7.2

a. S_n is Poisson with mean $n\mu$.

b.

$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n+1) \\ &= P(S_n \leq t) - P(S_{n+1} \leq t) \\ &= \sum_{k=0}^{\lfloor t \rfloor} e^{-n\mu} \frac{(n\mu)^k}{k!} - \sum_{k=0}^{\lfloor t \rfloor} e^{-(n+1)\mu} \frac{((n+1)\mu)^k}{k!}. \end{aligned}$$

7.4

a. No. Suppose, for instance, that the interarrival times of the first renewal process are identically equal to 1. Let the second be a Poisson process with rate λ. If the first interarrival time of the process $\{N(t), t \geq 0\}$ is equal to 3/4, then we can be certain that the next one is less than or equal to 1/4.

b. No. Use the same processes as in a for a counterexample. For instance, the first interarrival will equal 1 with probability $e^{-\lambda}$. The probability will be different for the next interarrival.

c. No, because of a or b.

7.5. The random variable N is equal to $N(1)+1$ where $\{N(t, t \geq 0\}$ is the renewal process whose interarrival distribution is uniform on $(0, 1)$. By the result of Example 7.3,

$$E[N(t)] = m(1) + 1 = e.$$

7.10. Yes, p/μ.

7.11.

$$\frac{N(t)}{t} = \frac{1}{t} + \frac{\text{number of renewals in } (X_1, t]}{t}.$$

Since $X_1 < \infty$, Proposition 7.1 implies that, as $t \to \infty$,

$$\frac{\text{number of renewals in } (X_1, t]}{t} \to \frac{1}{\mu}.$$

7.12. Let X be the time between successive d-events. Conditioning on the time until the next event following a d-event gives

$$E[X] = \int_0^d x\lambda e^{-\lambda x}dx + \int_d^\infty (x + E[X])\lambda e^{-\lambda x}dx = 1/\lambda + E[X]e^{-\lambda d}.$$

Therefore,

$$E[X] = \frac{1}{\lambda(1 - e^{-\lambda d})}.$$

a. $\frac{1}{E[X]} = \lambda(1 - e^{-\lambda d})$.

b. $1 - e^{-\lambda d}$.

7.15.

a. X_i is the amount of time he has to travel after his ith choice (we will assume that he keeps making choices even after becoming free). N is the number of choices he makes until becoming free.

b.

$$E[T] = E\left[\sum_1^N X_i\right] = E[N]E[X].$$

N is a geometric random variable with $p = 1/3$, so $E[N] = 3$, $E[X] = \frac{1}{3}(2{+}4{+}6) = 4$. Hence, $E[T] = 12$.

c.

$$E\left[\sum_1^N X_i | N = n\right] = (n-1)\frac{1}{2}(4+6) + 2 = 5n - 3,$$

since, given $N = n$, $X_1, \ldots, X_{n-1}$ are equally likely to be either 4 or 6, $X_n = 2$. Further, $E[\sum_1^n X_i] = 4n$.

d. From c,

$$E\left[\sum_1^N X_i\right] = E[5N - 3] = 15 - 3 = 12.$$

7.19. Since, from Example 7.3, $m(t) = e^t - 1$, $0 < t \le 1$, we obtain upon using the identity $t + E[Y(t)] = \mu(m(t) + 1)$ that $E[Y(1)] = e/2 - 1$.

7.21. This is an alternative renewal process, which is *on* when the server is busy and *off* when the server is vacant. The long-run proportion of time that the server is busy is

$$\frac{E[\textit{on} \text{ time in a cycle}]}{E[\text{cycle length}]} = \frac{\mu_G}{\mu_G + 1/\lambda},$$

where μ_G is the mean of the distribution G and the average cycle length follows from the memory-less property of the Poisson process (see also Examples 7.7, 7.11).

7.22. See the solution in the book.
7.26. The long-run average cost is

$$\frac{[c+2c+\cdots+(N-1)c]/\lambda+KNc+\lambda K^2c/2}{N/\lambda+K}=\frac{c(N-1)N/2\lambda+KNc+\lambda K^2c/2}{N/\lambda+K}.$$

7.32. Say that the system is on at t if $X_{N(t)+1}$, the interarrival time at t, is less than c (and off otherwise). Hence, the proportion of time that $X_{N(t)+1}$ is less than c is

$$\frac{E[\text{on time in a renewal cycle}]}{E[\text{cycle time}]}=\frac{\int_0^c tf(t)dt}{E[X]}.$$

7.37.

a. This is an alternating renewal process, with the mean off time obtained by conditioning on which machine fails to cause the off period.

$$\begin{aligned}E[\text{off}] &= \sum_{i=1}^{3} E[\text{off}|i \text{ fails}]P(i \text{ fails})\\ &= \frac{1}{5}\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}+2\frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}+\frac{3}{2}\frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}.\end{aligned}$$

As the on time in a cycle is exponential with rate equal to $\lambda_1+\lambda_2+\lambda_3$, we obtain that p, the proportion of time that the system is working is

$$p=\frac{1/(\lambda_1+\lambda_2+\lambda_3)}{E[C]},$$

where

$$E[C]=E[\text{cycle time}]=\frac{1}{\lambda_1+\lambda_2+\lambda_3}+E[\text{off}].$$

b. Think of the system as a renewal reward process by supposing that we earn 1 per unit of time that machine 1 is being repaired. Then, r_1, the proportion of time that machine 1 is being repaired is

$$r_1=\frac{\frac{1}{5}\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}}{E[C]}.$$

c. By assuming that we earn 1 per unit time when machine 2 is in a state of suspended animation, shows that, with s_2 being the proportion of time that 2 is in a state of suspended animation,

$$s_2 = \frac{\frac{1}{5}\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} + \frac{3}{2}\frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}}{E[C]}.$$

