parameters i, p. The result follows since starting with an initial population of i is equivalent to having i independent Yule processes, each starting with a single individual.

12. (a) If the state is the number of individuals at time t, we get a birth and death process with

$\lambda_n = n\lambda + \theta \qquad n < N$

$\lambda_n = n\lambda \qquad n \geq N$

$\mu_n = n\mu.$

(b) Let P_i be the long-run probability that the system is in state i. Since this is also the proportion of time the system is in state i, we are looking for $\sum_{i=3}^{\infty} P_i$.

We have $\lambda_k P_k = \mu_{k+1} P_{k+1}$.

This yields

$$P_1 = \frac{\theta}{\mu} P_0$$

$$P_2 = \frac{\lambda+\theta}{2\mu} P_1 = \frac{\theta(\lambda+\theta)}{2\mu^2} P_0$$

$$P_3 = \frac{2\lambda+\theta}{2\mu} P_2 = \frac{\theta(\lambda+\theta)(2\lambda+\theta)}{6\mu^3} P_0.$$

For $k \geq 4$, we get

$$P_k = \frac{(k-1)\lambda}{k\mu} P_{k-1},$$

which implies

$$P_k = \frac{(k-1)(k-2)\ldots(3)}{(k)(k-1)\ldots(4)} \left[\frac{\lambda}{\mu}\right]^{k-3} P_3 = \frac{3}{k}\left[\frac{\lambda}{\mu}\right]^{k-3} P_3;$$

therefore $\displaystyle\sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3 P_3 \sum_{k=3}^{\infty} \frac{1}{k}\left[\frac{\lambda}{\mu}\right]^k$,

but $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\left[\frac{\lambda}{\mu}\right]^k = \log\left[\frac{1}{1-\frac{\lambda}{\mu}}\right] = \log\left[\frac{\mu}{\mu-\lambda}\right]$ if $\frac{\lambda}{\mu} < 1$.

So $\displaystyle\sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3 P_3\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]$

$$\sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]$$

$$\frac{\theta(\lambda+\theta)(2\lambda+\theta)}{6\mu^3}P_0.$$

Now $\sum_0^{\infty} P_i = 1$ implies

$$P_0 = \left[1 + \frac{\theta}{\mu} + \frac{\theta(\lambda+\theta)}{2\mu^2} + \frac{1}{2\lambda^3}\theta(\lambda+\theta)(2\lambda+\theta)\right.$$

$$\left. \times \left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]\right]^{-1}.$$

And finally,

$$\sum_{k=3}^{\infty} P_k = \left[\left[\frac{1}{2\lambda^3}\right]\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]\right.$$

$$\theta(\lambda+\theta)(2\lambda+\theta)\Big] \Big/ \Big[1 + \frac{\theta}{\mu} + \frac{\theta(\lambda+\theta)}{2\mu^2} + \frac{\theta(\lambda+\theta)(2\lambda+\theta)}{2\lambda^3}$$

$$\times \left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]\Big].$$

13. With the number of customers in the shop as the state, we get a birth and death process with

$\lambda_0 = \lambda_1 = 3 \quad \mu_1 = \mu_2 = 4.$

Therefore

$P_1 = \frac{3}{4} P_0 \quad P_2 = \frac{3}{4} P_1 = \left[\frac{3}{4}\right]^2 P_0.$

And since $\Sigma_0^2 P_i = 1$, we get

$$P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2\right]^{-1} = \frac{16}{37}.$$

(a) The average number of customers in the shop is

$$P_1 + 2P_2 = \left[\frac{3}{4} + 2\left[\frac{3}{4}\right]^2\right] P_0$$

$$= \frac{30}{16}\left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2\right]^{-1} = \frac{30}{37}.$$

(b) The proportion of customers that enter the shop is

$$\frac{\lambda(1-P_2)}{\lambda} = 1 - P_2 = 1 - \frac{9}{16} \cdot \frac{16}{37} = \frac{28}{37}.$$

(c) Now $\mu = 8$, and so

$$P_0 = \left[1 + \frac{3}{8} + \left[\frac{3}{8}\right]^2\right]^{-1} = \frac{64}{97}.$$

So the proportion of customers who now enter the shop is

$$1 - P_2 = 1 - \left[\frac{3}{8}\right]^2 \frac{264}{97} = 1 - \frac{9}{97} = \frac{88}{97}.$$

The rate of added customers is therefore

$$\lambda\left[\frac{88}{97}\right] - \lambda\left[\frac{28}{37}\right] = 3\left[\frac{88}{97} - \frac{28}{37}\right] = .45.$$

The business he does would improve by 0.45 customers per hour.

