-2, 3] we find:

1. $g'(x) = 0 \Leftrightarrow$ *critical values* $x = 0$ or $x = 2$ or $x = -2$: all within [-2, 3]. See table for $f(x)$.
2. We add the function values $f(x)$ at the boundaries $x = -2$ and $x = 3$ in the following table:
3. The largest of these values is the **absolute maximum $f(2) = 64$** and the smallest is the **absolute minimum $f(3) = -189$**

x	-2	0	2	3
$f(x)$	-64	0	64	-189

b. $g''(x) = 120x - 60x^3 = 60x(2-x^2) = 60x(\sqrt{2}-x)(\sqrt{2}+x) = 0$

$\Rightarrow x = 0$ or $x = \sqrt{2}$ or $x = -\sqrt{2}$. Since $x = 0$ and $x = \pm\sqrt{2} \approx \pm 1.41$ are all within the interval and we have a **sign change of f''** at all of these values, we have three inflection points: $(0, 0)$, $\left(\sqrt{2}, f(\sqrt{2})\right) = (\sqrt{2}, 28\sqrt{2})$ and $(-\sqrt{2}, -28\sqrt{2})$.

c. $g(x) = 20x^3 - 3x^5 \Longrightarrow g(-x) = -20x^3 + 3x^5 = -g(x)$:
$g(x)$ is an **odd function**. (the graph can be reflected about the origin.)

4. **a.** $f(x) = x^4e^x \Rightarrow f'(x) = 4x^3e^x + x^4e^x = (x^4+4x^3)e^x = x^3(x+4)e^x = 0$,
if $x = 0$ or $x = -4$ $\Longrightarrow$ **0 and -4 are the critical values**

b. Since the sign of $f'(x)$ is changing from positive to negative f has a **local maximum** at x = -4:

$f(-4) = 256e^{-4} \approx 4.69.$

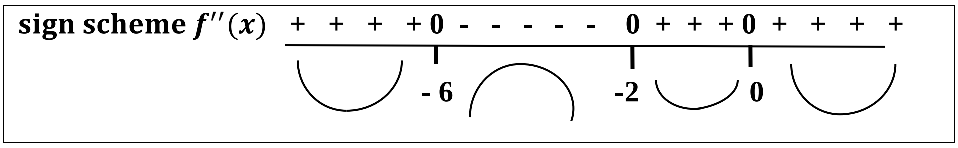
sign scheme $f'(x)$ + + **0** - - - - - **0** + + +
↗ **-4** ↘ **0** ↗

At x = 0: the sign of $f'(x)$ is changing from negative to positive, so f has a **local minimum $f(0) = 0$**

c. $f'(x) = (x^4+4x^3)e^x \Longrightarrow f''(x) = (4x^3+12x^2)e^x + (x^4+4x^3)e^x$

$$= (x^4+8x^3+12x^2)e^x = x^2(x+2)(x+6)e^x$$

$f''(x) = 0$ if $\Leftrightarrow x = 0$ *or* $x = -2$ or $x = -6$



$f''(x) < 0$ if $-6 < x < -2$ the function is concave downward on (-6, -2)

....

Some notes on the last exercise (in view of a full investigation and curve sketching)**:**

- In exercise a., to proof that a critical point *c* where $f'(c)$ is a maximum or minimum you have to check whether the sign of $f'(x)$ is changing at *x* = *c:* if f' is changing from + to - , it is a local maximum, from – to + a local minimum.
  Similarly in c. to show that a point where $f''(c)$ is an IP, $f''(x)$ must change its sign at *x* = *c*. e.g. if $f(x) = x^4$, then $f''(x) = 12x^2 = 0$ if *x* = 0, but (0, 0) is not an IP (check the graph!), as $f''(x) \geq 0$ near $x = 0$
- If you would use the second derivative test in b. you will find the following, using the second derivative found in c: at $x = -4$: $f''(-4) = 16 \cdot (-2) \cdot (+2)\mathrm{e}^{-2} < 0 \Rightarrow f$ is concave downward at $x = -4$, so $f(-4) = 256\mathrm{e}^{-4} \approx 4.69$ is a local maximum
  At $x = 0$: $f''(0) = 0$, but near $\mathrm{x} = 0$ we have $f''(x) > 0 \Rightarrow f(0) = 0$ is a local minimum
- Since $f''(x)$ has **no sign change** at *x* = 0 ("double root"),**(0, 0) is not an Inflection Point,** IP's are (-6, $f(-6)$) and $(-2, f(-2))$, where $f(-6) \approx 3.21$ and $f(-2) \approx 2.17$
- The behaviour of the function at infinity and negative infinity:
  $\lim_{x\to\infty} x^4 e^x = +\infty$ (both x^4 and e^x approach ∞) and
  $\lim_{x\to-\infty} x^4 e^x = \lim_{x\to-\infty} \frac{x^4}{e^{-x}} = 0$ since after substituting $y = -x$ we find:
  $\lim_{x\to-\infty} \frac{x^4}{e^{-x}} = \lim_{y\to\infty} \frac{y^4}{e^y} = 0$ (see exercise 3.14 applying L'Hopital's rule)
- Since $f(x) \geq 0$ and the local minimum $f(0) = 0$, this is an absolute minimum:
  There is no absolute maximum since $\lim_{x\to\infty} x^4 e^x = \infty$