THE TRAVEL TIME IN CAROUSEL SYSTEMS
UNDER THE NEAREST ITEM HEURISTIC

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Abstract

A carousel is an automated warehousing system consisting of a large number of drawers rotating in a closed loop. In this paper, we study the travel time needed to pick a list of items when the carousel operates under the nearest item heuristic. We find a closed form expression for the distribution and all moments of the travel time. We also analyse the asymptotic behaviour of the travel time when the number of items tends to infinity. All results follow from probabilistic arguments based on properties of uniform order statistics.

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1. Introduction

A carousel is an automated warehousing system consisting of a large number of drawers rotating in a closed loop in either direction. Such systems are mostly used for the storage and retrieval of small and medium sized goods, which are requested moderately often. The picker has a fixed position in front of the carousel, which rotates the required items to the picker. For a recent review of literature on carousels, as part of a general overview of planning and control of warehousing systems, we refer the reader to [9].

An important performance characteristic is the total time needed to pick a list of items. This consists of the pure pick time plus the travel time. Only the latter depends on the pick strategy. In this paper we consider the nearest item heuristic, where the next item to be picked is always the nearest one. We will study the statistical properties of the travel time under this heuristic.

We model the carousel as a circle of length 1. For ease of presentation, we act as if the picker travels to the items, instead of the other way around. The picker travels at unit speed and has to pick \(n\) (\(>0\)) items under the nearest item heuristic. Their positions are uniformly distributed on the circle. Using probabilistic arguments based on properties of uniform order statistics, we derive closed form expressions for the distribution and all moments of the travel time needed to pick \(n\) items. We also investigate the asymptotic behaviour of the travel time as \(n\) tends to infinity.

The performance of the nearest item heuristic has also been investigated by Bartholdi and Platzman [3]. They prove that the travel time under the nearest item heuristic is never greater...
than one rotation of the carousel. Litvak et al. [6] improve this upper bound and show that the new upper bound is tight. Using an analytical approach, they find the mean and variance of the remaining travel time under the nearest item heuristic, i.e., the travel time, when there is an empty space at one side of the picker’s position. They also introduce a probabilistic approach to determining the mean total travel time. In fact, in the present paper we elaborate this probabilistic approach and we show that it enables us to completely analyse the travel time.

The paper is organized as follows. In Section 2 we prove that the travel time under the nearest item heuristic can be represented as a linear combination of uniform spacings, or equivalently as a linear combination of independent normalized exponential random variables. From this representation we derive a closed form expression for the distribution of the travel time. In Section 3 we obtain all moments of the travel time. Finally, in Section 4 we give an exhaustive analysis of the limiting behaviour of the travel time distribution.

2. Distribution of the travel time

Let the random variable \( U_0 \) be the picker’s starting point and the random variable \( U_i \), where \( i = 1, 2, \ldots, n \), be the position of the \( i \)th item. We suppose that the variables \( U_i \), \( i = 0, 1, \ldots, n \), are independent and uniformly distributed on the interval \([0, 1)\). Let \( U_{0:n+1}, U_{1:n+1}, \ldots, U_{n:n+1} \) denote the order statistics of the random variables \( U_0, \ldots, U_n \) on \([0, 1)\). These order statistics partition the circle into \( n + 1 \) spacings with lengths

\[
D_1 = U_{1:n+1} - U_{0:n+1}, \quad \ldots, \quad D_n = U_{n:n+1} - U_{n-1:n+1},
\]

\[
D_{n+1} = 1 - U_{n:n+1} + U_{0:n+1}. \tag{1}
\]

To find the distribution of the travel time under the nearest item heuristic, we use the following very useful property of these spacings. If \( Y_1, \ldots, Y_{n+1} \) are independent exponentials with the same mean, then the random vectors \((D_1, \ldots, D_{n+1})\) and \((Y_1 / \sum_{i=1}^{n+1} Y_i, \ldots, Y_{n+1} / \sum_{i=1}^{n+1} Y_i)\) are identically distributed (cf. [7], [8], or [5, Section 13.1]). Hence, the uniform spacings are normalized exponentials.

