Renewal Reward Processes

\( X_1, X_2, \ldots \) – i.i.d. inter-arrival times, d.f. \( F \)

\( R_n \) – reward at the time of the \( n \)th renewal

\( R_1, R_2, \ldots \) – i.i.d; \( (X_n, R_n) \) – i.i.d. vectors, \( n \geq 1 \);

\( R_n \) may (and usually will) depend on \( X_n \)

\( R(t) = \sum_{n=1}^{N(t)} R_n \) – reward earned by time \( t \)

**Proposition 7.3** If \( E(R) < \infty, E(X) < \infty \), then:

(i) \( \frac{R(t)}{t} \to \frac{E(R)}{E(X)} \) as \( t \to \infty \), w.p. 1

(ii) \( \frac{E(R(t))}{t} \to \frac{E(R)}{E(X)} \) as \( t \to \infty \)
Example 7.11

Single-server bank. Arrivals of customers: Poisson process ($\lambda$). A customer enters the bank if the server is available. Otherwise, the customer leaves. The service time has a distribution $G$. A customer who entered the bank makes a deposit, the deposits are i.i.d. with distribution $H$. The rate at which deposits accumulate is

\[
\frac{E[\text{deposits during a cycle}]}{E[\text{cycle length}]} = \frac{\mu_H}{\mu_G + 1/\lambda},
\]

where $\mu_H$ is the mean of distribution $H$. 
Example similar to 7.12

$X_1, X_2, \ldots$ - i.i.d. units’ life-times, distribution function $F$

A unit is replaced either upon a failure or at age $T$

$Y_n$ – cost of the $n$th replacement

Replacement cost: $c_1$ at age $T$ and $c_2$ upon a failure

[cycle]= [time between replacements]

$E[\text{cost per cycle}] = E(Y_n) = c_1(1 - F(T)) + c_2F(T)$

$E[\text{cycle length}] = E(\min\{X_n, T\}) = \int_0^T (1 - F(x))dx$

The long-run cost per unit time is

$$\frac{c_1(1 - F(T)) + c_2F(T)}{\int_0^T (1 - F(x))dx}$$

From that, one can find $T$ which minimizes the long-run costs
Example 7.13

Customers arrive at a train depot according to a renewal process, mean interarrival time $\mu$. When there are $N$ customers, train leaves. The depot incurs a cost of $nc$ per unit time whenever $n$ customers are waiting. With each train, the depot incurs the cost of $K$ time units. What is the depot’s long-run cost per unit time?

$E[\text{cycle length}] = N\mu$

$E[\text{cost per cycle}] = K + c\mu + 2c\mu + \cdots + (N - 1)c\mu = K + c\mu\frac{N(N - 1)}{2}$

The long-run cost per unit time is

$$\frac{K}{N\mu} + \frac{c(N - 1)}{2}$$

Which value of $N$ minimizes the cost? Calculus:

$N_{optimal} = \sqrt{\frac{(2K)}{(c\mu)}}$. 

Examples 7.16-7.17 (The average age and excess)

Suppose that at any time \( s \) we earn at rate \( A(s) \)

Cycle is time between renewals, \( E[\text{cycle length}] = \mu \)

\[
\frac{\int_0^t A(s) \, ds}{t} \xrightarrow{t \to \infty} \frac{E(\text{reward during a cycle})}{E(\text{cycle length})}
\]

\[\text{[reward during a cycle]} = \int_0^X s \, ds = \frac{X^2}{2}. \text{ Hence}\]

\[
\lim_{t \to \infty} \frac{\int_0^t A(s) \, ds}{t} \xrightarrow{t \to \infty} \frac{E(X^2)}{2E(X)} \quad \text{w.p. 1}
\]

Now, suppose that at time \( s \) we earn at rate \( Y(s) \)

\[
\frac{\int_0^t Y(s) \, ds}{t} \xrightarrow{t \to \infty} \frac{E(\text{reward during a cycle})}{E(\text{cycle length})} = \frac{E\left[\int_0^X (X - s) \, ds\right]}{E(X)} = \frac{E(X^2)}{2E(X)}
\]
**Example.** $X_1, X_2, \ldots$ have a Pareto distribution:

$$f(x) = \frac{\nu}{x^{\nu+1}}, \quad \overline{F}(x) = x^{-\nu}, \quad \nu > 0, x \geq 1$$

- Appeared as description of incomes. Similar ‘power laws’ are also used for file sizes, number of in-links of a web-page, number of social contacts, city sizes, etc.

- High variability

$$\mu = E(X) = \frac{\nu}{\nu - 1}, \quad var(X) = \frac{\nu}{(\nu - 1)^2(\nu - 2)}$$

Then

$$\text{long-run average excess} = \frac{\mu^2 + var(X)}{2E(X)} = \frac{\nu - 1}{2(\nu - 2)}$$

Take $\nu = 2.001$. Then $E(X) = 2.001/1.001 = 1.999$ but the long-run average excess is $1.001/(2 \times 0.001) = 500.5 >> 1.999!$
**Inspection paradox**

Now, look at the long-run average *total life*

\[
\int_{0}^{t} \frac{X_{N(s)+1}}{t} \, ds = \int_{0}^{t} \frac{A(s)}{t} \, ds + \int_{0}^{t} \frac{Y(s)}{t} \, ds
\]

\[
\rightarrow \frac{E(X^2)}{2E(X)} + \frac{E(X^2)}{2E(X)} = \frac{E(X^2)}{E(X)} \geq E(X)
\]

with equality only if \( X \equiv \text{const} \) (Why?)

