Lecture Notes on Derivatives and Curve Sketching

The derivative $f'(x)$ of the function $f(x)$: $$\boldsymbol{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}.$$

(for all x for which f is differentiable the limit exists)

Property: **if f is differentiable at x, then f is continuous at x**
("continuity is a condition for differentiability")

Power rule (derivatives of powers of x can be found using the definition above, $n \neq 0$):

$f(x)$	x	x^2	x^3	$\frac{1}{x} = x^{-1}$	$\frac{1}{x^2} = x^{-2}$	$\sqrt{x} = x^{\frac{1}{2}}$	$\sqrt[3]{x} = x^{\frac{1}{3}}$
$f'(x)$	1	$2x$	$3x^2$	$-x^{-2} = -\frac{1}{x^2}$	$-2x^{-3} = -\frac{2}{x^3}$	$\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$	$\frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

$$\boldsymbol{In\ general: f(x) = x^n \Rightarrow f'(x) = nx^{n-1}}$$

Differentiation rules:

Function	Derivative	Short notations	
$f(x) = c$	$f'(x) = 0$		
$cf(x)$	$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$	$(cf)' = cf'$	
$f(x) + g(x)$ $f(x) - g(x)$	$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$	$(f \pm g)' = f' \pm g'$	Addition rule
$f(x)g(x)$	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$	$(fg)' = f'g + fg'$	Product rule
$\frac{f(x)}{g(x)}$	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$	Quotient rule

Examples for $f(x) = 2\sqrt{x}$ and $g(x) = x^3 + 4$,
then $f'(x) = 2\frac{d}{dx}\sqrt{x} = \frac{1}{\sqrt{x}}$ and $g'(x) = \frac{d}{dx}x^3 + \frac{d}{dx}4 = 3x^2 + 0$

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x) = \frac{1}{\sqrt{x}} \pm 3x^2$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f`(x)g(x)-f(x)g`(x)}{[g(x)]^2} = \frac{\frac{1}{\sqrt{x}}\times(x^3+4)-2\sqrt{x}\times 3x^2}{(x^3+4)^2} = \frac{4-x^3}{(x^3+4)^2\sqrt{x}}$$

Derivatives of common functions

$f(x) = c$	$\Rightarrow$	$f'(x) = 0$
$f(x) = x^n$	$\Rightarrow$	$f'(x) = nx^{n-1}$
$f(x) = e^x$	$\Rightarrow$	$f'(x) = e^x$
$f(x) = \ln(x)$	$\Rightarrow$	$f'(x) = \frac{1}{x}$
$f(x) = \sin(x)$	$\Rightarrow$	$f'(x) = \cos(x)$
$f(x) = \cos(x)$	$\Rightarrow$	$f'(x) = -\sin(x)$
$f(x) = \tan(x)$	$\Rightarrow$	$f'(x) = \frac{1}{\cos^2(x)}$
$f(x) = a^x$	$\Rightarrow$	$f'(x) = a^x\ln(a)$
$f(x) = \log_a(x)$	$\Rightarrow$	$f'(x) = \frac{1}{x\ln(a)}$

and some examples

$$f(x) = x^4\sqrt[3]{x} \Longrightarrow f'(x) = \frac{d}{dx}\left(x^{\frac{13}{3}}\right) = \frac{13}{3}x^{\frac{10}{3}} = \frac{13}{3}x^3\sqrt[3]{x}$$

$$\frac{d}{dx}\big(x\cos(x)\big) = 1 \times \cos(x) + x \times -\sin(x)$$
$$= \cos(x) - x\sin(x)$$

$$f(x) = \tan x = \frac{\sin x}{\cos x} \Longrightarrow (quotient\ rule)$$
$$f'(x) = \frac{\cos x \times \cos x - \sin x \times -\sin x}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{d}{dx}(2^x) = 2^x\ln(2) \quad \text{and} \quad \frac{d}{dx}\log_2(x) = \frac{1}{x ln(2)}$$

The chain rule for composite functions $f \circ g(x) = f(g(x))$: $\quad \boldsymbol{\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)}$

Or: in the Leibniz notation for $u = g(x)$ and $v = f(u)$: $\quad \boldsymbol{\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx}}$

Note that $\frac{dv}{du} = f'(u) = f'\big(g(x)\big)$ *and* $\frac{du}{dx} = g'(x)$

Some examples of applying the chain rule:

- If $k(x) = \sin^2 x$, then $k(x)$ has the form $f(g(x)) = (\sin x)^2$, where $g(x) = \sin x$ and $f(u) = u^2$
  Since $g'(x) = \cos x$ and $f'(u) = 2u$, $\quad k'(x) = \frac{d}{dx}[f(g(x))] = f'\big(g(x)\big)g'(x) = 2\sin x \times \cos x$
- If $f(x) = e^{x^2-5x+7} \implies f'(x) = e^{x^2-5x+7} \times \frac{d}{dx}(x^2 - 5x + 7) = (2x-5)e^{x^2-5x+7}$, since $\frac{d}{du}e^u = e^u$
- If $k(y) = \frac{1}{\ln(y)+y^5}$, we have $k(y) = (\ln(y) + y^5)^{-1} = f\big(g(y)\big)$, with $g(y) = \ln(y) + y^5$ and $f(u) = u^{-1}$. Since $f'(u) = -u^{-2}$ and $g'(y) = \frac{1}{y} + 5y^4$.
  it follows $k'(y) = -(\ln(y) + y^5)^{-2}\left(\frac{1}{y} + 5y^4\right) = -\frac{\frac{1}{y}+5y^4}{(\ln(y)+y^5)^2} = -\frac{1+5y^5}{y(\ln(y)+y^5)^2}$
- If we have $m(x) = \sqrt{e^{x^2-5x+7} + 9}$, we can choose different approaches:
  1. $m(x) = \sqrt{e^{x^2-5x+7} + 9}$ is of the form $f\big(g(x)\big)$, where $f(y) = \sqrt{y}$ and $g(x) = e^{x^2-5x+7} + 9$:
  Since $f'(y) = \frac{1}{2\sqrt{y}}$ and $g'(x) = \frac{d}{dx}\left[e^{x^2-5x+7}\right] = (2x-5)e^{x^2-5x+7}$ (found above using the chain rule!): $\quad m'(x) = \frac{1}{2\sqrt{e^{x^2-5x+7}+9}} \times (2x-5)e^{x^2-5x+7} = \frac{(2x-5)e^{x^2-5x+7}}{2\sqrt{e^{x^2-5x+7}+9}}$
  2. $m(x) = \sqrt{e^{x^2-5x+7} + 9}$ is of the form $f(g[h(x)]\,)$, where $f(u) = \sqrt{u}$ and $g(y) = e^y + 9$ and $h(x) = x^2 - 5x + 7$. Chain rule for $f \circ g \circ h$ with $f'(u) = \frac{1}{2\sqrt{u}}$, $g'(y) = e^y$ and $h'(x) = 2x - 5$:
  $$m'(x) = \frac{d}{dx}f(g[h(x)]\,) = f'(g[h(x)\,]\,)g'[h(x)]h'(x) = \frac{1}{2\sqrt{e^{x^2-5x+7}+9}} \times e^{x^2-5x+7} \times (2x-5)$$