14. Letting the number of cars in the station be the state, we have a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 20, \lambda_i = 0, i > 2, \mu_1 = \mu_2 = 12.$$

Hence,

$$P_1 = \frac{5}{3} P_0, P_2 = \frac{5}{3} P_1 = \left[\frac{5}{3}\right]^2 P_0, P_3 = \frac{5}{3} P_2 = \left[\frac{5}{3}\right]^3 P_0.$$

and as $\Sigma_0^3 P_i = 1$, we have

$$P_0 = \left[1 + \frac{5}{3} + \left[\frac{5}{3}\right]^2 + \left[\frac{5}{3}\right]^2\right]^{-1} = \frac{27}{272}.$$

(a) The fraction of the attendant's time spent servicing cars is equal to the fraction of time there are cars in the system and is therefore $1 - P_0 = 245/272$.

(b) The fraction of potential customers that are lost is equal to the fraction of customers that arrive when there are three cars in the station and is therefore

$$P_3 = \left[\frac{5}{3}\right]^3 P_0 = 125/272.$$

15. With the number of customers in the system as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 3 \quad \lambda_i = 0, i \geq 4, \mu_1 = 2 \quad \mu_2 = \mu_3 = 4.$$

Therefore, the balance equations reduce to

$$P_1 = \frac{3}{2} P_0 \quad P_2 = \frac{3}{4} P_1 = \frac{9}{8} P_0 \quad P_3 = \frac{3}{4} P_2 = \frac{27}{32} P_0.$$

And therefore,

$$P_0 = \left[1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right]^{-1} = \frac{32}{143}.$$

(a) The fraction of potential customers that enter the system is

$$\frac{\lambda(1-P_3)}{\lambda} = 1 - P_3 = 1 - \frac{27}{32} \cdot \frac{32}{143} = \frac{116}{143}.$$

(b) With a server working twice as fast we would get

$$P_1 = \frac{3}{4} P_0 \quad P_2 = \frac{3}{4} P_1 = \left[\frac{3}{4}\right]^2 P_0 \quad P_3 = \left[\frac{3}{4}\right]^3 P_0,$$

$$\text{and } P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2 + \left[\frac{3}{4}\right]^3\right]^{-1} = \frac{64}{175}.$$

So that now

$$1 - P_3 = 1 - \frac{27}{64} = 1 - \frac{64}{175} = \frac{148}{175}.$$

16. Let the state be

0: an acceptable molecule is attached

1: no molecule attached

2: an unacceptable molecule is attached.

Then this is a birth and death process with balance equations

$$P_{12} = \frac{\mu}{\lambda} P_0$$

$$P_2 = \frac{\lambda(1-\alpha)}{\mu_1} P_1 = \frac{(1-\alpha)}{\alpha} \frac{\mu_2}{\mu_1} P_0.$$

Since $\Sigma_0^2 P_i = 1$, we get

$$P_0 = \left[1 + \frac{\mu_2}{\lambda\alpha} + \frac{1-\alpha}{\alpha}\frac{\mu_2}{\mu_1}\right]^{-1} = \frac{\lambda\alpha\mu_1}{\lambda\alpha\mu_1 + \mu_1\mu_2 + \lambda(1-\alpha)\mu_2}.$$

P_0 is the percentage of time the site is occupied by an acceptable molecule.

The percentage of time the site is occupied by an unacceptable molecule is

$$P_2 = \frac{1-\alpha}{\alpha}\frac{\mu_2}{\mu_1}P_0 = \frac{\lambda(1-\alpha)\mu_2}{\lambda\alpha\mu_1 + \mu_1 + \lambda(1-\alpha)\mu_2}.$$

17. Say the state is 0 if the machine is up, say it is i when it is down due to a type i failure, i=1,2. The balance equations for the limiting probabilities are as follows.

$$\lambda P_0 = \mu_1 P_1 + \mu_2 P_2$$
$$\mu_1 P_1 = \lambda p P_0$$
$$\mu_2 P_2 = \lambda(1-p)P_0$$
$$P_0 + P_1 + P_2 = 1.$$

These equations are easily solved to give the results

$$P_0 = (1 + \lambda p/\mu_1 + \lambda(1-p)/\mu_2)^{-1}$$
$$P_1 = \lambda p P_0/\mu_1, \qquad P_2 = \lambda(1-p)P_0/\mu_2 .$$

18. There are k+1 states; state 0 means the machine is working, state i means that it is in repair phase i, i=1,...,k. The balance equations for the limiting probabilities are

$$\lambda P_0 = \mu_k P_k$$
$$\mu_1 P_1 = \lambda P_0$$
$$\mu_i P_i = \mu_{i-1} P_{i-1}\,, \quad i=2,\ldots,k$$
$$P_0 + \ldots + P_k = 1\,.$$

To solve, note that

$$\mu_i P_i = \mu_{i-1} P_{i-1} = \mu_{i-2} P_{i-2} = \ldots = \lambda P_0.$$

Hence,

$$P_i = (\lambda/\mu_i) P_0,$$

and, upon summing,

$$1 = P_0 [1 + \sum_{i=1}^{k} (\lambda/\mu_i)].$$

Therefore,

$$P_0 = [1 + \sum_{i=1}^{k} (\lambda/\mu_i)]^{-1}, \quad P_i = (\lambda/\mu_i) P_0, \; i=1,\ldots,k.$$

The answer to part (a) is P_i and to part (b) is P_0.