**Inspection paradox:** Suppose, the lifetimes of bulbs are i.i.d. \( X_1, X_2, \ldots \). The bulb that we observe at arbitrary moment \( t \) typically has a longer lifetime than \( X_1 \). Formally,

\[
P(X_{N(t)+1} > x) \geq P(X_1 > x)
\]

More detail in Section 7.7
Regenerative Processes

**Def.** \( \{X(t), \ t \geq 0\} \) – stochastic process with state space \( \{0, 1, 2, \ldots\} \). There exist time points \( S_1, S_2, \ldots \) where this process *probabilistically restarts itself*. Such process is called a *regenerative process*.

**Example:** Recurrent Markov chain

**Proposition 7.4.** \( E(S_1) < \infty \Rightarrow \text{w.p.1, as } t \to \infty \)

\[
\frac{E[\text{time in } j \text{ during } (0, t)]}{t} \to \frac{E[\text{time in state } j \text{ in a cycle}]}{E[\text{cycle length}]}
\]

**Proof.** Use the Renewal Reward Theorem. \( \square \)

**Example:** Indeed, in a recurrent Markov chain, \( \pi_j = 1/E[\text{time b/w 2 successive visits to } j] \)
Example 7.19 (Queue with Renewal Arrivals.)

Customers arrive according to a renewal process and are served one at a time according to an arbitrary service time distribution. If we suppose that at time 0 the initial customer has just arrived then \( \{X(t), t \geq 0\} \) is a regenerative process, where \( X(t) \) denoted the number of customers in the system at time \( t \). The process regenerates each time the customer arrives and finds the server free.
Example 7.20

Machine and a back-up. When a machine fails, the repair starts immediately, and the other machine is put in use. If the second machine fails before the repair of the 1st machine is finished then the 2nd machine waits until the repair of the 1st one is finished. What are the proportions $P_0, P_1, P_2$ of time when 0, 1, or 2 machines are in working condition?

Solution. $X(t)$ # machines in working conditions at time.

Regeneration epoch: a machine is put in use after repair. Denote: $Y$ is a life-time, $R$ is a repair time. Cycle length is $S = \max(Y, R)$.

$T_i$ is the time in a cycle when $i$ machines are functioning. Then $T_0 = (R - Y)_+, T_1 = \min(Y, R), T_2 = S - T_0 - T_1$, so

$$P_0 = \frac{E[(R - Y)_+]}{E[\max(Y, R)]}, \quad P_1 = \frac{E[\min(Y, R)]}{E[\max(Y, R)]}, \quad P_2 = 1 - P_0 - P_1.$$
Alternating renewal process

\( Z_1, Z_2, \ldots \) – on times; \( Y_1, Y_2, \ldots \) – off times

\( E(Z + Y) < \infty \), \( Z + Y \) has a distribution \( F \)

Regenerating process, two states: on and off.

Regenerating epochs: e.g. beginning of on-period

From Proposition 7.4., as \( t \to \infty \), with probability 1,

\[
\frac{\text{amount on time in } [0, t]}{t} \to \frac{E(Z)}{E(Z) + E(Y)}
\]
Example 7.21 (A Production Process)

A machine works for time $Z_1$, then breaks down and has to be repaired (which takes time $Y_1$), then works for a time $Z_2$, then is down for a time $Y_2$, and so on. If we suppose that the machine is as good as new after each repair, then this constitutes an alternative renewal process. Note that here it is reasonable to assume that on- and off- times are dependent.
Example 7.23 (*Age and excess time distribution*)

Cycle=[time between successive renewals]

The process is *on* for the *first* $x$ units. That is, after an arrival: *on*, if $[A(t) \leq x]$; *off* otherwise

\[
\text{proportion of time age is less than } c = \frac{E(\min\{X, c\})}{E(X)}
\]
\[
= \int_0^\infty P(\min\{X, c\} > y)dy = \int_0^c (1 - F(y))dy
\]

If we want to derive the proportion of time age is less than $c$ then we assume that the process is *on* for the *last* $c$ units.

\[
\text{proportion of time excess is less than } c = \frac{E(\min\{X, c\})}{E(X)}
\]

Note that the answer is the same!
**Example 7.26 (An Inventory example)**

$X_1, X_2, \ldots$ - i.i.d. inter-arrival times of customers, distribution $F$

$D_1, D_2, \ldots$ - i.i.d. demands of customers 1, 2, $\ldots$, distribution $G$

$(s, S)$-policy: inventory is $x \leq s$, then the order $S - x$ is placed

Find the long-run proportion of time that the inventory is $\geq y$.

**Solution.** The process is *on* if inventory $\geq y$; *off* otherwise

\[
\text{long-run proportion of time inventory } \geq y = \frac{E[\text{on time in a cycle}]}{E[\text{time of a cycle}]}
\]

Denote $N_x = \min\{n : D_1 + \cdots + D_n > S - x\}$

*on time in a cycle* $= \sum_{i=1}^{N_y} X_i$, *time of a cycle* $= \sum_{i=1}^{N_s} X_i$

\[
\text{long-run proportion of time inventory } \geq y = \frac{E\left(\sum_{i=1}^{N_y} X_i\right)}{E\left(\sum_{i=1}^{N_s} X_i\right)} = \frac{E(N_y)E(X)}{E(N_s)E(X)}
\]
Consider a renewal process with interarrival times $D_1, D_2, \ldots$

$N_x - 1 = \lceil \text{# renewals by time } S - x \rceil$

$$m_G(t) = \sum_{n=1}^{\infty} G_n(t)$$

long-run proportion of time inventory $\geq y = \frac{E(N_y)}{E(N_s)} = \frac{m_G(S - y) + 1}{m_G(S - s) + 1}$

**Note:** The inter-arrival distribution does not play any role here!