Applications of the chain rule: **1.** extended power rule: $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$

$$\textbf{2.}\quad \frac{d}{dx}a^x = \frac{d}{dx}e^{\ln a^x} = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\ln a$$

$$\text{So, if } f(x) = a^x, \text{then } f'(x) = a^x \ln(a)$$

<u>**Maxima and minima**</u>: **extreme values** $f(c)$ of f at c:

- $f(c)$ is an **absolute** maximum if $f(x) \le f(c)$ for all x in D_f
- $f(c)$ is an **absolute** minimum if $f(x) \ge f(c)$ for all x in D_f
- The maximum or minimum is **local** if the above is only true for "x near c", not for all x in D_f.

<u>**The Extreme Value Theorem:**</u> If $f(x)$ is **continuous at a closed interval** $[a, b]$, then f attains an absolute maximum and absolute minimum at $[a, b]$.

<u>**Fermat's theorem**</u>**:** if f has a **local maximum** or **minimum at** c and f is differentiable at c, then $\boldsymbol{f'(c) = 0}$.

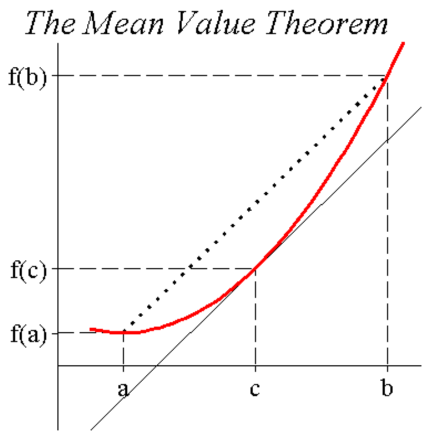
<u>Definition</u>: c is called a **critical value** of a function f if $f'(c) = 0$ or $f'(c)$ does not exist.

The closed interval method: finding extreme values of a continuous function on a closed interval **[*a*, *b*]:**

1. Compute $f(c)$ for all critical numbers c of f within the interval.
2. Compute $f(a)$ and $f(b)$.
3. The largest and the smallest of the function values at 1. and 2. are the absolute maximum and minimum

Example - applying the closed interval method on $f(x) = x^4 - 16x^2$ on the interval $[-1, 5]$:
Note that f meets the condition of continuity on the interval (f is continuous on $\mathbb{R}$).

1. For the critical values $f'(x) = 4x^3 - 32x = 4x(x^2 - 8) = 0 \Leftrightarrow x = 0$ or $x = 2\sqrt{2}$ or $x = -2\sqrt{2}$
   $-2\sqrt{2}$ is outside the interval, so we compute $\boldsymbol{f(0) = 0}$ and $\boldsymbol{f(2\sqrt{2}) = -80}$
2. For the interval boundaries we find: $\boldsymbol{f(-1) = -15}$ and $\boldsymbol{f(5) = 225}$
3. On the interval $[-1, 5]$ is $\boldsymbol{f(5) = 225}$ **the absolute maximum and** $\boldsymbol{f(2\sqrt{2}) = -80}$ **the absolute minimum**



<u>The Mean Value Theorem:</u>

If **1.** f is continuous on $[a, b]$ and
2. f is differentiable on (a, b),

then: there is a c in (a, b) such that $$\boldsymbol{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

Example – applying the Mean Value Theorem on the function $f(x) = x^2$, using the interval [0, 3].
Of course, f is continuous and differentiable on [0, 3].
According to the Mean Value Theorem, there must be a value of c between 0 and 3 such that
$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{3^2-0^2}{3-0} = 3.$ Since $f'(c) = 2c,$ there is such a $c: c = 1.5$
Note: for $f(x) = \frac{1}{x}$ and the interval $[-1, 1]$ the points $(-1, -1)$ and $(1, 1)$ have a line with slope $+1$, but there is no c such that $f'(c) = +1$ (since $f'(x) < 0$ for all $x \neq 0$), due to the fact that f is discontinuous at 0!

Tests to describe the shape ("behaviour") of the graph of the function:

<u>The increasing/decreasing test:</u>
If for all x in an interval I: **-** $\boldsymbol{f'(x) > 0}$ => f is an **increasing** function on I
- $\boldsymbol{f'(x) < 0}$ => f is a **decreasing** function on I

<u>The first derivative test:</u>
If, for a critical value c, $\boldsymbol{f'(c) = 0}$ and f' changes from positive to negative at c, then $f(c)$ is a local maximum, and $f(c)$ is a local minimum, if f'changes from negative to positive at c.
Note: $f'(c) = 0$ implies that f is continuous at $x = c$ and $f(c)$ is defined.

<u>The concavity test</u> If for x on an interval I: **-** $\boldsymbol{f''(x) > 0}$ => f is **concave upward** (**CU**) on I
- $\boldsymbol{f''(x) < 0}$ => f is **concave downward** (**CD**) on I
Note: The graph of $f(x) = ax^2 + bx + c$ is a parabola: $f'(x) = 2ax + b$ and $f''(x) = 2a$
*The parabola is **opening upward/concave upward if a > 0** and **opening/concave downward if a < 0.***

<u>The second derivative test:</u> for continuous f'' near c
- If $f'(c) = 0$ and $\boldsymbol{f''(c) < 0}$ => f has a local maximum at c.
- If $f'(c) = 0$ and $\boldsymbol{f''(c) > 0}$ => f has a local minimum at c.

Example: consider the function $f(x) = x^4 - 16x^2$ again, now on $\mathbb{R} = (-\infty, \infty)$
We found the critical values $f'(x) = 4x^3 - 32x = 4x(x^2 - 8) = 4x(x - \sqrt{8})(x + \sqrt{8})$
$$f'(x) = 0 \Leftrightarrow x = 0 \text{ or } x = 2\sqrt{2} \text{ or } x = -2\sqrt{2}$$
The increasing/decreasing test: $f'(x) > 0$ if $x > 2\sqrt{2}$ or if $-2\sqrt{2} < x < 0$
and: $f'(x) < 0$ if $x < -2\sqrt{2}$ or if $0 < x < 2\sqrt{2}$

The first derivative test: $f'(x)$ changes from positive to negative at $x = 0$ and from negative to positive at $x = -2\sqrt{2}$ and $x = -2\sqrt{2} \Rightarrow f$ has a local maximum $f(0) = 0$ at $x = 0$ and local minima $f(\sqrt{8}) = f(-\sqrt{8}) = -64$ at $x = 2\sqrt{2}$ and $x = -2\sqrt{2}$

$f''(x) = 12x^2 - 32 \Rightarrow f$ is **concave upward** if $f''(x) > 0 \Leftrightarrow x > \sqrt{\frac{32}{12}}$ or $x < -\sqrt{\frac{32}{12}}$

The second derivative test (*confirming the results of the* 1*st* *derivative test!):*
$f''(0) = -32 < 0 \Rightarrow f$ has a local maximum at 0
and $f''(\sqrt{8}) = f''(-\sqrt{8}) = 64 > 0$, so f has a local minima at $2\sqrt{2}$ and $-2\sqrt{2}$.

