Continuous Time Markov Chains (CTMCs)

A continuous time, discrete state Markov process is a process $\{X(t), t \geq 0\}$ with values in $\{0, 1, 2, \ldots\}$, such that "given the present, the past and future are independent", i.e., for all $s, t \geq 0$ and all states $i, j, x(u)$,

$$P(X(t+s) = j | X(s) = i, X(u) = x(u), 0 \leq u < s)) = P(X(t+s) = j | X(s) = i)$$

stationary transition probability function (independent of s):

$$P_{ij}(t) = P(X(t+s) = j | X(s) = i)$$

Compare:
A discrete time, discrete state Markov process is a process $\{X_n, n = 0, 1, 2, \ldots\}$ with values in $\{0, 1, 2, \ldots\}$, such that

$$P(X_{n+m} = j | X_m = i, X_{m-1} = i_{m-1}, \ldots, X_0 = i_0) = P(X_{n+m} = j | X_m = i)$$

n-step stationary transition probabilities: $P_{ij}^{(n)}$

If initial distribution $P(X(0) = i)$ and $P_{ij}(\cdot)$ are known then the distribution of $X(t)$ is known for all t:

$$P(X(t) = i) = \sum_k P(X(t) = i | X(0) = k) \cdot P(X(0) = k)$$

Compare: $P(X_n = i) = \sum_k P_{ki}^{(n)} \cdot P(X_0 = k)$.

How does it work?

Suppose process enters state i at time 0. Let

$$\begin{aligned} T_i &= \text{time till next transition, out of state } i \\ &= \text{sojourn time / holding time in state } i \end{aligned}$$

Suppose $X(u) = i$ for $0 \leq u \leq s$, then due to definition:

$$P(T_i > s + t | T_i > s) = P(T_i > t)$$

So T_i is memoryless, hence $T_i \sim \exp(v_i)$ for some v_i.
At time T_i, the process leaves state i and jumps to a state $j \neq i$ with some probability P_{ij} as in DTMC (independent of T_i).

Hence alternative definition:
A CTMC is a process that...

(i) ...stays in state i for a time $T_i \sim \exp(v_i)$, after which it...

(ii) ...moves to some other state j with some probability P_{ij}.

Here $P_{ii} = 0$ and $\sum_j P_{ij} = 1$ for all i.
(Note that P_{ij} and $P_{ij}(t)$ are not the same!)

Examples

- Poisson process (only $i \to i+1$)
  State space $\{0, 1, 2, \ldots\}$

$$
\begin{aligned}
T_i &\sim \exp(\lambda) \quad \text{so } v_i = \lambda, \quad i = 0, 1, 2, \ldots \\
P_{ij} &= 1, \qquad j = i+1 \\
&= 0, \qquad j \neq i+1
\end{aligned}
$$

- Pure birth process:
  Same P_{ij} as in Poisson process, but with $v_i = \lambda_i$.
  E.g. Yule process (linear birth process): $\lambda_i = i\lambda$.

- Birth and death process (only $i \to i+1$ or $i \to i-1$)
  State space $\{0, 1, \ldots\}$ or $\{0, 1, \ldots, N\}$
  Time till next 'birth' $i \to i+1$ is $B_i \sim \exp(\lambda_i)$
  Time till next 'death' $i \to i-1$ is $D_i \sim \exp(\mu_i)$
  Time till next transition is $\min(B_i, D_i) \sim \exp(v_i)$
  with $v_i = \lambda_i + \mu_i$

$$
\begin{aligned}
P_{i,i+1} &= P(B_i < D_i) = \frac{\lambda_i}{\lambda_i + \mu_i} \\
P_{i,i-1} &= P(B_i > D_i) = \frac{\mu_i}{\lambda_i + \mu_i}
\end{aligned}
$$

  State 0 has $\mu_0 = 0$, hence $v_0 = \lambda_0$ and $P_{0,-1} = 0, P_{0,1} = 1$

First entrance times in birth-death process

Let T_{ij} be the time to reach j from i. Then

$$\begin{aligned} ET_{i,i+1} &= \frac{1}{v_i} + P_{i,i+1}\cdot 0 + P_{i,i-1}ET_{i-1,i+1} \\ &= \frac{1}{\lambda_i+\mu_i} + \frac{\mu_i}{\lambda_i+\mu_i}(ET_{i-1,i} + ET_{i,i+1}) \\ ET_{i,i+1} &= \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i}ET_{i-1,i} \end{aligned}$$

This can be solved, starting from $ET_{0,1} = 1/\lambda_0$.

