The University of The Gambia
School of Arts and Sciences

Division of Physical and Natural Sciences
MTH 101. Calculus 1

Exam Calculus I, 23rd of May 2018, 9.00-11.00 hrs.
Groups of lecturers Ceesay and Meijer

The use of a calculator (without memory for formulas) is allowed, not the use of a phone. Motivate all your solutions adequately!

1. Given is the function $f(x) = \frac{2x+1}{2-x}$.

a. Determine the inverse function of $f(x)$.
b. Determine the domain and the range of $f(x)$.
c. Show that $f'(x) = \frac{5}{(2-x)^2}$ and determine the intervals at which f is increasing.
d. Determine the equation of the tangent line at the point (1, 3) of the graph of f .

2. Determine the following limits, if they exist. Show your solution (including ∞ or -∞), in some appropriate steps. If a limit does not exist, state why not.

a. $\lim_{x\to 4} \frac{x^2+3x-28}{4x-x^2}$

b. $\lim_{x\to 2} \frac{x-2}{\sqrt{x^2+5}-3}$

c. $\lim_{x\to \frac{1}{6}\pi} \frac{\sin(x)-\frac{1}{2}}{\cos(x)-\frac{1}{2}}$

d. $\lim_{h\to 0} \frac{\frac{1}{3-h}-\frac{1}{3}}{h}$

3. Find the derivatives of the following functions:

a. $f(x) = (x + \sin x)^5$

b. $h(x) = e^{\sqrt{x^2+1}}$

c. $k(x) = \dfrac{\cos(x)}{\sin(x)}$

d. $l(x) = x^x$

4. The function f is a piecewise defined function: $f(x) = \begin{cases} x^2 - 2x\,, & \text{if } x \leq 2 \\ 2x - 4\,, & \text{if } x > 2 \end{cases}$

a. Give the derivative of f on the interval $(-\infty, 2)$.

b. Use the definition of derivative to show that the derivative of f, given in a. is correct.

c. Use the definition of continuity to show that f is continuous at $x = 2$.

d. Is f differentiable at $x = 2$? Why (not)?

5. **a**. The function h is defined as follows:

$$h(x) = 2x^3 - 3x^2 - 12x - 4$$

Find the absolute maximum and absolute minimum of h on the interval $[-3, 3]$.

b. Find the horizontal and vertical asymptotes of the function $f(x) = \dfrac{2x^2-6x}{x^2- 8x + 15}$

c. Show that the line $y = x - 3$ is a slant asymptote of $l(x) = \dfrac{x^3-3x^2}{x^2+1}$

6. Given is the function $f(x) = (x^2 - 3)e^x$.

a. Find the *y*-intercept and the *x*-intercept(s) of f.

b. Does the function $f(x)$ have special properties: is it even, odd or periodic?

c. Determine the derivative of f and all local minima and maxima.

d. Determine the second derivative and the intervals at which the f is concave downward and concave upward, respectively.

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Grade $= 1 + \dfrac{\#\ of\ points}{54} \times 9$

1				2				3				4				5			6			Total
a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	a	b	c	d	
3	2	2	2	2	2	2	2	2	2	2	2	1	3	2	2	4	4	3	2	2	4	2	54

Solutions

Exercise 1

a. $y = f(x) = \frac{2x+1}{2-x} \Rightarrow (2-x)y = 2x+1 \Rightarrow 2y-1 = yx+2x \Rightarrow x = \frac{2y-1}{y+2}$.

Interchanging x and y we find $f^{-1}(x) = y = \frac{2x-1}{x+2}$.

b. Domain $D_f = \mathbb{R}\backslash\{2\}$ (other notation $\{x \in \mathbb{R} | x \neq 2\}$ or $(-\infty, 2) \cup (2, \infty)$)

Range $R_f = D_{f^{-1}} = \mathbb{R}\backslash\{-2\}$

c. $f'(x) = \frac{2\times(2-x)-(2x+1)\times(-1)}{(2-x)^2} = \frac{4-2x+2x+1}{(2-x)^2} = \frac{5}{(2-x)^2} > 0$ on the domain of f.

Hence f is increasing on $(-\infty, 2)$ and on $(2, \infty)$)

d. The tangent line at the point (1, 3) of the graph of f has slope $m = f'(1) = \frac{5}{(2-1)^2} = 5$

Substituting (1, 3) in the equation $y = 5x + b$, we find $3 = 5 \times 1 + b$ or $b = -2$.

The equation of the requested tangent line is: $y = 5x - 2$.

(or use $\frac{y-y_1}{x-x_1} = m$, where $(x_1, y_1) = (1,3)$, to find the same equation.)