L' Hopital's Rule, an additional method for solving **indeterminate limits** (form $\frac{0}{0}$ or $\frac{\infty}{\infty}$):

If $f(x)$ and $g(x)$ are differentiable at a, then: $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$

How and when to apply L' Hopital's Rule:

1. This rule is also valid for limits at infinity ($x \to \pm\infty$)
2. Solving limits of form $0\times\infty$, using L' Hopital's Rule: $\lim\limits_{x\to a} f(x)\, g(x) = \lim\limits_{x\to a} \frac{f(x)}{1/g(x)}$ (now of form $\frac{0}{0}$)
3. Only apply the rule if other methods (direct substitution, factoring, root trick) do not work.

Some applications of L' Hopital's Rule

- Form $\frac{0}{0}$: $\lim\limits_{x\to 0} \frac{\sin x}{x} = \lim\limits_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim\limits_{x\to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$
- Form $\frac{\infty}{\infty}$: $\lim\limits_{x\to\infty} \frac{x}{e^x} = \lim\limits_{x\to\infty} \frac{1}{e^x} = 0$ and $\lim\limits_{x\to\infty} \frac{x^3}{e^x} = \lim\limits_{x\to\infty} \frac{3x^2}{e^x} = \lim\limits_{x\to\infty} \frac{6x}{e^x} = \lim\limits_{x\to\infty} \frac{6}{e^x} = 0$
  In general: $\lim\limits_{x\to\infty} \frac{polynomial}{e^x} = 0$
- Form $\frac{\infty}{\infty}$: $\lim\limits_{x\to\infty} \frac{\ln x}{x} = \lim\limits_{x\to\infty} \frac{1/x}{x} = \lim\limits_{x\to\infty} \frac{1}{x^2} = 0$. ***In general:*** $\lim\limits_{x\to\infty} \frac{ln\,(x)}{polynomial} = 0$
- Form $0\times\infty$: $\lim\limits_{x\to\infty} x\, sin\frac{1}{x} = \lim\limits_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim\limits_{x\to\infty} \frac{-\frac{1}{x^2}\cos\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \cos\frac{1}{x} = \cos 0 = 1$
- Form $0\times-\infty$: $\lim\limits_{x\to-\infty} xe^x = \lim\limits_{x\to-\infty} \frac{x}{e^{-x}} = (substitute\ y = -x) \lim\limits_{y\to\infty} -\frac{y}{e^y} = 0$

Reviewing the Methods of limit solving (a summary including L'Hopital):

1. Always try first **substitution: the substitution rule** can be applied to every function that is continuous on its domain.
2. If substitution leads to the form $\frac{c\neq 0}{0}$, we have a vertical asymptote: find the left and the right hand limit, by **reasoning (**∞ or $-\infty$)

If substitution leads to the **indeterminate** form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $0\times\infty$ or $\infty - \infty$, try:

3. **Factoring**, especially for rational functions or
4. **The root trick** for function with roots.
5. Limits at infinity of rational functions: **divide by the largest power of x in the denominator**.
   (for quotients of exponential functions: divide by the highest power of e^x or a^x in the denominator).
6. **L'Hopital's Rule**.
7. **The Squeeze Theorem.** For special cases such as functions of cos(x) and sin(x) (use $-1 \leq \sin x \leq 1$)

Curve Sketching: an extensive investigation of the curve of f should always cover the following 6 steps:

1. The **domain** D_f of the function f.

2. The ***x*-intercepts**: roots of the equation $f(x) = 0$, and the ***y*-intercept** $f(0)$.
3. Is f **even** ($f(-x) = f(x) \rightarrow$ the graph can be reflected about the *y*-axis), **odd** ($f(-x) = -f(x) \rightarrow$ reflection about the origin) or **periodic (**state the period**)**?
4. Find the equations of the asymptotes:

   Vertical asymptotes (VA) ***x* = *a***: e.g. $\ln(x)$ at $x = 0$, $\tan x$ at $\frac{1}{2}\pi$ or rational functions if $f(a)$ has form $\frac{c\neq 0}{0}$: find in that case both $\lim_{x\downarrow a} f(x)$ and $\lim_{x\uparrow a} f(x)$ (*reason if it is ∞ or -∞)*

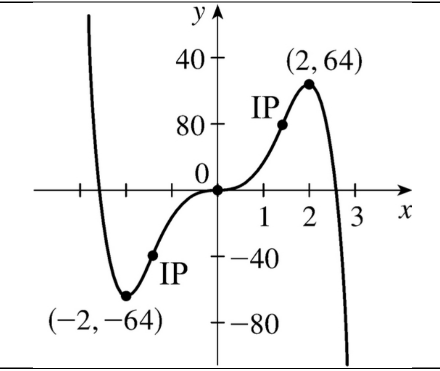
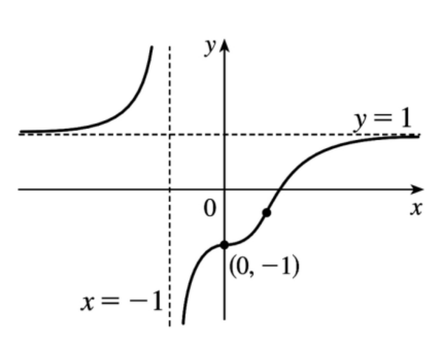
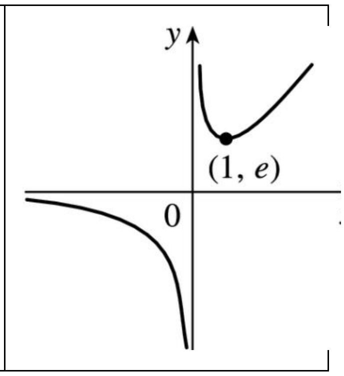
   Compute $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$: **horizontal asymptotes (HA) *y* = *L*:** $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$
5. Compute the derivative $f'(x)$
   - Solve $f'(x) = 0$ and give all critical values, including the values for which f is not defined.
   - Use the sign scheme of $f'(x)$ or the **second** derivative test to find all **local and absolute minima and maxima (***extremes***):** *do not forget to take all critical values (eg VA) into account.*
6. **Compute the second derivative** $f''(x)$ and find the **Inflection Points (IP)**: points *c* where the graph of f changes from upward to downward concavity, or reverse. These inflection points can be found by:

   1. Solving $f''(x) = 0$

   2. Making a sign scheme of $f''(x)$:

   if the sign changes at $x = a$ and $f(a)$ exists, $(a, f(a))$ is an inflection point of f.