7.41. This is a renewal process where a renewal happens when a machine is replaced by a new one. A machine is not older than one year iff the age of the renewal process is not greater than one. By Example 7.23, we have:

(a)

$$\frac{\int_0^1 (1-x/2)\,dx}{1} = 3/4$$

(b)

$$\frac{\int_0^1 e^{-x}\,dx}{1} = 1 - e^{-1} \approx 0.6321$$

7.44. Let T be the time it takes the shuttle to return. Now, given T, X is Poisson with mean λT. Thus,

$$E[X|T] = \lambda T, \quad \mathrm{Var}(X|T) = \lambda T.$$

Consequently,

a. $E[X] = E[E[X|T]] = \lambda E[T]$.

b. $\mathrm{Var}(X) = E[\mathrm{Var}(X|T)] + \mathrm{Var}(E[X|T])$(see Proposition 3.1) $= \lambda E[T] + \lambda^2 \mathrm{Var}(T)$.

c. Assume that a reward of 1 is earned each time the shuttle returns empty. Then, from the renewal reward theory, r, the rate at which the shuttle returns empty, is

$$\begin{aligned} r &= \frac{P(\text{empty})}{E[T]} \\ &= \frac{\int_0^\infty P(\text{empty}|T=t)f(t)dt}{E[T]} \\ &= \frac{\int_0^\infty e^{-\lambda t}f(t)dt}{E[T]} \\ &= \frac{E[e^{-\lambda T}]}{E[T]}. \end{aligned}$$

d. Assume that a reward of 1 is earned each time that a customer writes an angry letter. Then, with N_a equal to the number of angry letters written in a cycle, it follows that

r_a, the rate at which angry letters are written, is

$$\begin{aligned} r_a &= E[N_a]/E[T] \\ &= \int_0^\infty E[N_a|T=t]f(t)dt/E[T] \\ &= \int_c^\infty \lambda(t-c)f(t)dt/E[T] \\ &= \lambda E[\max\{0, T-c\}]/E[T]\,. \end{aligned}$$

Since passengers arrive at rate λ, this implies that the proportion of passengers that write angry letters is r_a/λ.

e. Because passengers arrive at a constant rate, the proportion of them that have to wait more than c will equal the proportion of time that the age of the renewal process (whose event times are the return times of the shuttle) is greater than c. It is thus equal to $1 - F_e(c)$.

7.45.

(a) . Let $\{X_n, n \geq 0\}$ be a Markov chain in a discrete time $n = 0, 1, 2, \ldots$, with transition matrix P. Let π_1, π_2, π_3 denote the stationary probabilities that the Markov chain is in states $1, 2$ or 3, respectively. Solving

$$\pi_1 = (1/2)\pi_2 + \pi_3,\ \pi_2 = \pi_1,\ \pi_3 = (1/2)\pi_2,\ \pi_1 + \pi_2 + \pi_3 = 1,$$

we get $\pi_1 = 2/5, \pi_2 = 2/5, \pi_3 = 1/5$. Now, the long-run proportion of transitions that brings the system to state 1 is equal to the long-run fraction of visits of the Markov chain $\{X_n, n \geq 0\}$ to state 1, that is, 2/5.

(b) Assume that a renewal happens when the system makes a transition to state 1. The long-run fraction of time in state 1 is

$$\frac{E[\text{time in state 1 in a cycle}]}{E[\text{cycle length}]} = \frac{\mu_1}{\mu_1 + \sum_{i=2,3} E[\#\text{ visits to i in a cycle}]\mu_i}$$

Note that

$$\begin{aligned} \text{long-run fraction of visits to } 2 = \pi_2 &= \frac{E[\#\text{ visits to 2 in a cycle}]}{\#\text{ transitions in a cycle}} \\ &= \frac{E[\#\text{ visits to 2 in a cycle}]}{1/\pi_1}. \end{aligned}$$

Thus,

$E[\#\text{ visits to 2 in a cycle}] = \pi_2/\pi_1$ and similarly $E[\#\text{ visits to 3 in a cycle}] = \pi_3/\pi_1$.

The answer is:

$$\text{long-run fraction of time the system is in state 1} = \frac{\mu_1}{\mu_1 + \mu_2\pi_2/\pi_1 + \mu_3\pi_3/\pi_1}.$$

Further,

$$\text{long-run fraction of time the system is in state 2} = \frac{\mu_2\pi_2/\pi_1}{\mu_1 + \mu_2\pi_2/\pi_1 + \mu_3\pi_3/\pi_1},$$

$$\text{long-run fraction of time the system is in state 3} = \frac{\mu_3\pi_3/\pi_1}{\mu_1 + \mu_2\pi_2/\pi_1 + \mu_3\pi_3/\pi_1}.$$

7.51. What we observe here is the inspection paradox. Let t be the time of an inspection in a hotel room. Consider arrivals of visitors to a hotel room as a renewal process where interarrival times $X_1, X_2, \ldots$ are the lengths of stay with a common distribution function F and expectation μ. Then the interarrival time containing the instant t is typically larger than an arbitrary interarrival time. In other words, we are more likely to find a visitor who stays for a longer time. If we inspect many hotel rooms, then, by the law of large numbers, we will obtain a larger average length of stay than μ. At the airport, however, we simply have a sample from F. Again, by the law of large numbers, the average length of stay in this sample is approximately μ.