Under the nearest item heuristic the picker does not have to know all spacings at once. He first considers the two spacings adjacent to his starting position, and then travels to the nearest item. Next he also looks at the other spacing adjacent to that item and compares the distance to the item located at the endpoint of that spacing and the distance to the first item in the other direction, which is the sum of the spacings previously considered. Then he travels again to the nearest item, and so on. Furthermore, we may act as if the picker faces non-normalized exponential spacings, and afterwards divide the travel time by the sum of all spacings. Then it is clear that each new spacing faced by the picker is independent of the ones already observed.

Now let \( X_i \), where \( i = 1, \ldots, n + 1 \), denote the \( i \)th non-normalized exponential spacing faced by the picker, so the spacings are numbered as observed by the picker operating under the nearest item heuristic (see Figure 1). Let

\[
S_0 = 0; \quad S_i = \sum_{j=1}^{i} X_j, \quad i = 1, 2, \ldots,
\]

and let the random variable \( T_n \) denote the travel time needed to pick \( n \) items under the nearest item heuristic. Then \( T_n \) can be expressed as

\[
T_n = \sum_{i=1}^{n} \frac{\min(S_i, X_{i+1})}{S_{n+1}}. \tag{2}
\]
The travel time under the nearest item heuristic

In this section we introduce a quite simple random variable, which has the same distribution as the right-hand side of (2). Further, we shall use the common notation \( X \overset{d}{=} Y \) to indicate that the random variables \( X \) and \( Y \) have the same distribution. Now we are going to prove our main result.

**Theorem 1.** Let \( X_1, X_2, \ldots \) be independent exponentials with mean 1. Then for \( n = 1, 2, \ldots \)

\[
\sum_{i=1}^{n} \frac{\min(S_i, X_{i+1})}{S_{n+1}} \overset{d}{=} \sum_{j=1}^{n} \left( 1 - \frac{1}{2^{i-j}} \right) \frac{X_j}{S_{n+1}}. \tag{3}
\]

**Proof.** Let us assume that for some \( i = 1, 2, \ldots, n \)

\[
\sum_{j=1}^{n} \frac{\min(S_j, X_{j+1})}{S_{n+1}} \overset{d}{=} \sum_{j=1}^{i-1} \left( 1 - \frac{1}{2^{i-j}} \right) \frac{X_j}{S_{n+1}} + \sum_{j=i}^{n} \frac{\min(S_j, X_{j+1})}{S_{n+1}}. \tag{4}
\]

By expanding the term \( \min(S_j, X_{j+1}) \), that appears on the right-hand side, we will show that (4) then also holds for \( i + 1 \). Since (4) trivially holds for \( i = 1 \), this would complete the proof.

Given the event

\[ E_{i+1,k} = [S_{k-1} \leq X_{i+1} < S_k], \]

for some \( k = 1, \ldots, i \), the random variables \( X_1, \ldots, X_{n+1} \) can be coupled as

\[
X_j = \begin{cases} 
\frac{1}{2} Y_{l}, & l = 1, \ldots, k - 1, \\
\frac{1}{2} Y_k + Y_{k+1}, & l = k, \\
Y_{k+1}, & l = k + 1, \ldots, i, \\
\sum_{l=1}^{i} \frac{1}{2} Y_{l}, & l = i + 1, \\
Y_{i+1}, & l = i + 2, \ldots, n + 1, 
\end{cases}
\]

where \( Y_1, Y_2, \ldots \) are independent exponentials with mean 1. This follows by observing that, given \( E_{i+1,k} \), the random variable \( X_1 \) is the minimum of \( X_1 \) and \( X_{i+1} \), and thus it is exponential.

**Figure 1:** The nearest item route of the picker facing 5 exponential spacings.
with mean $\frac{1}{2}$. Since the overshoot of $X_{i+1}$ is again exponential with mean 1, we can repeat the argument for $X_2$, and so on. Eventually $X_{i+1} - S_k$ is less than $X_k$, so it is exponential with mean $\frac{1}{2}$. The random variable $X_k$ is then the sum of two exponentials, one with mean $\frac{1}{2}$ and the other part (i.e., the overshoot) with mean 1 (see also Figure 2). Since the event $E_{i+1,k}$ does not provide any information on the other random variables, they remain exponential with mean 1.