Function (3 examples)	$f(x) = 20x^3 - 3x^5$	$f(x) = \frac{x^3-1}{x^3+1}$	$f(x) = \frac{e^x}{x}$
1. Domain	$D = \mathbb{R}$	$D = \mathbb{R}\backslash\{-1\}$ =(-∞,-1) ∪ (-1, ∞)	D =(-∞, 0) ∪ (0, ∞)
2. Intercepts	*y*-intercept: $f(0) = 0$ $20x^3 - 3x^5 = x^3(20 - 3x^2) = 0$ ⇔ *x*-intercepts 0, $\sqrt{\frac{20}{3}}$ or $-\sqrt{\frac{20}{3}}$	*y*-intercept: $f(0) = -1$ $f(x) = 0 \Leftrightarrow x^3 = 1 \Leftrightarrow$ $x = 1$ is the *x*-intercept	No *y*-intercept and no *x*-intercept
3. Even/odd/ periodic	$f(-x) = -20x^3 + 3x^5$ $= -f(x) \Rightarrow$ the graph can be reflected about the origin	$f(-x) = \frac{-x^3-1}{-x^3+1}$: f is even nor odd.	$f(-x) = -\frac{e^{-x}}{x}$ f is even nor odd
4. Asymp-totes	No asymptotes: $\lim_{x\to\infty} x^3(20 - 3x^2)$ = -∞ and $\lim_{x\to-\infty} x^3(20 - 3x^2)$ = +∞	HA *y*=1: $\lim_{x\to\infty}\frac{x^3-1}{x^3+1} = \lim_{x\to\infty}\frac{1-\frac{1}{x^3}}{1+\frac{1}{x^3}}=1$ VA $x = -1$: $\lim_{x\downarrow-1}\frac{x^3-1}{x^3+1} = -\infty$ and $\lim_{x\uparrow-1}\frac{x^3-1}{x^3+1} = \infty$	$\lim_{x\to\infty}\frac{e^x}{x} = \lim_{x\to\infty}\frac{e^x}{1} = \infty$ $\lim_{x\to-\infty}\frac{e^x}{x} = 0 \Rightarrow HA\ y = 0$ VA $x = 0$: $\lim_{x\downarrow 0}\frac{e^x}{x} = \infty$ and $\lim_{x\uparrow 0}\frac{e^x}{x} = -\infty$
5. $\boldsymbol{f'(x)}$ **and Extreme values**	$f'(x) = 60x^2 - 15x^4$ $= 15x^2(2-x)(2+x) = 0 \Leftrightarrow$ $x = 0$ or $x = 2$ or $x = -2$ f is increasing on (-2, 0) and (0, 2) and decreasing if $x < -2$ or $x > 2 \Rightarrow$ Abs. and local max. $f(2) = 64$ and abs. and local min. $f(2) = -64$	$f'(x) = \frac{3x^2(x^3+1)-(x^3-1)3x^2}{(x^3+1)^2}$ $= \frac{6x^2}{(x^3+1)^2} = 0 \Leftrightarrow x = 0$ No sign change of f' at *x* = 0 or at the second critical point *x* = -1. No (absolute or local) extreme values	$f'(x) = \frac{e^x x - e^x}{x^2}$ $= \frac{e^x(x-1)}{x^2} = 0 \Leftrightarrow$ $x = 1$, where f' changes sign: f is increasing for $x > 1$ and decreasing for $x < 1 \Rightarrow$ local min $f(1)=e$
6. $\boldsymbol{f''(x)}$ **and Inflection Points (IP)**	$f''(x) = 120x - 60x^3$ $= 60x(\sqrt{2}-x)(\sqrt{2}+x) = 0$ $x = 0\ or\ x = \sqrt{2}\ or\ x = -\sqrt{2}$ $f''(x)$ changes sign at all 3 values, so the inflection points are (0, 0), $(\sqrt{2}, 28\sqrt{2})$, $(-\sqrt{2}, -28\sqrt{2})$ f is concave upward for $x > \sqrt{2}$ or $\sqrt{2} < x < 0$ and concave downward on $(-\infty, -\sqrt{2})$ and $(0, \sqrt{2})$	$f''(x) = \frac{12x(x^3+1)^2 - 6x^2 \times 2\cdot(x^3+1)\cdot 3x^2}{(x^3+1)^4}$ $= \frac{12x-24x^4}{(x^3+1)^3} = \frac{12x(1-2x^3)}{(x^3+1)^3} = 0$ $\Leftrightarrow x = 0$ or $x = \sqrt[3]{\frac{1}{2}}$ $f''(x)$ is changing sign at -1 (VA), 0 and $\sqrt[3]{\frac{1}{2}}$, so IP (0, -1) and $(\sqrt[3]{\frac{1}{2}}, -\frac{1}{3})$ f is CU on (-∞, -1) and $(0, \sqrt[3]{\frac{1}{2}})$ and CD on (-1, 0) and $(\sqrt[3]{\frac{1}{2}}, \infty)$	$f''(x) = \frac{e^x(x^2-2x+2)}{x^3}$ $f''(x) = 0$ has no roots, so no inflection points. $(x^2 - 2x + 2 =$ $(x-1)^2 + 1 > 0\ )$ $f''(x) > 0\ CU\ if\ x > 0$ $f''(x) < 0\ CD\ if\ x < 0$

The sketch of the graph			

Optimization of a function *f*

An application of the procedure above is to find all minima and maxima of *f* on its domain.

Slant asymptotes (an **"extra"** for the 6 points curve sketching above**):**

The graph of the function $f(x)$ has **a slant asymptote $\boldsymbol{y = mx + b}$** if $\lim_{x\to\infty}[f(x) - (mx + b)] = 0$

or if $\lim_{x\to-\infty}[f(x) - (mx + b)] = 0$

Note that rational functions $f(x) = \frac{P(x)}{Q(x)}$ have a slant asymptote if : degree of $P(x)$ = 1+ degree of $Q(x)$

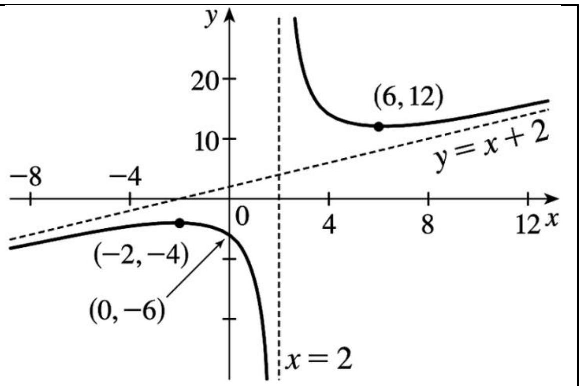
Example: Consider the function $f(x) = \frac{x^2+12}{x-2}$.

