Homework assignment 2 – Applied Statistics 2018
Hand in ***your own solutions*** *at the start of the lecture on 23/3 or (Kanifing, 11.00 h)).*

A random variable X has a density function f with unknown parameter $\theta > 0$, defined as follows:

$$f(x) = \begin{cases} \frac{2x}{\theta^2}, & \text{if voor } 0 \le x \le \theta \\ 0, & \text{elsewhere} \end{cases}$$

A random sample $X_1, \dots, X_n$ of X is available.

- **a.** Determine the first three moments $E(X), E(X^2), E(X^3)$ and $var(X)$
- **b.** Sketch the graph of the density function, state whether the distribution is skewed (to the left or the right?) and verify that the skewness coefficient of the distribution has a corresponding value (use a.)
- **c.** Use the result of a. to determine the moment estimator of θ.
- **d.** Show that $\frac{3}{2}\overline{X}$ is an unbiased and consistent estimator of θ.
- **e.** Consider all estimators $T = a \cdot \overline{X}$, with real constant $a \in \mathbb{R}$. For which value of a is T the best estimator of θ.
- **f.** Determine the maximum likelihood estimator of θ.

a	b	c	d	e	f	Total
2	1.5	1	2	2	1.5	**10**

Solutions:

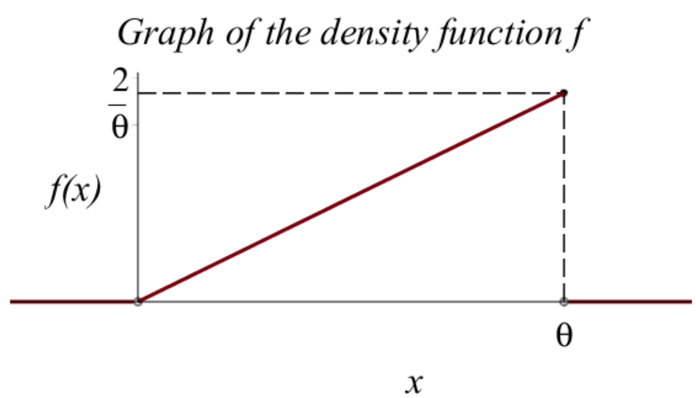
a. $E(X) = \int_{-\infty}^{\infty} x f(x)dx = \int_0^{\theta} x \cdot \frac{2x}{\theta^2} dx = \left[\frac{2x^3}{3\theta^2}\right]_{x=0}^{\theta} = \frac{2}{3}\theta$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^{\theta} x^2 \cdot \frac{2x}{\theta^2} dx = \left[\frac{2x^4}{4\theta^2}\right]_{x=0}^{\theta} = \frac{1}{2}\theta^2$

$E(X^3) = \int_{-\infty}^{\infty} x^3 f(x)dx = \int_0^{\theta} x^3 \cdot \frac{2x}{\theta^2} dx = \left[\frac{2x^5}{5\theta^2}\right]_{x=0}^{\theta} = \frac{2}{5}\theta^3$

$var(X) = E(X^2) - (EX)^2 = \frac{1}{2}\theta^2 - \frac{4}{9}\theta^2 = \frac{1}{18}\theta^2$

b. The graph is skewed to the left ("tail" on the left hand side).



The skewness coefficient is $\gamma_1 = E(X - \mu)^3/\sigma^3$.
In a. we found $\sigma^2 = \frac{1}{18}\theta^2$ ($\sigma = \frac{\theta}{\sqrt{18}}$).