'Similar' for Variance (see book), but also (not in book):

$$\begin{aligned} Ee^{-sT_{i,i+1}} &= \frac{v_i}{v_i+s}\left(P_{i,i+1}\cdot 1 + P_{i,i-1}Ee^{-sT_{i-1,i+1}}\right) \\ &= \frac{\lambda_i}{\lambda_i+\mu_i+s} + \frac{\mu_i}{\lambda_i+\mu_i+s}Ee^{-sT_{i-1,i}}Ee^{-sT_{i,i+1}} \\ Ee^{-sT_{i,i+1}} &= \frac{\lambda_i}{\lambda_i+\mu_i+s-\mu_i Ee^{-sT_{i-1,i}}} \end{aligned}$$

This can be solved, starting from $Ee^{-sT_{0,1}} = \lambda_0/(\lambda_0+s)$.

Finally, if $i<j$ we have $T_{i,j} = T_{i,i+1} + \cdots + T_{j-1,j}$, all terms independent!

Transition probabilities

We did: starting from i, when does process reach (fixed) j?
Answer: expectation/distribution of first entrance time T_{ij}

Now: starting from i, where is process after (fixed) time t?
Answer: transition probability function

$$P_{ij}(t) = P(X(t) = j | X(0) = i)$$

Known for Poisson process with rate λ (let $j \geq i$):

$$\begin{aligned} P_{ij}(t) &= P(j - i \text{ jumps in } (0,t] | X(0) = i) \\ &= P(j - i \text{ jumps in } (0,t]) \\ &= e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!} \end{aligned}$$

More general, pure birth process (let $j \geq i$):

$$\begin{aligned} P_{ij}(t) &= P(X(t) = j | X(0) = i) \\ &= P(X(t) \leq j | X(0) = i) - P(X(t) \leq j - 1 | X(0) = i) \\ &= P(X_i + \cdots + X_j > t) - P(X_i + \cdots + X_{j-1} > t) \end{aligned}$$

with $X_i \sim \exp(\lambda_i)$ independent.
Rest is technical. Suppose λ_i are different, then

$$Ee^{-s(X_i + \cdots + X_j)} = \prod_{k=i}^{j} \frac{\lambda_k}{\lambda_k + s} = \sum_{k=i}^{j} \alpha_k \frac{\lambda_k}{\lambda_k + s}$$

with

$$\alpha_k = \prod_{r \neq k} \frac{\lambda_r}{\lambda_r - \lambda_k}$$

So

$$P(X_i + \cdots + X_j > t) = \sum_{k=i}^{j} \alpha_k e^{-\lambda_k t}$$

But what if $P_{i,i+1} \neq 1$?

In DTMC $P_{ik}^{(n)}$ were found from difference equations, based on

- Chapman-Kolmogorov and
- 'short-term behaviour' $P_{ij}^{(1)}$

$$P_{ij}^{(n)} = \sum_k P_{ik} P_{kj}^{(n-1)} \Rightarrow P^{(n)} = P P^{(n-1)}$$

In CTMC, $P_{ik}(t)$ are found similarly from differential equations. So we need

- Chapman Kolmogorov: $P_{ij}(t+s) = \sum_k P_{ik}(t) P_{kj}(s)$
  (in matrix form: $P(t+s) = P(t)P(s)$)
  Proof: ...
- Short-term behaviour for $h \downarrow 0$ (as in Poisson Process):

$$\begin{aligned} P_{ii}(h) &= 1 - v_i h + o(h), \\ P_{ij}(h) &= v_i P_{ij} h + o(h), \quad j \neq i \end{aligned}$$

  Proof: ...