$\boldsymbol{y = x + 2}$ ***is a slant asymptote,*** since

$$\lim_{x\to\infty}[f(x) - (x + 2)] = \lim_{x\to\infty}\left[\frac{x^2+12}{x-2} - (x + 2)\right]$$

$$= \lim_{x\to\infty}\left[\frac{x^2+12}{x-2} - \frac{x^2-4}{x-2}\right] = \lim_{x\to\infty}\left[\frac{16}{x-2}\right] = 0$$

For $x \to -\infty$ we find the same result.



Note *that* $y = \frac{x^2+12}{x-2} = \frac{x^2-4+4+12}{x-2} = x - 2 + \frac{16}{x+2}$

This form can in general be found by applying "long division".

To sketch the curve we found furthermore:

- The domain is $\mathbb{R}\backslash\{x = 2\}$ and $x = 2$ is a VA: $\lim_{x\downarrow 2}\frac{x^2+12}{x-2} = \infty$ and $\lim_{x\uparrow 2}\frac{x^2+12}{x-2} = -\infty$
- The *y*-intercept $f(0) = -6$. There is no *x*-intercept: no solutions for $f(x) = 0$
- $f'(x) = \frac{2x(x-2)-1(x^2+12)}{(x-2)^2} = \frac{x^2-4x-12}{(x-2)^2} = \frac{(x-6)(x+2)}{(x-2)^2} = 0$ if $x = 6$ *or* $x = -2$

  f is increasing if $x < -2$ or $x > 6$ and decreasing if $-2 < x < 6$, *the asymptote* $x = 2$ excluded: sign changes of f' at $x = -1$ and 6: a local maximum $f(-2) = -4$ and a local minimum $f(6) = 12$

Implicit differentiation:

If the relation of *y* and *x* is given by an equation (not by a function), find the derivative $y' = \frac{dy}{dx}$ by:

- Differentiating the equation with respect to *x*.
- Solve y' from the resulting equation.

An example with implicit and explicit differentiation for finding the slope of the tangent line at an arbitrary point (*x*, *y*) on the unit circle $x^2 + y^2 = 1$.

Implicit differentiation: $\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}1 \iff \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1 \iff 2x + 2y\frac{dy}{dx} = 0$

So $\frac{dy}{dx} = -\frac{x}{y}$ For the point $\left(\frac{3}{5}, \frac{4}{5}\right)$ on the unit circle we find a slope $m = -\frac{3}{4}$ of the tangent line at this point

Explicit differentiation: $x^2 + y^2 = 1 \Leftrightarrow y^2 = 1 - x^2 \Leftrightarrow y = \pm\sqrt{1-x^2}$ (two functions $y > 0$ or $y < 0$)
$\Longrightarrow (y > 0)\, \frac{dy}{dx} = \frac{d}{dx}[1-x^2]^{\frac{1}{2}} = \frac{1}{2}[1-x^2]^{-\frac{1}{2}} \times -2x = -\frac{x}{\sqrt{1-x^2}}$: for $\left(\frac{3}{5}, \frac{4}{5}\right)$ we find $\frac{dy}{dx} = -\frac{3}{4}$

Some applications of Implicit differentiation

- The **derivative of** $\boldsymbol{f(x) = \log_a(x)}$ using the rule $\frac{d}{dx}(a^x) = a^x \ln(a)$ (see applications of chain rule)
  $y = \log_a(x) \Leftrightarrow x = a^y \quad \Rightarrow \frac{d}{dx}x = \frac{d}{dx}a^y \Leftrightarrow 1 = a^y \ln(a)\frac{dy}{dx} \Leftrightarrow f'(x) = \frac{dy}{dx} = \frac{1}{a^y \ln(a)} = \boldsymbol{\frac{1}{x\ln(a)}}$
- The **derivative of** $\boldsymbol{f(x) = \sin^{-1} x}$: $y = \sin^{-1} x \Leftrightarrow x = \sin y \quad \Rightarrow \frac{d}{dx}x = \frac{d}{dx}\sin y = \cos y \frac{dy}{dx}$
  $$f'(x) = \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$
  the derivatives of $\cos^{-1} x$ and $\tan^{-1} x$ can be found in a similar way

Derivatives of the inverse trigonometric functions:

$$f(x) = \sin^{-1}(x) \quad \Rightarrow \quad f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f(x) = \cos^{-1}(x) \quad \Rightarrow \quad f'(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$f(x) = \tan^{-1}(x) \quad \Rightarrow \quad f'(x) = \frac{1}{1+x^2}$$

The **anti derivative** of a function, a differentiation technique preparing for integrals (Calculus II):
$F(x)$ is an anti derivative of the function if $F'(x) = f(x)$.

Since the derivative of a constant *c* is 0, we can always add an arbitrary constant to $F(x)$:

$$\frac{d}{dx}[F(x) + c] = f(x)$$

Some common functions, their derivatives and anti derivatives (the constants *c* in F are omitted):

Anti derivative *F*	**Function *f***	**Derivative *f'***
$F(x) = ax$	$f(x) = a$	$f'(x) = 0$
$F(x) = \frac{1}{n+1}x^{n+1}$	$f(x) = x^n$	$f'(x) = nx^{n-1}$
$F(x) = e^x$	$f(x) = e^x$	$f'(x) = e^x$
$F(x) = xln(x) - x$	$f(x) = \ln(x)$	$f'(x) = \frac{1}{x}$
$F(x) = -\cos(x)$	$f(x) = \sin(x)$	$f'(x) = \cos(x)$
$F(x) = \sin(x)$	$f(x) = \cos(x)$	$f'(x) = -\sin(x)$
$F(x) = \tan(x)$	$f(x) = \frac{1}{\cos^2(x)}$	
$F(x) = \frac{a^x}{\ln(a)}$	$f(x) = a^x$	$f'(x) = a^x \ln(a)$

For $f(x) = \ln(|x|)$ the derivative is
$$f'(x) = \frac{1}{x}$$
$for\ x < 0\ \ f(x) = \ln(-x)$ and
$(chain\ rule!)\ f(x) = -1 \times \frac{1}{-x} = \frac{1}{x}$,
So the anti derivative of $f(x) = \frac{1}{x}$
on its domain is $F(x) = \ln(|x|)$

Exercises Derivatives Calculus I (Stewart chapter 3 and 4)