Now, given the event $E_{i+1,k}$, it follows that
\[
\min(S_i, X_{i+1}) = X_{i+1} = \sum_{l=1}^{k} \frac{1}{2} Y_l,
\]
and for any $j = i + 1, \ldots, n + 1$ we have
\[
S_j = \frac{1}{2} Y_1 + \cdots + \frac{1}{2} Y_k + Y_{k+1} + \cdots + Y_{i+1} + \sum_{l=1}^{k} \frac{1}{2} Y_l + Y_{i+2} + \cdots + Y_j
= Y_1 + \cdots + Y_j.
\]
Hence, given $E_{i+1,k}$, we can replace the $X_j$ in the right-hand side of (4) by $Y_j$, yielding
\[
\sum_{j=1}^{n} \left(1 - \frac{1}{2^{j+1-j}}\right) \frac{Y_j}{Y_1 + \cdots + Y_{n+1}} + \sum_{j=i+1}^{n} \frac{\min(\sum_{l=1}^{j} Y_l, Y_{j+1})}{Y_1 + \cdots + Y_{n+1}}.
\]
(5)
Note that (5) does not depend on $k$.

Along the same lines, it can be verified that, given the event $[X_{i+1} \geq S_i]$, the right-hand side of (4) has again the same distribution as (5). Now, it immediately follows from the law of full probability that
\[
\sum_{j=1}^{n} \frac{\min(S_j, X_{j+1})}{S_{n+1}} = \sum_{j=1}^{i} \left(1 - \frac{1}{2^{j+1-j}}\right) \frac{X_j}{S_{n+1}} + \sum_{j=i+1}^{n} \frac{\min(S_j, X_{j+1})}{S_{n+1}},
\]
where the $Y_l$ in (5) are replaced again by $X_l$. Thus, by subsequently expanding $\min(S_1, X_2)$, $\min(S_2, X_3)$, ..., $\min(S_n, X_{n+1})$ we finally obtain:
\[
\sum_{j=1}^{n} \frac{\min(S_j, X_{j+1})}{S_{n+1}} \geq \sum_{j=1}^{n} \left(1 - \frac{1}{2^{n+1-j}}\right) \frac{X_j}{S_{n+1}},
\]
which is exactly (3).

We can also rewrite (3) in terms of uniform spacings $D_1, D_2, \ldots, D_{n+1}$. Together with (2) it gives
\[
T_n = \sum_{i=1}^{n} \left(1 - \frac{1}{2^i}\right) D_i.
\]
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Note that it immediately follows from (6) that

\[ P(T_n < x) = 1, \quad x \geq 1 - \frac{1}{2^n}. \]

which agrees with the upper bound of [6] (see Corollary 3.4).

Now we use the following result presented in Theorem 2 of [2]. Actually, this theorem has been proved in [1].

**Theorem 2.** (Ali.) Let \( D_1, D_2, \ldots, D_{n+1} \) be the uniform spacings as defined by (1). Then

\[ P\left( \sum_{i=1}^{n} a_i D_i < x \right) = \frac{1}{n} \sum_{k=0}^{n} f(a_k) \prod_{j=0}^{n} (a_k - a_j)^{-1}. \]  

(7)

where \( a_0 = 0, x_+ = x \) if \( x < 0, x_- = 0 \) if \( x \geq 0 \), and the divided difference of a function \( f(z) \) with distinct arguments \( a_0, a_1, \ldots, a_n \) is given by

\[ [ f(z) | z = a_0, a_1, \ldots, a_n ] = \sum_{k=0}^{n} f(a_k) \prod_{j=0}^{n} (a_k - a_j)^{-1}. \]  

(8)

Applying (7) and (8) to (6) we immediately obtain the following result.

**Theorem 3.** For all \( n = 1, 2, \ldots \)

\[ P(T_n < x) = \sum_{k=0}^{n} (-1)^k c_{n-k} (2^k x - 2^k + 1)^{n-k} \prod_{j=1}^{k} \frac{1}{2^{j-1}}, \quad 0 < x \leq 1, \]  

(9)

where \( x_+ = x \) if \( x > 0, x_- = 0 \) if \( x \leq 0 \), and the coefficients \( c_m \) are defined to be

\[ c_0 = 1; \quad c_m = 2 \times \frac{4}{3} \times \cdots \times \frac{2^m}{2^{m-1}}, \quad m \geq 1. \]  

(10)

