

LNMB EXAM Introduction to Stochastic Processes (ISP)  
Saturday October 7, 2006, 11.00-14.00 hours.

**EXERCISE 1**

- a. [3 pt.]  $\{1, 2, 5\}, \{3, 4\}, \{6\}$ .  
 b. [3 pt.] Let  $T_{ij} = P(\text{ever in } j | X_0 = i)$ . Then  $T_{41} = \frac{1}{10}T_{31} + \frac{1}{2}1 + \frac{2}{5}0$  and  $T_{31} = \frac{2}{5} + \frac{1}{2} + \frac{1}{10}T_{41}$ . Hence  $T_{31} = \frac{95}{99}$  and  $T_{41} = \frac{59}{99}$ .  
 c. [4 pt.] Starting from  $\{1, 2, 5\}$ :  $V_1 = \{3/10, 5/10, 0, 0, 2/10, 0\}$ .  
 Starting from 6:  $V_6 = \{0, 0, 0, 0, 0, 1\}$ .  
 Starting from 3:  $\frac{95}{99}V_1 + \frac{4}{99}V_6$ .  
 Starting from 4:  $\frac{59}{99}V_1 + \frac{40}{99}V_6$ .

**EXERCISE 2**

- a. [2 pt.]  $e^{-(200+100)*0.01} = e^{-3}$ .  
 b. [2 pt.]  $\frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{1}{3}$ .  
 c. [2 pt.]  $1 - e^{-\lambda_1 t} - \lambda_1 t e^{-\lambda_1 t} = 1 - 3e^{-2}$ .  
 d. [2 pt.]  $\frac{\lambda}{\lambda + s}$  is the LST of the time until one task has arrived, so  $(\frac{\lambda}{\lambda + s})^K$  is the LST of the time until  $K$  tasks have arrived. Now use that  $E[T_K^i]$  equals  $(-1)^i$  times the  $i$ -th derivative of the latter LST to derive that the variance equals  $\frac{K}{90000}$ .  
 e. [2 pt.]  $\int_0^\infty \mu_1 e^{-\mu_1 t} \sum_{n=0}^\infty e^{-\lambda_2 t} \frac{(\lambda_2 t)^n}{n!} z^n dt = \frac{\mu_1}{\mu_1 + \lambda_2(1-z)}$  with  $\mu_1 = \lambda_2 = 200$ .

**EXERCISE 3**

- a. [2 pt.]  $E[Z] = \frac{1-p}{p}$  and  $E[X_i] = E[Z]^i = \frac{(1-p)^i}{p^i}$ .  
 b. [2 pt.] Clearly  $\pi_0 = 1$  for  $\frac{1}{2} \leq p \leq 1$  (since then  $E[Z] \leq 1$ ). If  $0 \leq p < \frac{1}{2}$ , then  $\pi_0$  is the root in  $[0, 1)$  of

$$\pi_0 = \sum_{i=0}^{\infty} P[Z = i] \pi_0^i = \frac{p}{1 - (1-p)\pi_0},$$

yielding

$$\pi_0 = \frac{p}{1-p}.$$

**EXERCISE 4**

- a. [2 pt.] The transition rate matrix  $Q$  is

$$\begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -\lambda - \mu & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}.$$

- b. [2 pt.] Let  $T_i$  ( $i = 1, 2$ ) denote the time, starting from state  $i$ , it takes to enter state 0.

Then

$$E[T_1] = \frac{1}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} E[T_2], \quad E[T_2] = \frac{1}{2\mu} + E[T_1].$$

Solving these equations gives

$$E[T_1] = \frac{2\mu + \lambda}{2\mu^2}.$$

c. [2 pt.] The balance equations are:

$$\begin{aligned} 2\lambda P_0 &= \mu P_1, \\ \lambda P_1 &= 2\mu P_2. \end{aligned}$$

from which, together with the normalization equation, follows that

$$P_0 = \left( \frac{\mu}{\lambda + \mu} \right)^2.$$

Alternatively, this can be obtained by noting that both workers are working independently, and the probability that a working is 'working' is  $\mu/(\lambda + \mu)$ .

d. [2 pt.] Now only two states are possible, state 0 and 1. Balance of flow gives

$$2\lambda P_0 = \mu P_1,$$

and thus

$$P_0 = \frac{\mu}{\mu + 2\lambda}.$$

### EXERCISE 5

a. [2 pt.] The mean time between replacements is  $\frac{7}{6}$  year. Hence the system is replaced  $\frac{6}{7}$  times per year.

b. [2 pt.] The long-run average cost per year is  $1400 \cdot \frac{6}{7} = 1200$  euro.

c. [2 pt.] The mean time between two events is

$$\int_{t=0}^{\frac{2}{3}} \left(t + \frac{1}{6}\right) e^{-t} dt + \frac{5}{6} \int_{t=\frac{2}{3}}^{\infty} e^{-t} dt = \frac{7}{6} - e^{-2/3} = 0.653.$$

d. [2 pt.] The mean cost of an event is  $1400 \cdot P[L < \frac{2}{3}] + 200 \cdot P[L > \frac{2}{3}] + 600 \cdot P[\frac{2}{3} < L < \frac{5}{6}] = 831$  euro. Hence the mean cost per year is

$$\frac{831}{0.653} = 1273 \text{ euro.}$$