Answers of the exercises of ISP (September 2006), Week 1

1.

$$\mathbb{E}[X^2] = \int_{v=0}^{\infty} f(v) \int_{u=0}^{v} 2u\mathrm{d}u\mathrm{d}v = \int_{u=0}^{\infty} 2u\mathbb{P}(X > u)\mathrm{d}u.$$

2. a: $\mathrm{e}^{-\lambda_1 v}$.
b: $\frac{\lambda_1}{\lambda_1+\lambda_2}$.
c: $1 - \mathrm{e}^{-(\lambda_1+\lambda_2)x}$.

3. a: $1/2$.
b: $\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{2\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2}$.

4. $\mathbb{P}(Y > x) = p\mathrm{e}^{-\lambda_1 x} + (1-p)\mathrm{e}^{-\lambda_2 x}$.
Density: $p\lambda_1\mathrm{e}^{-\lambda_1 x} + (1-p)\lambda_2\mathrm{e}^{-\lambda_2 x},\ x > 0$.

5. a: $\lambda^2 t\mathrm{e}^{-\lambda t}$.

6. a: $(1/5, 2/5, 2/5)$.
b: $m_i = 1/\pi_i,\ i = 1, 2, 3$.

7. $k = 1 : 0$. $k = 2 : (1/4)(2/5) = 1/10$. $k = 3 : (1/4)(3/5) = (3/20)$. $k = 4 : 3/4$.

8. a: $1 - \mathrm{e}^{-12} - 12\mathrm{e}^{-12}$.
b: $\mathrm{e}^{-12/4} = \mathrm{e}^{-3}$.
c: $32\mathrm{e}^{-32}$ (notice that clicking students form a Poisson process with intensity $(2/5) \times 10$)

9. Equations: $\pi_p = 0.7\pi_p + \pi_b$; $\pi_g = 0.2\pi_p + 0.6\pi_g$; $\pi_r = 0.1\pi_p + 0.2\pi_g + 0.5\pi_r$; $\pi_b = 0.2\pi_g + 0.5\pi_r$; $\sum \pi_i = 1$.
Solution: $(\frac{10}{22}, \frac{5}{22}, \frac{4}{22}, \frac{3}{22})$.

10. a: $\{1, 5\}$; $\{2\}$; $\{3, 4, 6\}$.
b: When starting in 1 or 5: $x_A = (3/5, 0, 0, 0, 2/5, 0)$. When starting in 3,4 or 6: $x_B = (0, 0, 4/14, 5/14, 0, 5/14)$. When starting in 2: $(1/3)x_A + (2/3)x_B$.
c: 1/3

11. a: $\mathrm{e}^{-\lambda_R t}$.
b: $\frac{\lambda_R}{\lambda_R+\lambda_W}$.
c: $\sum_{j=0}^{3} \mathrm{e}^{-\lambda_w t}\frac{(\lambda_w t)^j}{j!}$
d: $1 - \mathrm{e}^{-\lambda t} - \lambda t\mathrm{e}^{-\lambda t}$, with $\lambda = \lambda_R + \lambda_W$.
e: $\mathbb{P}(j\ read|n) = \binom{n}{j}\ (\frac{\lambda_R}{\lambda_R+\lambda_W})^j(\frac{\lambda_W}{\lambda_R+\lambda_W})^{n-j}$.

12. a: $a_{03} = 1 + \frac{2}{p^2}$. Use equations like $a_{03} = 1 + a_{13}$.
b: Use local balance equations like $\pi_0 = (1/2)\pi_1$ and $\pi_1 = \pi_2$ to obtain: $(\frac{1}{2N}, \frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}, \frac{1}{2N})$.

13. a: All states can reach each other; $P_{00} > 0$ implies aperiodicity; negative drift beyond N implies positive recurrence.
b: Use local balance equations: $\pi_i = \pi_{i+1}$ for $i = 0, \ldots, N-1$ and $(1/2)\pi_N = (1-p)\pi_{N+1}$ and $\pi_{N+j+1} = \frac{p}{1-p}\pi_{N+j}$ for $j = 1, 2, \ldots$, plus normalization.
c: $a_{03} = 12$.