Formula (9) gives different expressions for the distribution function of \( T_n \) in the sequence of intervals \((0, \frac{1}{2}), (\frac{1}{2}, \frac{3}{4}), \ldots, (1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n})\). The expression in \((0, \frac{1}{2})\) is the simplest one:

\[ P(T_n < x) = 2 \times \frac{4}{3} \times \cdots \times \frac{2^n}{2^{n-1}} x^n, \quad 0 < x \leq \frac{1}{2}. \]

In the next interval we get one additional term:

\[ P(T_n < x) = 2 \times \frac{4}{3} \times \cdots \times \frac{2^n}{2^{n-1}} x^n - 2 \times \frac{4}{3} \times \cdots \times \frac{2^{n-1}}{2^{n-1} - 1} (2x - 1)^n, \quad \frac{1}{2} < x \leq \frac{3}{4}. \]

Furthermore, an extra term appears in each of the following intervals.

In Figure 3 we show the distribution of the travel time for several values of \( n \).

The distribution of \( T_n \) can also be written in another form:

\[ P(T_n < x) = 1 - \sum_{k=0}^{n} (-1)^k c_{n-k} (2^k x - 2^k + 1)^{n-k} \prod_{j=1}^{k} \frac{1}{2^{j-1}}, \quad 0 < x \leq 1, \]  

(11)
which yields an expression with fewer terms than in (9) for $1 - \frac{k-1}{2} < x \leq 1 - \frac{k}{2}$, when $k$ is greater than $\frac{1}{2}n$. To establish (11), we note that for any $x$

$$
\sum_{k=0}^{n} (-1)^k e_{n-k}(2^k x - 2^k + 1)^n \prod_{j=1}^{k} \frac{1}{2^j - 1} = 1.
$$

(12)

Indeed, let us consider the function

$$
f(s, x) = \frac{1}{1-s} \times \frac{2}{2-s} \times \cdots \times \frac{2^n}{2^n-s} (sx-s+1)^n.
$$

This function can be expanded in rational fractions of $s$ as

$$
f(x, s) = \frac{1}{1-s} e_n x^n - \frac{2}{2-s} e_{n-1} (2x-1)^n + \frac{4}{4-s} \frac{1}{3} e_{n-2} (4x-3)^n - \cdots + (-1)^n \frac{2^n}{2^n-s} \left( \prod_{j=1}^{n} \frac{1}{2^j-1} \right) c_0 (2^n x - 2^n + 1)^n.
$$

Putting $s = 0$ we get (12). Combining (9) with (12) we immediately obtain (11).

3. Moments of the travel time

In this short section, we shall use the representation (6) to directly calculate the moments of the travel time $T_n$. From (6) we find that for the $k$th moment of the travel time,

$$
E[T_n^k] = E\left[ \left( \sum_{i=1}^{n} \left( 1 - \frac{1}{2} \right) D_i \right)^k \right] = \sum_{k_1, k_2, \ldots, k_n \geq 0} \frac{k!}{k_1!k_2!\cdots k_n!} E(D_1^{k_1} D_2^{k_2} \cdots D_n^{k_n}) \prod_{j=1}^{n} \left( 1 - \frac{1}{2^j} \right)^{k_j}.
$$

(13)
For any collection $k_1, k_2, \ldots, k_{n+1}$ of non-negative integers

$$E(D_1^{k_1} D_2^{k_2} \cdots D_{n+1}^{k_{n+1}}) = \frac{k_1! k_2! \cdots k_{n+1}! n!}{(n + k_1 + k_2 + \cdots + k_{n+1})!}. \quad (14)$$

Indeed, under the condition that $D_3 = d_3, \ldots, D_{n+1} = d_{n+1}$, the random variable $D_1$ is uniform on the interval $[0, 1 - d_3 - \cdots - d_{n+1}]$ (cf. [5, Section 13.1]). Hence, by conditioning and partial integration, we obtain

$$E\left(\frac{D_1^{k_1}}{k_1!} \frac{D_2^{k_2}}{k_2!} \cdots \frac{D_{n+1}^{k_{n+1}}}{k_{n+1}!}\right) = E\left(\frac{D_1^{k_1+1}}{(k_1 + 1)!} \frac{D_2^{k_2-1}}{(k_2 - 1)!} \cdots \frac{D_{n+1}^{k_{n+1}}}{k_{n+1}!}\right).$$

