Master's thesis

Draft report

v1.4

June 5, 2009

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Contents

1 Introduction

1.1 Motivation

Water waves in oceans or streams can have a serious impact on the human environment. Waves such as tsunamis, hurricane enduces waves and tidal waves can damage offshore and coastal constructions, ships and when flooding of the land occurs even constructions on land itself can be damaged and the safety of people can be endangered. In order to prevent damage to such constructions or to guarantee the safety of the people information about these waves is of great importance. Predicting such water waves is therefor an important area of research. Modeling water waves can be done mathematically and physically. Experiments versus simulations. Testing constructions at Offshore Basin at MARIN. Setting up takes time and money. First do numerical simulations save time and money. It is therefore interesting to develop a numerical model that describes the wave motion in the basin.

1.2 Water waves in a wave tank

Describe wave tank
Why a coupled model?
Describe physics in relation with mathematical models (to justify approach).

The Maritime Research Institute Netherlands (MARIN) facilitates a basin where offshore models are tested in a realistic environment (source: MARIN site). In this Offshore Basin waves are generated using wave makers. In order to prevent reflected waves from interfering with the waves in the basin, passive wave absorbers are installed opposite to the wave makers. The current computational models at MARIN describing the waves in the Offshore Basin do not incorporate the simulation of the fluid at these wave absorbers. The goal of this research is to develop a numerical model of the surface waves in the Offshore Basin included simulating of the behaviour of the fluid at the passive wave absorbers. The approach used in the research is coupling a linear potential flow model with a shallow water model. The linear potential flow model to be used is developed by Ambati/Bokhove (more?) and describes surface waves in a incompressible, irrotional fluid when no wave breaking is present. The shallow water model describing the wave motion at the wave absorbers can handle these breaking waves.

1.3 Research objectives

Describe objectives and how they are met.
Based on the motivation and situation in the wave tank, the following main objective is set:
-....

1.4 Outline

In chapter 2 the mathematical model is described. Chapter 3 involves the presentation of the numerical schemes that are used for the simulation of water waves in the Offshore Basin. The numerical results for each of the two schemes are compared with exact solutions. In chapter 4 simulation are compared with real data of the Offshore Basin. This report ends with conclusions and recommendations which can be found in chapter 5.

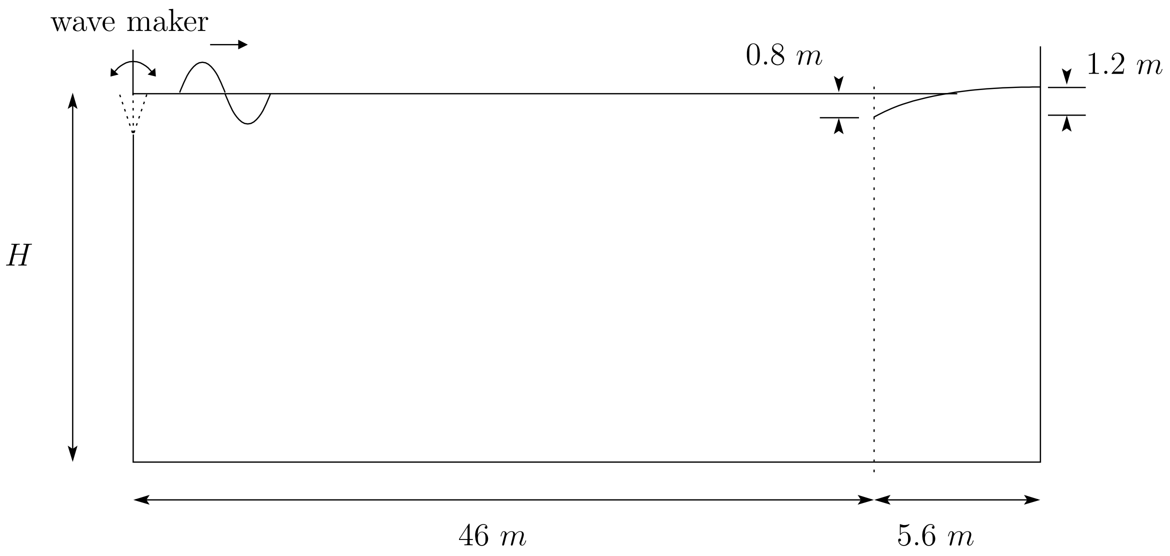

Figure 1: Sketch of the wave tank

2 Mathematical models

2.1 Introduction

In this chapter the mathematical models that describe the wave motion in a wave tank are presented. As can been seen from the cross section of a wave tank in figure 1 (chapter 1), the OB consists of a deep water part and a shallow water part, the latter being located at the wave absorbers at the right end of the domain. The wave motion in the wave tank is described by a two dimensional linear potential flow model in the deep waters and a one dimensional depth-averaged shallow water model in the shallow waters. A sketch of the modeled domain is depicted in figure 2, with domain I being the deep water domain and domain II the shallow water domain. In the next two sections (sections 2.2 and 2.3) each of these models are treated. In section 2.4 the coupling of these two models is treated.