Notation: let

$$q_{ij} = v_i P_{ij} \qquad \text{for } j \neq i$$

These q_{ij}'s characterize the process:

$$v_i = \sum_{j \neq i} q_{ij} \qquad \text{and} \qquad P_{ij} = \frac{q_{ij}}{\sum_{j \neq i} q_{ij}},$$

Kolmogorov's equations for $P_{ij}(t)$

$$\begin{aligned}P_{ij}(t+h) &= \sum_k P_{ik}(h)P_{kj}(t) \\ &= (1-v_ih+o(h))P_{ij}(t)+\sum_{k\neq i}(q_{ik}h+o(h))P_{kj}(t)\end{aligned}$$

$\Rightarrow$ Kolmogorov's backward equation:
(always hold)

$$P'_{ij}(t) = -v_iP_{ij}(t)+\sum_{k\neq i} q_{ik}P_{kj}(t)$$

Also:

$$\begin{aligned}P_{ij}(t+h) &= \sum_k P_{ik}(t)P_{kj}(h) \\ &= P_{ij}(t)(1-v_jh+o(h))+\sum_{k\neq j} P_{ik}(t)(q_{kj}h+o(h))\end{aligned}$$

$\Rightarrow$ Kolmogorov's forward equation:
(almost always hold)

$$P'_{ij}(t) = -P_{ij}(t)v_j+\sum_{k\neq j} P_{ik}(t)q_{kj}$$

Initial conditions:

$$\begin{aligned}P_{ii}(0) &= 1, \\ P_{ij}(0) &= 0, \qquad j\neq i\end{aligned}$$

Generator matrix

Let *generator matrix* Q be such that

$$\begin{aligned} Q_{ii} &= -v_i \\ Q_{ij} &= q_{ij}, \qquad i \neq j \end{aligned}$$

For finite-state Markov chain on $\{0, 1, \ldots, N\}$:

$$Q = \begin{pmatrix} -v_0 & q_{01} & q_{02} & \cdots & q_{0N} \\ q_{10} & -v_1 & q_{12} & \cdots & q_{1N} \\ q_{20} & q_{21} & -v_2 & \cdots & q_{2N} \\ \vdots & & & & \vdots \\ q_{N0} & & \cdots & & -v_N \end{pmatrix}$$

Also let

$$P(t) = [P_{ij}(t)]$$

Then Kolmogorov's equation in matrix form are

$$\begin{aligned} P'(t) &= Q \cdot P(t), \qquad \text{(backward)} \\ P'(t) &= P(t) \cdot Q, \qquad \text{(forward)} \end{aligned}$$

and initial condition: $P(0) = I$.

Note: row sums of Q are 0
row sums of $P(t)$ are 1 (stochastic matrix)

How to find Q or the (infinitesimal) transition rates?

Example birth-death process:

$$\begin{aligned}
v_i &= \lambda_i + \mu_i \\
q_{i,i+1} &= v_i P_{i,i+1} = (\lambda_i + \mu_i)\frac{\lambda_i}{\lambda_i + \mu_i} = \lambda_i \\
q_{i,i-1} &= v_i P_{i,i-1} = (\lambda_i + \mu_i)\frac{\mu_i}{\lambda_i + \mu_i} = \mu_i \\
q_{i,j} &= 0 \quad \text{for all other } j \neq i \quad (\text{since then} P_{ij} = 0)
\end{aligned}$$

Hence, for a finite-state birth-death process, Q is given by

$$\begin{pmatrix}
-\lambda_0 & \lambda_0 & & & & & 0 \\
\mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & & & & \\
 & \mu_2 & -(\lambda_2 + \mu_2 & \lambda_2 & & & \\
 & & \ddots & \ddots & & \ddots & \\
 & & & \mu_{N-1} & -(\lambda_{N-1} + \mu_{N-1}) & \lambda_{N-1} \\
0 & & & & \mu_N & -\mu_N
\end{pmatrix}$$