3.1 Differentiate: **a.** $f(x) = \sqrt{30}$ **b.** $g(x) = 5x - 1$ **c.** $h(x) = 2x^2 - x - 6$ **d.** $y = x^{-2/5}$

e. $f(t) = 6t^{-9}$ **f.** $y = \sqrt[3]{x}$ **g.** $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$ **h.** $f(x) = x^2 e^x$

i. $g(x) = \sqrt{x}e^x$ **j.** $y = \dfrac{e^x}{x^2}$ **k.** $y = \dfrac{e^x}{1 + x}$ **l.** $y = \dfrac{x^3}{1 - x^2}$

m. $y = (r^2 - 2r)e^r$ **n.** $f(x) = x - 3\sin(x)$ **o.** $y = \sin(x) + 10\tan(x)$ **p.** $g(t) = t^3\cos(t)$

q. $y = \sin(4x)$ **r.** $y = (1 - x^2)^{10}$ **s.** $y = e^{\sqrt{x}}$ **t.** $F(x) = (x^3 + 4x)^7$

u. $F(z) = \sqrt{\dfrac{z - 1}{z + 1}}$ **v.** $y = \dfrac{r}{\sqrt{r^2 + 1}}$ **w.** $y = \tan(\cos x)$ **x.** $f(x) = \ln(x^2 + 10)$

y. $f(x) = \sin(\ln x)$ **z.** $f(x) = \ln(\sin^2 x)$

3.2 Find **the equation of the tangent line** to the curve at the given point

a. $y = \sqrt[4]{x}$ at $(1, 1)$ **b.** $y = x^4 + 2e^x$ at $(0, 2)$ **c.** $y = 2xe^x$ at $(0, 0)$

3.3 Find the first and the second derivative:

a. $h(x) = x^4 - 3x^3 + 16x$ **b.** $g(x) = ax^2 + bx + c$ **c.** $y = xe^{2x}$ **d.** $h(x) = x^2\ln(2x)$

3.4 Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent line is horizontal.

3.5 At what numbers is *f* differentiable? Find out by graphically checking whether *f* is continuous and differentiable at -1 and at 1: Sketch the graphs of *f* and *f* ʻ

$$f(x) = \begin{cases} -1 - 2x & if \quad x < -1 \\ x^2 & if \ -1 \le x \le 1 \\ x & if \quad x > 1 \end{cases}$$

3.6 Differentiate **a.** $f(x) = 2^x$ **b.** $g(x) = 5^x$ **c.** $h(x) = 2^{\sin(x)}$

3.7 Use the identity $y = e^{\ln(y)}$ to differentiate: **a.** $y = x^x$ **b.** $g(x) = (\cos x)^x$

3.8 The reciprocals of sin, cos and tan are defined: $\csc(x) = (\sin x)^{-1}$, $\sec(x) = (\sin x)^{-1}$, $\cot(x) = (\tan x)^{-1}$ Find the derivatives of **a.** $\csc(x)$ **b.** $\sec(x)$ and **c.** $\cot(x)$

3.9 Sketch the graph of *f* on the given domain (use transformations) to find the absolute maximum and the abs. minimum. Check whether *f* is an increasing or decreasing function on this domain by computing $f'(x)$.

a. $f(x) = 8 - 3x,\ x \ge 1$ **b.** $f(x) = x^2,\ 0 \le x < 2$ **c.** $f(x) = x^2, -3 \le x \le 2$ **d.** $f(x) = 1 - \sqrt{x}$

3.10 Find the critical numbers of the function

a. $f(x) = 5x^2 + 4x$ **b.** $f(x) = x^3 + 3x^2 - 24x$ **c.** $g(y) = \dfrac{y - 1}{y^2 - y + 1}$ **d.** $f(x) = x^2e^{-3x}$

3.11 Find the absolute maximum and absolute minimum on the given interval

a. $f(x) = 3x^2 - 12x + 5$, [0,3] **b.** $f(x) = 2x^3 - 3x^2 - 12x + 1$, [-2, 3]

c. $f(x) = \dfrac{x}{x^2 + 1}$, $[0, 2]$ **d.** $f(x) = x - \ln(x)$, [½ , 2]

3.12 Verify that the function *f* satisfies the conditions of the Mean Value Theorem on the **interval [-1, 1]** and find all numbers *c* that satisfy the conclusion of the of the Mean Value Theorem: $f(x) = 3x^2 + 2x + 5$

3.13 Find the limit using L'Hospital's Rule, but only if a more elementary method is not possible.

a. $\lim\limits_{x\to 2} \dfrac{x^2 + x - 6}{x - 2}$ **b.** $\lim\limits_{x\to 2} \dfrac{x^9 - 1}{x^5 - 1}$ **c.** $\lim\limits_{t\to 0} \dfrac{e^{3t} - 1}{t}$ **d.** $\lim\limits_{x\to\infty} \dfrac{\ln(x)}{\sqrt{x}}$

e. $\lim\limits_{x\downarrow 0} \dfrac{\ln(x)}{x}$ **f.** $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2 + 2}}{\sqrt{2x^2 + 1}}$ **g.** $\lim\limits_{x\to-\infty} x^2e^x$ **h.** $\lim\limits_{x\to\infty} e^{-x}\ln(x)$

i. $\lim\limits_{x\to\infty} (xe^{1/x} - x)$ **j.** $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x}$ **k.** $\lim\limits_{x\to 0} \dfrac{1 - \cos(x)}{x}$

3.14 In general $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x} = 0$ is true for all integer $n > 0$ Proof it for *n* = 3 using L'Hospital's Rule

In general $\lim\limits_{x\to\infty} \dfrac{\ln(x)}{x^n} = 0$ is true for all $n > 0$ Proof it for *n* = 1½ using L'Hospital's Rule

3.15 a. Show that $f(x) = \frac{x^3}{x^2+1}$ has a slant asymptote $y = x$

b. Show that $f(x) = \sqrt{x^2 + 4x}$ has two slant asymptotes: $y = x + 2$ and $y = -x - 2$

3.16 Find the anti derivative (Check the correctness of the anti derivative by differentiating it!):

a. $f(x) = x^5$ **b.** $f(x) = x^3 - 3x^2 - 5x + 8$ **c.** $f(x) = x\sqrt[3]{x^2}$ **d.** $f(x) = \sin(x) + \cos(4x)$

e. $f(x) = \frac{1}{x^2+1}$ **f.** $f(x) = \frac{1}{2x}$ **g.** $f(x) = e^{-3x}$ **h.** $f(x) = \sqrt{2x + 1}$

3.17 Find $\frac{dy}{dx}$ using implicit differentiation: a. $x^3 + y^3 = 1$ b. $xe^y = x - y$

3.18 Sketch the graph of the following functions after investigating them: find
1. The domain, **2. *x*- and *y-intercept*,** **3. Even/odd/periodic,**
4. Vertical and horizontal asymptotes **5. Derivative and all extreme values**
6. 2nd Derivative and inflection points (Slant asymptotes are "extra")

a. $y = x^4 + 4x^3$ **b.** $y = \frac{x}{x-1}$ **c.** $y = \dfrac{x^2}{x^2+3}$ **d.** $y = x\sqrt{5-x}$

e. $y = 8x^2 - x^4$ **f.** $y = \dfrac{\ln(x)}{x^2}$ **g.** $f(x) = \frac{\cos x}{2+\sin x}$, $0 \le x \le 2\pi$ **h.** $g(x) = ln(4-x^2)$ **i.** $f(x) = xe^x$