19. There are 4 states. Let state 0 mean that no machines are down, state 1 that machine one is down and two is up, state 2 that machine one is up and two is down, and 3 that both machines are down. The balance equations are as follows.

$$(\lambda_1 + \lambda_2)P_0 = \mu_1 P_1 + \mu_2 P_2$$
$$(\mu_1 + \lambda_2)P_1 = \lambda_1 P_0 + \mu_1 P_3$$
$$(\lambda_1 + \mu_2)P_2 = \lambda_2 P_0$$
$$\mu_1 P_3 = \mu_2 P_1 + \mu_1 P_2$$
$$P_0 + P_1 + P_2 + P_3 = 1 .$$

These equations are easily solved and the proportion of time machine 2 is down is $P_2 + P_3$.

20. Letting the state be the number of down machines, this is a birth and death process with parameters

$$\lambda_i = \lambda,\ i = 0, 1$$
$$\mu_i = \mu,\ i = 1, 2 .$$

By the results of Example 3g, we have that

$$E[\text{time to go from 0 to 2}] = 2/\lambda + \mu/\lambda^2 .$$

Using the formula at the end of Section 3, we have that

$$\begin{aligned}
&\text{Var(time to go from 0 to 2)} \\
&\quad = \text{Var}(T_0) + \text{Var}(T_1) \\
&\quad = 1/\lambda^2 + \frac{1}{\lambda(\lambda + \mu)} + \mu/\lambda^3 + \frac{\mu}{\mu + \lambda}(2/\lambda + \mu/\lambda^2)^2 .
\end{aligned}$$

Using Equation (5.3) for the limiting probabilities of a birth and death process, we have that

$$P_0 + P_1 = \frac{1 + \lambda/\mu}{1 + \lambda/\mu + (\lambda/\mu)^2} .$$

21. Now we have a birth and death process with parameters

$$\lambda_i = \lambda, \quad i = 1, 2$$

$$\mu_i = i\mu, \quad i = 1, 2 .$$

Therefore,

$$P_0 + P_1 = \frac{1 + \lambda/\mu}{1 + \lambda/\mu + (\lambda/\mu)^2/2} ,$$

and so the probability that at least one machine is up is higher in this case.

22. The number in the system is a birth and death process with parameters

$$\lambda_n = \lambda/(n + 1), \quad n \geq 0$$

$$\mu_n = \mu , \quad n \geq 1 .$$

From Equation (5.3),

$$1/P_0 = 1 + \sum_{n=1}^{\infty} (\lambda/\mu)^n/n! = e^{\lambda/\mu}$$

and

$$P_n = P_0(\lambda/\mu)^n/n! = e^{-\lambda/\mu}(\lambda/\mu)^n/n! , \quad n \geq 0.$$

23. Let the state denote the number of machines that are down. This yields a birth and death process with

$$\lambda_0 = \frac{3}{10}, \lambda_1 = \frac{2}{10}, \lambda_2 = \frac{1}{10}, \lambda_i = 0, i \geq 3$$

$$\mu_1 = \frac{1}{8}, \mu_2 = \frac{2}{8}, \mu_3 = \frac{2}{8}.$$

The balance equations reduce to

$$P_1 = \frac{3/10}{1/8} P_0 = \frac{12}{5} P_0$$

$$P_2 = \frac{2/10}{2/8} P_1 = \frac{4}{5} P_1 = \frac{48}{25} P_0$$

$$P_3 = \frac{1/10}{2/8} P_2 = \frac{4}{10} P_3 = \frac{192}{250} P_0.$$

Hence, using $\sum_0^3 P_i = 1$, yields

$$P_0 = \left[1 + \frac{12}{5} + \frac{48}{25} + \frac{192}{250}\right]^{-1} = \frac{250}{1522}.$$

(a) Average number not in use

$$= P_1 + 2P_2 + 3P_3 = \frac{2136}{1522} = \frac{1068}{761}.$$

(b) Proportion of time both repairmen are busy

$$= P_2 + P_3 = \frac{672}{1522} = \frac{336}{761}.$$