In general situation:

First determine state space S, then for all $i \in S$:

- v_i
- P_{ij}
- q_{ij}

$$\Rightarrow Q$$

Easy example: the two-state process (on-off source)

Often used, e.g. in telecommunications.
Some source alternates between off- and on-state.
Subsequent off- and on-times are independent, with
Off-time $\sim \exp(\alpha)$, On-time $\sim \exp(\beta)$

Then $\{X(t)\}$ is a CTMC (birth-death process) with state space $\{0, 1\}$ (0=off, 1=on), and

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

Solution of $P'(t) = P(t) \cdot Q$, $P_{ij}(0) = \delta_{ij}$:

$$\begin{aligned} P'_{00}(t) &= -\alpha P_{00}(t) + \beta P_{01}(t) \\ &= -\alpha P_{00}(t) + \beta(1 - P_{00}(t)) \end{aligned}$$

$$\Rightarrow P_{00}(t) = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} e^{-(\alpha+\beta)t}$$

$\Rightarrow P_{01}(t), P_{10}(t), P_{11}(t)$ also known.

Limiting probabilities:

$$\begin{aligned} \lim_{t\to\infty} P_{00}(t) &= \lim_{t\to\infty} P_{10}(t) = \frac{\beta}{\alpha+\beta} \\ \lim_{t\to\infty} P_{01}(t) &= \lim_{t\to\infty} P_{11}(t) = \frac{\alpha}{\alpha+\beta} \end{aligned}$$

Less easy example: n identical on-off sources

Let $X(t) = \#$ active sources at time t.

How to find Q?

Suppose $X(t) = i$

• each source that is on turns off after time $\sim \exp(\beta)$

$\Rightarrow$ time until the first turns off $\sim \exp(i\beta)$

• each source that is off turns on after $\sim \exp(\alpha)$

$\Rightarrow$ time until the first turns on $\sim \exp((n-i)\alpha)$

So: $\{X(t)\}$ is a birth-death process.

Next jump after time $\sim exp(i\beta + (n-i)\alpha)$,

to state $i+1$ w.p. $\frac{(n-i)\alpha}{(n-i)\alpha+i\beta}$ or

to state $i-1$ w.p. $\frac{i\beta}{(n-i)\alpha+i\beta}$

$$Q = \cdots$$

Finding $P_{ij}(t)$ is difficult, but for $t \to \infty$?

Next example:

Two independent machines are on (1) or off (0).
Let $X_i(t) =$ state of machine i at time t. $i = 1, 2$
and $X(t) = (X_1(t), X_2(t))$

If $\{X_1(t)\}$ and $\{X_2(t)\}$ are CTMC's as before, with parameters (α, β) and (λ, μ) respectively,

then also $\{X(t)\}$ is a continous time Markov chain.
E.g. in state $(0, 0)$,
• machine 1 turns on after time $\sim \exp(\alpha)$, or
• machine 2 turns on after time $\sim \exp(\lambda)$,
whichever happens first. Hence:

$$\begin{aligned} v_{00} &= \alpha + \lambda \\ q_{00,01} &= v_{00} P_{00,01} = (\alpha + \lambda) \frac{\lambda}{\alpha + \lambda} = \lambda \end{aligned}$$

and

$$Q = \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix} \begin{pmatrix} -(\alpha+\lambda) & \lambda & \alpha & 0 \\ \mu & -(\alpha+\mu) & 0 & \alpha \\ \beta & 0 & -(\beta+\lambda) & \lambda \\ 0 & \beta & \mu & -(\beta+\mu) \end{pmatrix}$$

Limiting behaviour

Accessibility, communicating states, communicating class, irreducibility, transience, recurrence, ...

...are all defined similarly as in DTMCs, with transition probabilities replaced by transition rates q_{ij}.
or:
... are defined via the *embedded* Markov chain with transition probabilities $P_{ij} = q_{ij}/v_i$.