By symmetry and repeatedly applying this equality we find that

$$E\left(\frac{D_1^{k_1}}{k_1!} \frac{D_2^{k_2}}{k_2!} \cdots \frac{D_{n+1}^{k_{n+1}}}{k_{n+1}!}\right) = E\left(\frac{D_1^{k_1+\cdots+k_{n+1}}}{(k_1 + \cdots + k_{n+1})!}\right).$$

Using $E(D_1^k/k!) = n!/(n + k)!$ yields (14). Then, by substituting (14) into (13), we finally arrive at

$$E[T_n^k] = \binom{n+k}{k}^{-1} \sum_{k_1, k_2, \ldots, k_n \geq 0} \prod_{j=1}^{n} \left(1 - \frac{1}{2^j}\right)^{k_j}.$$ 

For example, for $k = 1$ we have

$$E(T_n) = \frac{n}{n+1} - \frac{1}{n+1} \left(1 - \frac{1}{2^n}\right). \quad (15)$$

This formula has already been derived in [6].

### 4. Asymptotic results

In this section we analyse the distribution of the travel time under the nearest item heuristic, when the number of items $n$ tends to infinity. In fact, we consider $1 - T_n$, which is the difference between the travel time under the nearest item heuristic and one complete rotation of the carousel. It is clear that $1 - T_n$ converges in distribution to zero as $n \to \infty$. However, since (15) gives

$$E(1 - T_n) = \frac{2}{n+1} \left(1 - \frac{1}{2^{n+1}}\right),$$

we may expect that $(n+1)(1 - T_n)$ has a proper limiting distribution.

We will use the common notation $Z_n \overset{\text{d}}{\to} Z$ if the sequence $Z_1, Z_2, \ldots$ converges in distribution to $Z$. The limiting distribution of $(n+1)(1 - T_n)$ is presented in the following theorem.

**Theorem 4.** Let $X_1, X_2, \ldots$ be independent exponentials with mean 1. Then

$$(n+1)(1 - T_n) \overset{\text{d}}{\to} \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} X_j.$$ 

(16)
and the limiting distribution is given by

\[ P(\sum_{j=1}^{\infty} \frac{1}{2^{j-1}} X_j < x) = c_{\infty} \sum_{j=0}^{\infty} (-1)^j (1 - \exp(-2^j x)) \prod_{l=1}^{j} \frac{1}{2^l - 1}, \quad x > 0, \]  

where \( c_{\infty} \) is defined as (cf. (10))

\[ c_{\infty} = \prod_{l=1}^{\infty} \frac{2^l}{2^l - 1}. \]  

**Proof.** Let

\[ \xi_n = \sum_{j=1}^{n} \frac{1}{2^{j-1}} X_j, \quad \xi = \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} X_j. \]

According to the monotone convergence theorem we have \( E(\xi) = \lim_{n \to \infty} E(\xi_n) = 2 \), which in particular implies that \( P(\xi < \infty) = 1 \).

To prove (16) we only need to rewrite (3) as

\[ (n + 1)(1 - T_n) = \frac{(n + 1)\xi_{n+1}}{S_{n+1}}. \]

By definition, the sequence \( \{\xi_n\} \) converges almost surely to \( \xi \). Furthermore, according to the strong law of large numbers, the sequence \( \{S_n/n\} \) converges almost surely to 1. Thus, the sequence \( \{n\xi_n/S_n\} \) converges almost surely to \( \xi \), which immediately gives (16).

The distribution of \( \xi \) can be determined via inversion of its Laplace–Stieltjes transform \( \alpha(s) \), which is given by

\[ \alpha(s) = E(e^{-s\xi}) = \prod_{j=0}^{\infty} \frac{2^j}{s + 2^j}. \]

It is readily verified that \( \alpha(s) \) is a meromorphic function with simple poles \( s_j = -2^j, j = 0, 1, \ldots \). The residues \( r_j \) at these poles are given by

\[ r_j = c_{\infty} (-2)^j \prod_{l=1}^{j} \frac{1}{2^l - 1}, \quad j = 0, 1, \ldots. \]

To invert \( \alpha(s) \) we first expand it in rational fractions of \( s \) as

\[ \alpha(s) = \sum_{j=0}^{\infty} \frac{r_j}{s - s_j}. \]  