$E(X-\mu)^3 = E[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3] = E(X^3) - 3\mu \cdot E(X^2) + 3\mu^2 \cdot E(X) - \mu^3$
$= E(X^3) - 3\mu \cdot E(X^2) + 2\mu^3 \overset{\text{a.}}{=} \frac{2}{5}\theta^3 - 3 \cdot \frac{1}{3}\theta^3 + 2 \cdot \frac{8}{27}\theta^3 = -\frac{1}{135}\theta^3$

$\gamma_1 = \frac{E(X-\mu)^3}{\sigma^3} = \frac{-\frac{1}{135}\theta^3}{(\theta/\sqrt{18})^3} = -\frac{2}{5}\sqrt{2} \approx -0.56$ (<0, as expected).

c. $E(X) = \frac{2}{3}\theta$, so $\theta = \frac{3}{2}\mu$ and $\overline{X}$ is the moment estimator of μ: $\frac{3}{2}\overline{X}$ is the moment estimator of θ.

d. $E(\overline{X}) = \mu = \frac{2}{3}\theta$ and $var(\overline{X}) = \frac{\sigma^2}{n} = \frac{\theta^2}{18n}$

$E\left(\frac{3}{2}\overline{X}\right) = \frac{3}{2}E(\overline{X}) = \frac{3}{2}\cdot\frac{2}{3}\theta = \theta \implies \frac{3}{2}\overline{X}$ is an unbiased estimator of θ.

Furthermore $var\left(\frac{3}{2}\overline{X}\right) = \frac{9}{4}var(\overline{X}) = \frac{\theta^2}{8n}$ and $\lim_{n\to\infty} var\left(\frac{3}{2}\overline{X}\right) = \lim_{n\to\infty}\frac{\theta^2}{8n} = 0$,

so unbiased and variance converges to 0, then $\frac{3}{2}\overline{X}$ is a consistent estimator of θ.

e. $E(T) = a \cdot E(\overline{X}) = \frac{2a\theta}{3}$ and $var(T) = a^2 \cdot E(\overline{X}) = \frac{a^2\theta^2}{18n}$

$MSE(T) = (ET-\theta)^2 + var(T) = \left(\frac{2a}{3}-1\right)^2\theta^2 + \frac{a^2\theta^2}{18n} = \theta^2\left[\left(\frac{2a}{3}-1\right)^2 + \frac{a^2}{18n}\right]$

Determine maximum of $f(a) = \left(\frac{2a}{3}-1\right)^2 + \frac{a^2}{18n}$:

$f`(a) = \frac{4}{3}\left(\frac{2a}{3}-1\right) + \frac{a}{9n} = 0 \iff \left(\frac{8}{9}+\frac{1}{9n}\right)a - \frac{4}{3} = 0 \iff \text{a} = \frac{4}{3}\cdot\frac{8n}{8n+1}$

Since $f``(a) = \frac{8}{9}+\frac{1}{n} > 0$, $f(a)$ attains a minimum at a $= \frac{12n}{8n+1}$

$T = \frac{12n}{8n+1}\overline{X}$ is the best estimator (*note that* $E(T) = \frac{12n}{8n+1}\cdot\frac{2}{3}\theta$ *is not unbiased, but*

$\lim_{n\to\infty} E(T) = \lim_{n\to\infty}\frac{12n}{8n+1}\cdot\frac{2}{3}\theta = \frac{12}{8}\cdot\frac{2}{3}\theta = \theta$*:*

$T = \frac{12n}{8n+1}\overline{X}$ *is asymptotically unbiased*).

f. $L(\theta) = \prod f(x_i|\theta) = 2^n(x_1\cdot\ldots\cdot x_n)\cdot\frac{1}{\theta^n}$, where $0 \le x_i \le \theta$ for all $i = 1,\ldots,n$,

so domain of L is: $\theta \ge \max(x_1,\ldots,x_n)$.

L is a decreasing function in θ ($L`(\theta) = -n2^n(x_1\cdot\ldots\cdot x_n)\cdot\frac{1}{\theta^{n+1}} < 0$) and attains its maximum in the bound $\theta = \max(x_1,\ldots,x_n)$.

Hence $\hat{\theta} = \max(X_1,\ldots,X_n)$ is the maximum likelihood estimator of θ.