Exercise 2.

a. [form $\frac{0}{0}$, factor] $\lim\limits_{x\to4} \frac{x^2+3x-28}{4x-x^2}) = \lim\limits_{x\to4} \frac{(x+7)(x-4)}{-x(x-4)} = \lim\limits_{x\to4} \frac{(x+7)}{-x} = -\frac{11}{4}$

b. [form $\frac{0}{0}$, root trick] $\lim\limits_{x\to2} \frac{x-2}{\sqrt{x^2+5}-3} \times \frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3} = \lim\limits_{x\to2} \frac{(x-2)(\sqrt{x^2+5}+3)}{x^2+5-9} = \lim\limits_{x\to2} \frac{\sqrt{x^2+5}+3}{x+2} = \frac{6}{4} = \frac{3}{2}$

c. [Direct substitution, $\frac{1}{6}\pi = 30°$] $\lim\limits_{x\to\frac{1}{6}\pi} \frac{\sin(x)-\frac{1}{2}}{\cos(x)-\frac{1}{2}} = \frac{\frac{1}{2}-\frac{1}{2}}{\frac{1}{2}\sqrt{3}-\frac{1}{2}} = 0$

d. [form $\frac{0}{0}$] $\lim\limits_{h\to0} \frac{\frac{1}{3-h}-\frac{1}{3}}{h} = \lim\limits_{h\to0} \frac{\frac{1}{3-h}-\frac{1}{3}}{h} \times \frac{3(3-h)}{3(3-h)} = \lim\limits_{h\to0} \frac{3-(3-h)}{3h(3-h)} = \lim\limits_{h\to0} \frac{1}{3(3-h)} = \frac{1}{9}$

Exercise 3

a. $f'(x) = 5(x+\sin x)^4 \times \frac{\mathrm{d}}{\mathrm{dx}}(x + sin\ x) = 5(1+\cos x)(x+\sin x)^4$

b. $h'(x) = e^{\sqrt{x^2+1}} \times \frac{d}{dx}\left[(x^2+1)^{\frac{1}{2}}\right] = e^{\sqrt{x^2+1}} \times 2x \times \frac{1}{2}(x^2+1)^{-\frac{1}{2}} = \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

c. $k'(x) = \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) = \frac{-\sin(x)\times\sin(x)-\cos(x)\times\cos(x)}{(\sin(x))^2} = \frac{-(\sin^2(x)+\cos^2(x))}{\sin^2(x)} = -\frac{1}{\sin^2(x)}$

(*or use* $k(x) = \frac{1}{\tan(x)}$ *and apply the chain rule*)

d. $l(x) = x^x = e^{\ln(x^x)} = e^{x\ln(x)}$, then $l'(x) = e^{x\ln(x)} \times \frac{d}{dx}(x\ln(x)) = x^x(\ln(x)+1)$

Exercise 4

a. If $0 < x < 2$, then $f'(x) = (x^2-2x)' = 2x-2$

b. $f'(x) = \lim\limits_{h\to0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to0} \frac{(x+h)^2-2(x+h)-(x^2-2x)}{h}$

$= \lim\limits_{h\to0} \frac{x^2+2hx+h^2-2x-2h-x^2+2x}{h} = \lim\limits_{h\to0} \frac{h(2x+h-2)}{h} = \lim\limits_{h\to0} \frac{2x+h-2}{1} = 2x-2$

c. $f(x)$ is continuous at $x = 2$ if $\lim_{x\to 2} f(x) = f(2)$

$f(2) = 2^2 - 2 \times 2 = 0 = \lim_{x\to 2} f(x)$, since both the left and right limit are 0:

$\lim_{x\uparrow 2} f(x) = \lim_{x\uparrow 2}(x^2 - 2x) = 0$ ("$\lim_{x\uparrow 2}$" means "$\lim_{x\to 2^-}$") and

$\lim_{x\downarrow 2} f(x) = \lim_{x\downarrow 2}(2x - 4) = 0$ ("$\lim_{x\downarrow 2}$" means "$\lim_{x\to 2^+}$") .

d. f is differentiable at $x = 2$ if $\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$ exists.

Since $\lim_{h\uparrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\uparrow 0} \frac{(2+h)^2 - 2(2+h) - 0}{h} = \lim_{h\uparrow 0} \frac{4+h-2}{1} = 2$

and $\lim_{h\downarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\downarrow 0} \frac{2(2+h)-4 - 0}{h} = \lim_{h\uparrow 0} \frac{2h}{h} = 2$ are the same: hence f is differentiable at $x = 2$.

(*If students solved this by differentiating f on the intervals $(-\infty, 2)$ and $(2, \infty)$, respectively, and compare the right hand and left hand limit of the derivative at $x = 2$, this is correct, provided that they note that the function is continuous at $x = 2$*).

Exercise 5

a. We will apply the closed interval method, since the polynomial h is continuous and differentiable.