To validate (19) we follow the approach in [10, Section 7.4]. This approach requires that \( |\alpha(s)| \) is uniformly bounded on a sequence of circles \( C_j \), with centre at 0 and radius \( R_j \), not passing through any poles, and such that \( R_j \to \infty \) as \( j \to \infty \). In this case, we can take \( R_j = -\frac{1}{2}(s_j + s_{j+1}) = 2^{j-1} + 2^j \) and it is straightforward to show that for all \( s \in C_j \),

\[ |\alpha(s)| \leq |\alpha(-R_j)| \leq 2 \prod_{l=0}^{\infty} \frac{2^{2^l}}{2^{2^l} - 3}. \]
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\[ \text{Figure 4: The distribution of } (n + 1)(1 - T_n) \text{ for } n = 2, 5, 10, 20 \text{ and the limiting distribution as } n \to \infty. \]

Since this upper bound does not depend on \( j \), the function \(|\alpha(s)| \) is indeed uniformly bounded on the sequence of circles \( C_j \). Now we can conclude from [10, Section 7.4] that

\[ \alpha(s) = \alpha(0) + \sum_{j=0}^{\infty} r_j \left[ \frac{1}{s - s_j} + \frac{1}{s_j} \right]. \]

It can be shown by combinatorial arguments (cf. [6, Remark 10.3]) that

\[ \sum_{j=0}^{\infty} r_j s_j = c_0 \sum_{j=0}^{\infty} (-1)^{j+1} \prod_{l=1}^{j} \frac{1}{2^l - 1} = -1 = -\alpha(0), \]

which implies (19). Inversion of (19) finally yields (17).

In Figure 4 we demonstrate the rate at which the distribution of \((n + 1)(1 - T_n)\) converges to its limiting distribution.

In our case convergence in distribution implies convergence in moments. Indeed, using (6), for any \( k = 1, 2, \ldots \) we have (cf. (14))

\[
\begin{align*}
\text{E}((n + 1)(1 - T_n)^k) &= (n + 1)^k \sum_{k_1, k_2, \ldots, k_{n+1} \geq 0 \atop k_1 + k_2 + \cdots + k_{n+1} = k} \frac{k!}{k_1! k_2! \cdots k_{n+1}!} \text{E}(D_1^{k_1} D_2^{k_2} \cdots D_{n+1}^{k_{n+1}}) \prod_{j=1}^{n+1} \left( \frac{1}{2^{j-1}} \right)^{k_j} \\
&= (n + 1)^k \left( \frac{n + k}{k} \right)^{-1} \sum_{k_1, k_2, \ldots, k_{n+1} \geq 0 \atop k_1 + k_2 + \cdots + k_{n+1} = k} \prod_{j=1}^{n+1} \left( \frac{1}{2^{j-1}} \right)^{k_j} \\
&\leq \frac{k!(n + 1)^k}{(n + 1)(n + 2) \cdots (n + k)} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \right)^k \leq k! 2^k.
\end{align*}
\]
Hence (see e.g. [4, Section 4.5]) for any \( k = 1, 2, \ldots \)

\[
\lim_{n \to \infty} E((n+1)(1-T_n)^k) = E(\xi^k) < \infty.
\]

To find \( E(\xi^k) \) we use (17) and then change the order of integration and summation. This yields

\[
E(\xi^k) = k! c_\infty \sum_{j=0}^{\infty} (-1)^j \frac{1}{2^j} \prod_{l=1}^{j} \frac{1}{2^l - 1}.
\]

Changing the order of integration and summation is allowed, since the sum above is absolutely convergent. Expression (21) can be simplified by using the equality

\[
\sum_{j=0}^{\infty} (-1)^j \frac{1}{2^j} \prod_{l=1}^{j} \frac{1}{2^l - 1} = \prod_{j=k+1}^{\infty} \left(1 - \frac{1}{2^j}\right),
\]

which holds for \( k = 0, 1, \ldots \). For \( k = 0 \) it is just (20). The case \( k > 0 \) can be proved along the same lines. Substituting the last equality into (21) and using (18) gives the simple expression

\[
E(\xi^k) = k! \prod_{j=1}^{k} \frac{2^j}{2^j - 1}.
\]

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