In particular:
$\{X(t)\}$ is irreducible $\Leftrightarrow$ $P_{ij}(t) > 0$ for all $i, j \in S$
(note: positive/null recurrence cannot be defined as in DTMCs)

Theorem (no proof):
If CTMC is irreducible and positive recurrent, then

$$P_j = \lim_{t\to\infty} P_{ij}(t) > 0$$

exists, independent of i.

Interpretations of $\{P_j\}$:

- Limiting distribution
- Long run fraction of time spent in j
- Stationary distribution

Last bullet: If $P(X(0) = i) = P_i$ for all $i \in S$,
then $P(X(t) = j) = P_j$ for all $j \in S$ *and* all $t > 0$.
Proof: on board

Note 1: no periodicity issue.
Note 2: P_i, P_{ij} and $P_{ij}(t)$ are all different.
Other notation for $P_j = \lim_{t\to\infty} P_{ij}(t)$ is π_j (as in DTMC).

Balance equations

To find the limiting distribution, take limit $t \to \infty$ in forward equations:

$$\lim_{t\to\infty} P'_{ij}(t) = \lim_{t\to\infty} \left(\sum_{k\neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t) \right)$$

becomes

$$0 = \sum_{k\neq j} q_{kj} P_k - v_j P_j$$

so limiting distribution satisfies

$$\begin{cases} v_j P_j = \sum_{k\neq j} q_{kj} P_k & \text{(balance equations)} \\ \sum_j P_j = 1 & \text{(normalization)} \end{cases}$$

Interpretation 'rate out = rate in':

Balance equation for set A (summing above over $i \in A$):

$$\sum_{j\in A} P_j \sum_{i\notin A} q_{ji} = \sum_{i\notin A} P_i \sum_{j\in A} q_{ij}$$

Balance equations in matrix form

Balance equations can also be found in matrix form from forward equation:

Let $t \rightarrow$ in $P'(t) = P(t)Q$:

$$0 = \begin{bmatrix} P_0 & P_1 & P_2 & \dots \\ P_0 & P_1 & P_2 & \dots \\ P_0 & P_1 & P_2 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} Q$$

where $P_i = \lim_{t\to\infty} P_{ki}(t)$ for all k

Hence, with $\mathbf{P} = (P_0, P_1, \dots)$ solve

$$\begin{cases} \mathbf{P}Q = \mathbf{0} & \text{(balance equations)} \\ \sum_j P_j = 1 & \text{(normalization)} \end{cases}$$

Compare with DTMC:

$$\begin{cases} \pi P = \pi & \text{(balance equations)} \\ \sum_j \pi_j = 1 & \text{(normalization)} \end{cases}$$

Birth death process:

$$\begin{aligned} -\lambda_0 P_0 + \mu_1 P_1 &= 0 \\ \lambda_0 P_0 - (\lambda_1 + \mu_1) P_1 + \mu_2 P_2 &= 0 \\ \lambda_1 P_1 - (\lambda_2 + \mu_2) P_2 + \mu_3 P_3 &= 0 \\ &\vdots \end{aligned}$$

or

$$\begin{aligned} \lambda_0 P_0 &= \mu_1 P_1 \\ (\lambda_1 + \mu_1) P_1 &= \lambda_0 P_0 + \mu_2 P_2 \\ (\lambda_2 + \mu_2) P_2 &= \lambda_1 P_1 + \mu_3 P_3 \\ &\vdots \end{aligned}$$

Summing first i equations (or applying balance equations to set $A = \{0, 1, \ldots, i-1\}$):

$$\lambda_{i-1} P_{i-1} = \mu_i P_i$$

Recursively:

$$P_i = \frac{\lambda_{i-1}}{\mu_i} P_{i-1} = \ldots = \frac{\lambda_{i-1}\lambda_{i-2}\cdots\lambda_0}{\mu_i\mu_{i-1}\cdots\mu_1} P_0$$

Two cases:

- Normalization $\sum_{i=0}^{\infty} P_i = 1$ is possible:

$$P_0 = \left[\sum_{i=0}^{\infty} \frac{\lambda_{i-1}\lambda_{i-2}\cdots\lambda_0}{\mu_i\mu_{i-1}\cdots\mu_1} \right]^{-1}$$

  Positive recurrent process

- Normalization $\sum_{i=0}^{\infty} P_i = 1$ is not possible:

$$\sum_{i=0}^{\infty} \frac{\lambda_{i-1}\lambda_{i-2}\cdots\lambda_0}{\mu_i\mu_{i-1}\cdots\mu_1} = \infty, \qquad P_i = 0$$

  Transient or null-recurrent process; no limiting distribution.