2.2 Linear potential flow model

2.2.1 Governing equations

In the deep waters of a wave tank, the assumping of irrotational flow is justified when describing free surface gravity water waves. This means that $\nabla \times \mathbf{u} = 0$, with $\mathbf{u}$ the vector containing the components of the velocity in different directions, so for two dimensional flow $\mathbf{u} = (u, v)$ with u the velocity in the x-direction and v the velocity in the y-direction. Furthermore for irrotational flow the velocity vector $\mathbf{u}$ can be expressed as the gradient of the velocity potential Φ, or

$$\mathbf{u} = \nabla \Phi \tag{2.1}$$

This type of flow is called potential flow.
From the continuity equation we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{2.2}$$

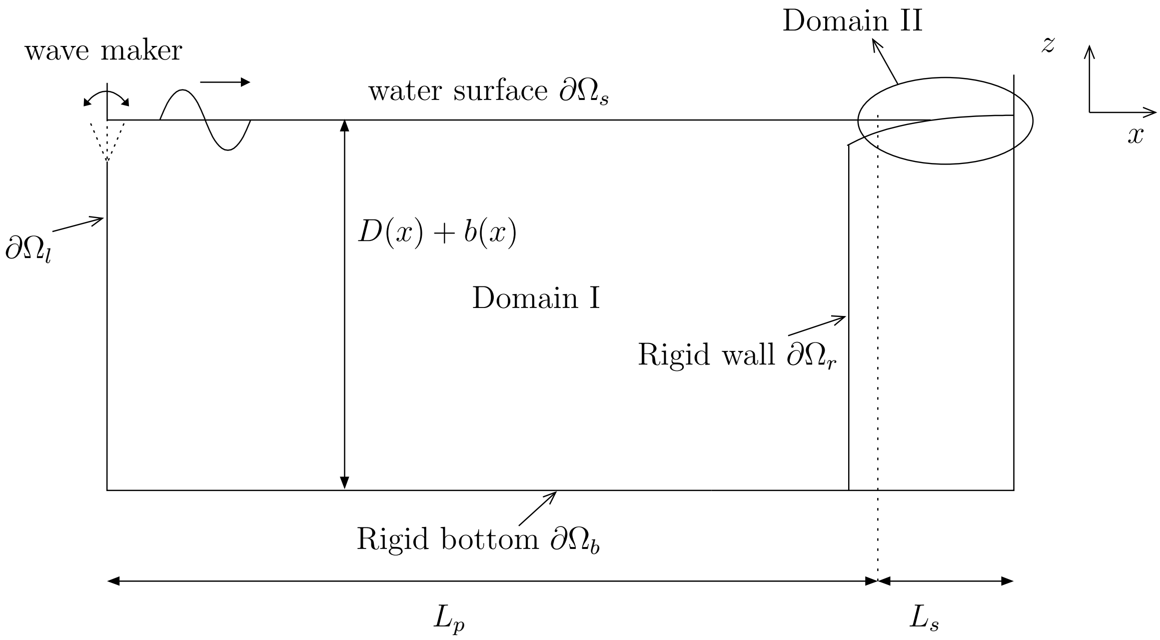

Figure 2: Domain of consideration.