Answers exercises derivatives:

3.1
a. $f'(x) = 0$ **b.** $f'(x) = 5$ **c.** $h'(x) = 4x - 1$ **d.** $y' = -\dfrac{2}{5}x^{-7/5}$

e. $f'(t) = -54t^{-10}$ **f.** $y' = \dfrac{1}{3}x^{-\frac{2}{3}} = \dfrac{1}{3\sqrt[3]{x^2}}$ **g.** $y' = \frac{3}{2}\sqrt{x} + \frac{2}{\sqrt{x}} - \frac{3}{2x\sqrt{x}}$ **h.** $f'(x) = (2x + x^2)e^x$

i. $g'(x) = (\dfrac{1}{2\sqrt{x}} + \sqrt{x})e^x$ **j.** $y' = \dfrac{(x-2)e^x}{x^3}$ **k.** $y' = \dfrac{xe^x}{(1+x)^2}$ **l.** $y' = \dfrac{3x^2 - x^4}{(1-x^2)^2}$

m. $y' = (r^2 - 2)e^r$ **n.** $f'(x) = 1 - 3\cos(x)$ **o.** $y' = \cos(x) + \dfrac{10}{cos^2 x}$ **p.** $g'(t) = 3t^2\cos(t) - t^3\sin(t)$

q. $y' = 4cos(4x)$ **r.** $y' = -20x(1-x^2)^9$ **s.** $y' = \dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$ **t.** $F'(x) = 7(3x^2+4) \times (x^3+4x)^6$

u. $F'(z) = \dfrac{1}{(z+1)^2}\sqrt{\dfrac{z+1}{z-1}}$ **v.** $y' = \dfrac{\sqrt{r^2+1} - \frac{r^2}{\sqrt{r^2+1}}}{r^2+1}$ **w.** $y' = \dfrac{-\sin(x)}{\cos^2(\cos x)}$ **x.** $f'(x) = \dfrac{2x}{x^2+10}$

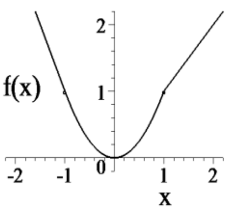
y. $f'(x) = \dfrac{\cos(\ln x)}{x}$ **z.** $f'(x) = \dfrac{2\sin(x)\cos(x)}{\sin^2 x} = \dfrac{2}{\tan(x)}$ [or use $f(x) = 2\ln(|\sin x|)$]

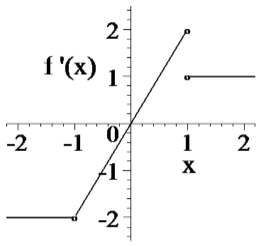
3.2 **a.** $y - 1 = \frac{1}{4}(x-1)$ or $y = \frac{1}{4}x + \frac{3}{4}$ **b.** $y - 2 = 2(x - 0)$ **c.** $y = 2x$

3.3 **a.** $h'(x) = 4x^3 - 9x^2 + 16$, $h''(x) = 12x^2 - 18x$ **b.** $g'(x) = 2ax + b$, $g``(x) = 2a$
c. $y' = (1+2x)e^{2x}$, $y'' = 4(1+x)e^{2x}$ **d.** $h'(x) = x\ln(2x) + \frac{1}{2}x$, $h``(x) = \ln(2x) + 1$

3.4 $y' = 6x^2 + 6x - 12 = 6(x+2)(x-1) = 0 \Leftrightarrow x = -2$ or $x = 1$:
a horizontal tangent line at (-2, 29) and (1,-6)

3.5 $f'(x) = \begin{cases} -2 & if \quad x < -1 \\ 2x & if \; -1 < x < 1 \\ 1 & if \quad x > 1 \end{cases}$





f is continuous at *x* = -1 and *x* = 1: at *x* = -1 the derivative from the left is equal to the derivative from the right. This is not the case at *x* = 1. *f* is differentiable at *x* = -1, but not at *x* = 1.

3.6 **a.** $f'(x) = 2^x\ln(2)$ **b.** $g'(x) = 5^x\ln(5)$ **c.** $h'(x) = 2^{sin(x)}\cos(x)\ln(\sin x)$

3.7 **a.** $y = x^x = e^{\ln(x^x)} = e^{x\ln(x)} => y` = e^{x\ln(x)}(\ln(x)+1) = x^x(\ln(x)+1)$
b. $g'(x) = (\cos x)^x(-x\tan x + \ln(\cos x))$

3.8 **a.** $\csc x = (\sin x)^{-1} => (\csc x)' = -(\sin x)^{-2}\cos(x)$ **b.** $(\sec x)' = (\cos x)^{-2}\sin(x)$ [$= \sec(x)\tan(x)$]
c. $\cot(x) = (\tan x)^{-1} = \frac{\cos(x)}{\sin(x)} => (\cot x)' = \frac{(\cos x)` \times \sin(x) - \cos(x) \times (\sin x)`}{\sin^2(x)} = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)}$

3.9 **a.** $f'(x) = -3$: decreasing on [0,∞) **b.** $f'(x) = 2x$: increasing on [0,2) **c.** $f'(x) = 2x$ (nor incr. or decr. on [-3,2]) **d.** $f'(x) = -\dfrac{1}{2\sqrt{x}}$: decreasing

3.10 **a.** -2/5 **b.** -4 and 2 **c.** 0 and 2 **d.** 0 and 2/3

3.11 **a.** *f*(0) = 5: abs. max *f*(2) = -7: abs. min. **b.** *f*(3) = 19: abs. max *f*(1) = -1: abs. min. **c.** *f*(±√2) = -2: abs. min. *f*(-3) = 47: abs. max. **d.** *f*(2) = 2-ln(2) ≈1.31 abs. max. *f*(1) = 1: abs. min.