1. Critical values: $h'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$

   $h'(x) = 0$ if $x = 2$ or $x = -1$, with function values $h(2) = -20$ and $h(-1) = 3$

2. Function values at the boundaries: $h(-3) = -49$ and $h(3) = -13$

In conclusion: the smallest function value, $h(-3) = -49$, is the absolute minimum on the interval and the largest, $h(-1) = 3$, is the absolute maximum.

b. $f(x) = \frac{2x^2 - 6x}{x^2 - 8x + 15} = \frac{2x(x-3)}{(x-3)(x-5)}$

Vertical asymptote: $x = 5$, since $f(5)$ has the form $\frac{20}{0}$

$x = 3$ is not a VA, since $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{2x(x-3)}{(x-3)(x-5)} = \lim_{x\to 3} \frac{2x}{x-5} = -4$,
(a removable discontinuity.)

Horizontal asymptote: $y = 2$

since $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{2x^2 - 6x}{x^2 - 8x + 15} = \lim_{x\to\pm\infty} \frac{2 - \frac{6}{x}}{1 - \frac{8}{x} + \frac{15}{x^2}} = 2$

c. $\lim_{x\to\pm\infty} [l(x) - (x-3)] = \lim_{x\to\pm\infty} \left[\frac{x^3 - 3x^2}{x^2+1} - (x-3) \times \frac{x^2+1}{x^2+1}\right]$

$$= \lim_{x\to\pm\infty} \frac{(x^3 - 3x^2) - (x^3 - 3x^2 + x - 3)}{x^2+1} = \lim_{x\to\pm\infty} \frac{-x\mp 3}{x^2+1} = \lim_{x\to\pm\infty} \frac{-\frac{1}{x} + \frac{3}{x^2}}{1 + \frac{1}{x^2}} = \frac{0+0}{1+0} = 0$$

Hence the line $y = x - 3$ is a slant asymptote of $l(x) = \frac{x^3 - 3x^2}{x^2+1}$

Exercise 6

a. x-intercepts: $f(x) = (x^2 - 3)e^x = (x + \sqrt{3})(x - \sqrt{3})e^x = 0$ if $x = \sqrt{3}$ or $x = -\sqrt{3}$.

y-intercept: $y = f(0) = (0^2 - 3)e^0 = -3$.

b. $f(-x) = (x^2 - 3)e^{-x} \Rightarrow f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$: f is neither odd nor even and not-periodic.

c. $f'(x) = 2x \times e^x + (x^2 - 3)e^x = (x^2 + 2x - 3)e^x = (x - 1)(x + 3)e^x$

$f'(x) = 0$ if $x = 1$ or $x = -3$

$f(-3) = 6e^{-3} \approx 0.3$ is a local maximum since $f'(x)$ changes from positive (increasing) to negative (decreasing) at $x = -3$.

(e.g. if $x < -3$, then $f'(x) = (x - 1)(x + 3)e^x$ is $- \times - \times += +$)

$f(1) = -2e \approx -5.4$ is a local minimum since $f'(x)$ changes from negative (decreasing) to positive (increasing) at $x = 1$.

(*Note 1*: *a sign scheme of f' could show the local maximum and minimum.*

Note 2: if you consider the limits at infinity and negative infinity, the absolute extrema can be determined: $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x^2 - 3) \times e^x = \infty$ $(form\ \infty \times \infty)$ *and*

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x^2 - 3)e^x$ *(indeterminate* $\infty \times 0$*)* $= \lim_{x\to-\infty} \frac{x^2-3}{e^{-x}} = \lim_{x\to\infty} \frac{x^2-3}{e^x} = 0$

since $\lim_{x\to\infty} \frac{polynomial}{e^x} = 0$ *(or apply L'Hopital twice on* $\lim_{x\to-\infty} \frac{x^2-3}{e^{-x}}$*)*

Conclusion: there is no absolute maximum and

$f(1) = -2e \approx 5.4$ *is an absolute minimum*.)

d. $f''(x) = (2x + 2)e^x + (x^2 + 2x - 3)e^x = (x^2 + 4x - 1)e^x = 0$ if

$x = \frac{-4\pm\sqrt{16+4}}{2}$. Hence: $x = -2 + \sqrt{5} \approx +0.2$ or $x = -2 - \sqrt{5} \approx -4.2$.

Between these two roots of the equation $f''(x) = 0$, $f''(x) < 0$, since $x^2 + 4x - 1$ is a parabola that opens upward: f is concave downward on $(-2 - \sqrt{5}, -2 + \sqrt{5})$ and concave downward on the intervals $(-\infty, -2 - \sqrt{5})$ and $(-2 + \sqrt{5}, \infty)$

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Extra: based on the information above and some extra points (see table) we can sketch the curve of f.

x	-3	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	2
y	0.3	0	-0.74	-3	-5.4	0	$e^2 \approx 7.4$

Sketch of the graph of f