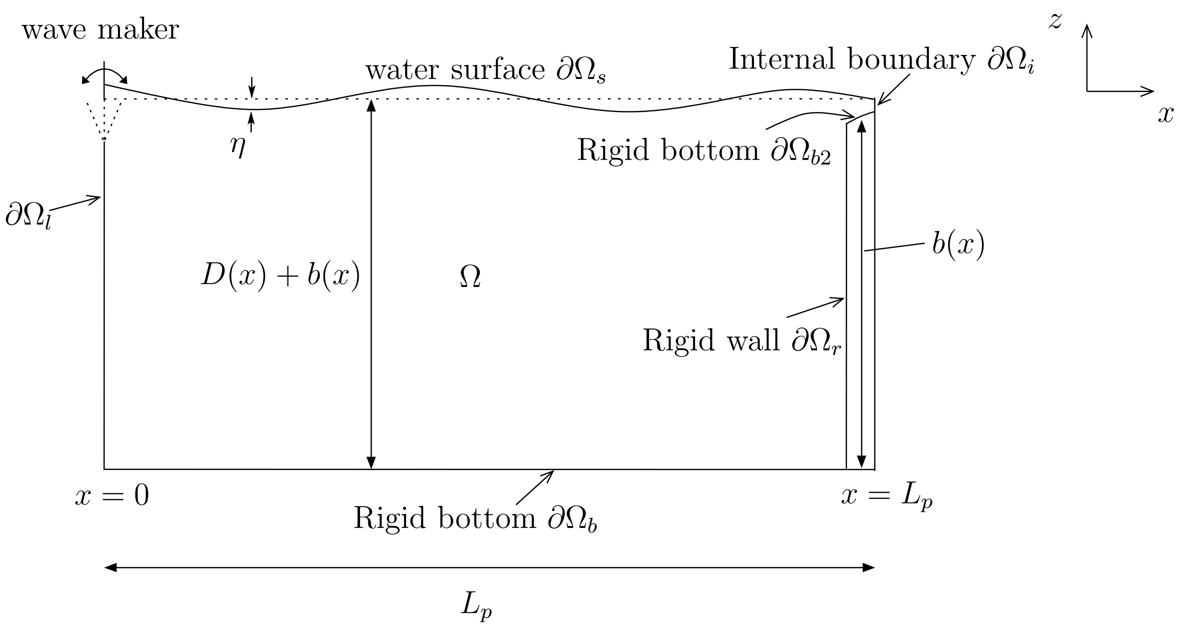

Figure 3: Potential flow domain (domain I)

or, using the chain rule

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho = 0 \tag{2.3}$$

A second assumption is to assume the fluid to be incompressible meaning the density ρ is constant. Then $\frac{\partial \rho}{\partial t} = 0$ and $\nabla \rho = \mathbf{0}$ so equation (2.3) reduces to:

$$\nabla \cdot \mathbf{u} = 0 \tag{2.4}$$

Combining equations (2.1) and (2.4) results in Laplace's equation:

$$\nabla^2 \Phi = 0 \tag{2.5}$$

Which holds for the interior Ω of the domain. For the boundary of the domain other conditions hold. At the free surface there is an interface between the liquid and the gas above the liquid. At this boundary a kinematic and dynamic condition holds. The kinematic condition states that fluid particles that are on the free surface at some time stay at the free surface (Currie). The free surface corresponds to $z - \eta(\mathbf{x_s}, t) = 0$, with $\mathbf{x_s}$ the vector in the free surface which is perpendicular to the z-direction. The material derivative describes the time rate of change of some quantity when following it, while moving (Wikipedia) with a velocity field. This means that while moving with a velocity field u, the position of the particles at the surface doesn't change, thus the material derivative of the location of the particles at the surface equals zeros:

$$\frac{D}{Dt}(z - \eta(\mathbf{x_s}, t)) = 0 \tag{2.6}$$

with $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ the material derivative. Expanding equation (2.6) gives:

$$w - (\frac{\partial \eta(\mathbf{x_s}, t)}{\partial t} - \nabla_s \eta(\mathbf{x_s}, t)) = 0, \tag{2.7}$$

or

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \eta}{\partial t} - \nabla_s \eta \quad \text{at} \quad z = \eta. \tag{2.8}$$

Because we are considering linear free surface waves, η is considering small, therefore u and $\frac{\partial \eta}{\partial x}$ are small. We then have (and by using equation (2.1)):

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at} \quad z = 0. \tag{2.9}$$

The dynamic condition involves the influence of the pressure of the gas above the liquid. At the free surface the pressure is always equal to the ambient pressure. At this free surface the unsteady Bernoulli equation for irrotational fluids applies:

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + \frac{p}{\rho} + gz = F(t), \tag{2.10}$$

with $F(t)$ the constant of integration which depends on t. Taking the ambient pressure to be zero:

$$p = 0 \quad \text{at} \quad z = \eta. \tag{2.11}$$

Substitution this into equation (2.10) gives

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + g\eta = 0. \tag{2.12}$$

Small amplitude waves are considered, therefore this equation results in:

$$\frac{\partial \Phi}{\partial t} + g\eta = 0 \quad \text{at} \quad z = 0. \tag{2.13}$$