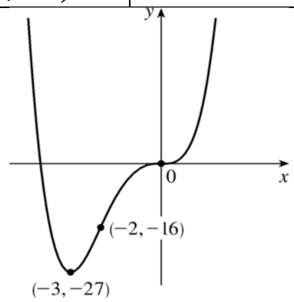
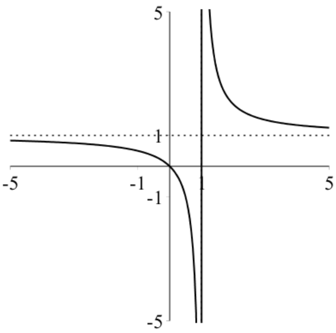
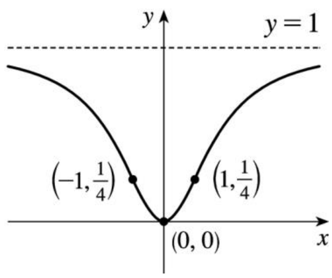
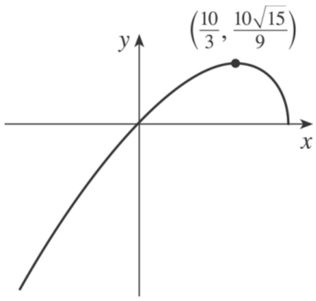
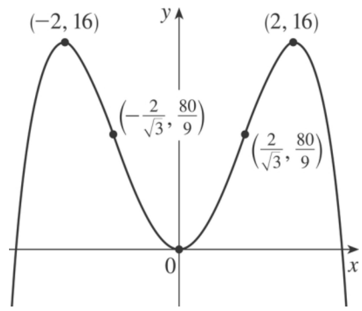
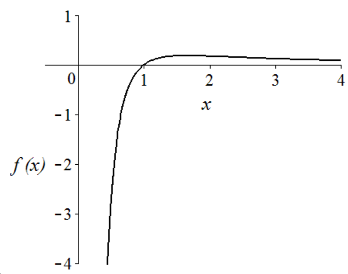
3.12 *c* = 0

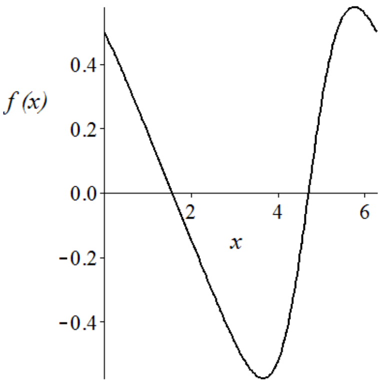
3.13 **a.** 5 (factor!) **b.** 9/5 (L'Hospital) **c.** 3 (L'Hospital) **d.** 0 (L'Hospital)
e. -∞ (Reasoning) **f.** ¼ (divide by $x = \sqrt{x^2}$) **g.** 0 (L'Hospital) **h.** 0 (L'Hospital)
i. 1 (fact.+ L'Hospital)

3.16 **a.** $\frac{1}{6}x^6$ **b.** $\frac{1}{4}x^4 - x^3 - 5x + 8$ **c.** $\frac{3}{8}x^{8/3} = \frac{3}{8}x^2\sqrt[3]{x^2}$ **d.** $-\cos x + \frac{1}{4}\sin 4x$

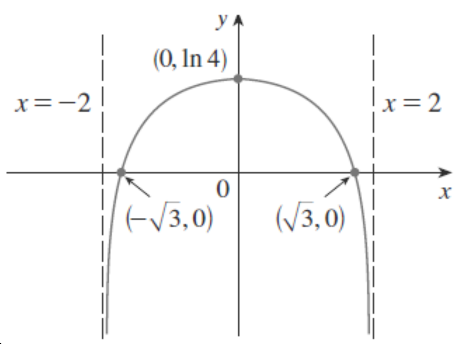
e. $\tan^{-1} x$ **f.** $\frac{1}{2} ln\,|2x|$ **g.** $-\frac{1}{3}e^{-3x}$ h. $\frac{1}{3}(2x+1)^{\frac{3}{2}}$

3.17 **a.** $y' = -\frac{x^2}{y^2}$ **b.** $\frac{dy}{dx} = \frac{1-e^y}{xe^y+1}$

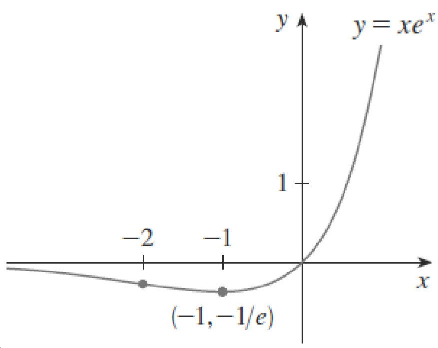
3.18	**a.**	**b.**	**c.**	**d.**	**e.**	**f.**
1. Domain	D = ℝ	D = $\{x \mid x \neq 1\}$	D = ℝ	D = (-∞, 5]	D = ℝ	D = (0, ∞)
2. intercepts	$f(0) = 0$, $f(x) = 0 \Leftrightarrow x = -4, 0$	$f(0) = 0$, $f(x) = 0 \Leftrightarrow x = 0$	$f(0) = 0$, $f(x) = 0 \Leftrightarrow x = 0$	$f(0) = 0$, $f(x) = 0 \Leftrightarrow x = 0, 5$	$f(0) = 0$, $f(x) = 0 \Leftrightarrow x = 0, \pm 2\sqrt{2}$	**$f(0)$ does not exist, $f(x)=0 \Leftrightarrow x=0$**
3 even/per.	No symmetry	No symmetry	f is even	no symmetry	f is even	**no symmetry**
4. Asymptote	No asymptotes	H.A. $y = 1$ V.A. $x = 1$	H.A. $y = 1$	No asymptote	No asymptote	**V.A. $x = 0$, H.A. $y = 0$**
5. $f'(x)$ and extreme values	$y' = 4x^3 + 12x^2$ Local **and absolute** minimum: $f(-3) = 27$	$f'(x) = -\frac{1}{(x-1)^2}$ no extreme values	$f'(x) = \frac{6x}{(x^2+3)^2}$ Absolute minimum: $f(0) = 0$	$f'(x) = \frac{10-3x}{2\sqrt{5-x}}$ Absolute maximum $f(\frac{10}{3}) \approx 4.3$	$f'(x) = 16x - 4x^3$ Local **and absolute** max. $f(\pm 2) = 16$, loc. min $f(0) = 0$	$f'(x) = \frac{1-2\ln x}{x^3}$ Local **and abs. max.:** $f\left(e^{\frac{1}{2}}\right) = \frac{1}{2e}$
6. $f''(x)$ and Inflection Points (IP)	$y'' = 12x^2 + 24x$ IP: (0, 0) and (-2, -16)	$f''(x) = \frac{2}{(x-1)^3}$ no IP	$f''(x) = \frac{-18(x+1)(x-1)}{(x^2+3)^3}$ 2 IP: (±1, ¼)	$f''(x) = \frac{3x-20}{4(5-x)^{\frac{3}{2}}}$ no IP: $\frac{20}{3}$ not in domain	$f''(x) = 16 - 12x^3$ 2 IP's: $(\pm\frac{2}{\sqrt{3}}, \frac{80}{9})$	$f''(x) = \frac{6\ln x - 5}{x^4}$ IP: $(e^{\frac{5}{6}}, \frac{5}{6e^{\frac{5}{3}}})$
The graphs	 a.  b.		 c.  d.		 e.  f.	



g.



h.



i.