Embedded Markov chain

Let $\{X(t), t \geq 0\}$ be some irreducible and positive recurrent CTMC with parameters v_i and $P_{ij} = q_{ij}/v_i$.
Let Y_n be the state of $X(t)$ after n transitions.

Then $\{Y_n, n = 0, 1, \ldots\}$ is DTMC with transition probs. P_{ij}:
Embedded chain. (Note: $P_{jj} = 0$).

Now let P_i satisfy

$$\sum_j P_j = 1 \quad \text{and} \quad P_j v_j = \sum_{i \neq j} P_i q_{ij}$$

and let π_i satisfy

$$\sum_j \pi_j = 1 \quad \text{and} \quad \pi_j = \sum_{i \neq j} \pi_i P_{ij}$$

Is $\pi_j = P_j$?

No, not in general
Yes, if $v_i \equiv v$ for all i.

Example: on board

Uniformization

Idea:

- if CTMC has different v_i's, choose a uniform $v \geq \sup_{i\in S} v_i$
- Compensate by adding 'self-transitions' $i \to i$
- Then define 'embedded' Markov chain

Suppose process is in state i. Then:

- Stay in i for $X \sim \exp(v)$
- Then either leave i w.p. $\frac{v_i}{v}$, or 'return' to i w.p. $1 - \frac{v_i}{v}$.

Consequences:

- Jumps occur according to Poisson Proces, rate v
- Sojourn time in i is $S = \sum_{k=1}^{N} X_i$ with $N \sim$ geom(v_i/v) and X_i i.i.d. $\sim \exp(v)$
  So (e.g. by considering LST $\phi_S(s) = g_N(\phi_X(s))$), we have $S \sim \exp(v_i)$!

New DTMC has

$$\begin{aligned} P^*_{ii} &= 1 - \frac{v_i}{v} \\ P^*_{ij} &= \frac{v_i}{v} P_{ij} \qquad (j \neq i) \end{aligned}$$

Again an embedded process, but at event times of a Poisson process rate v, with self transitions.

In this way almost (!) any CTMC (with parameters v_i, P_{ij}) can be viewed as a combination of a PP (with rate v) and a DTMC (with transition probabs. P^*_{ij})

Properties of uniformized Markov chain

Uniformizing DTMC has same stationary distribution as the original CTMC:

$$\begin{aligned}
\pi_j &= \sum_{i \in S} \pi_i P^*_{ij} \\
\pi_j &= \pi_j(1 - \frac{v_j}{v}) + \sum_{i \neq j} \pi_i P_{ij} \frac{v_i}{v} \\
v_j \pi_j &= \sum_{i \neq j} v_i P_{ij} \pi_i \\
v_j \pi_j &= \sum_{i \neq j} q_{ij} \pi_i
\end{aligned}$$

Same set of equations as for P_i; result follows by uniqueness.

Relation between transition probabilities:

Let $N(t)$ be the Poisson process, at rate v, then:

$$\begin{aligned}
P_{ij}(t) &= P(X(t) = j | X(0) = i) \\
&= \sum_{n=0}^{\infty} P(X(t) = j | X(0) = i, N(t) = n) P(N(t) = n | X(0) = i) \\
&= \sum_{n=0}^{\infty} P^{*(n)}_{ij} e^{-vt} \frac{(vt)^n}{n!}
\end{aligned}$$

See example 6.21 for uniformization of two-state CTMC (parameters λ and μ) at rate $\lambda + \mu$ to find $P_{00}(t)$, etcetera.

Other numerical applications: see Section 6.8
(but not for the exam!)