The domain under consideration is shown in figure 2. At the free surface, $\partial\Omega_S$, the following boundary conditions apply:

$$\partial_t \Phi + g\eta = 0 \text{ and} \tag{2.14}$$

$$\partial_t \eta - \partial_z \Phi = 0. \tag{2.15}$$

The boundaries $\partial\Omega_b, \partial\Omega_r$ and $\partial\Omega_{b_2}$ are modeled as fixed impermeable walls, therefor a no normal flow boundary condition applies on both of these boundaries:

$$\begin{aligned} \mathbf{n_b} \cdot \nabla\Phi &= 0 \quad \text{on } \partial\Omega_b, \\ -\mathbf{n_r} \cdot \nabla\Phi &= 0 \quad \text{on } \partial\Omega_r, \\ \mathbf{n}_{b_2} \cdot \nabla\Phi &= 0 \quad \text{on } \partial\Omega_{b_2}. \end{aligned}$$

Because of the flat (horizontal) bottom at $\partial\Omega_b$ and the vertical wall at $\partial\Omega_r$ the boundaries conditions reduce to:

$$\partial_z \Phi = 0 \quad \text{on } \partial\Omega_b, \tag{2.16}$$

$$-\partial_x \Phi = 0 \quad \text{on } \partial\Omega_r, \tag{2.17}$$

$$\mathbf{n}_{b_2} \cdot \nabla\Phi = 0 \quad \text{on } \partial\Omega_{b_2}. \tag{2.18}$$

The boundary $\partial\Omega_i$ is the boundary that is linked to the shallow water model. The condition for this boundary is derived in section 2.4. In this section it is treated as a fixed impermeable vertical wall:

$$\partial_x \Phi = 0 \quad \text{on } \partial\Omega_i. \tag{2.19}$$

At the left boundary, $\partial\Omega_l$, the wave maker is located, which is modeled as a prescribed horizontal velocity at this boundary (Westhuis):

$$\partial_x \Phi = g(z)T(t) \quad \text{on } \partial\Omega_l. \tag{2.20}$$

The type of wave maker is governed by the function $g(z)$, which is defined as (Schaffer (1996))

$$g(z) = \begin{cases} 1 + \frac{z}{h+l} & \text{for } -(h-d) \le z \le 0, \\ 0 & \text{for } -h \le z \le -(h-d). \end{cases} \tag{2.21}$$

where l is the length of the wave board, $z = -(h+l)$ is the center of rotation of the board and $d > 0$ the elevation of the rotation point over the bottom of the domain. The wave maker in the OB at MARIN is a flap-type wave maker, which is a hinged board that oscillates around a rotation point (see Westhuis p 86). This type of wave maker corresponds to the case where $l = -d$ (see figure 4), with $l = 1.2m$.

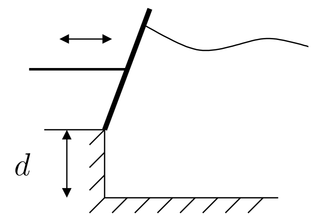

Figure 4: MARIN flap-type wave board

Summerizing the above, the equations for the wave motion in the deep waters of the wave tank are:

$$\begin{aligned}
\nabla^2\Phi &= 0 \quad \text{in } \Omega, && (2.22)\\
-\partial_t\Phi - g\eta &= 0 \quad \text{on } \partial\Omega_s, && (2.23)\\
\partial_t\eta - \nabla\Phi\cdot n_s &= 0 \quad \text{on } \partial\Omega_s, && (2.24)\\
\partial_z\Phi &= 0 \quad \text{on } \partial\Omega_b, && (2.25)\\
\partial_x\Phi &= g(z)T(t) \quad \text{on } \partial\Omega_l, && (2.26)\\
-\partial_x\Phi &= 0 \quad \text{on } \partial\Omega_r, && (2.27)\\
-\nabla\Phi\cdot n_{b2} &= 0 \quad \text{on } \partial\Omega_{b2}, && (2.28)\\
-\partial_x\Phi &= 0 \quad \text{on } \partial\Omega_i. && (2.29)
\end{aligned}$$

2.2.2 Variational formulation

The Lagrangian functional for the modeling of linear potential flow is stated as:

$$\mathcal{L}(\Phi,\eta) = \int_{t_0}^{t_1}\int_\Omega -\frac{1}{2}|\nabla\Phi|^2 \,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\partial\Omega_s}\Big\{-\frac{1}{2}g\eta^2 + \Phi\partial_t\eta\Big\}\,\mathrm{d}x\,\mathrm{d}t, \tag{2.30}$$

with $\Phi = \Phi(x,z,t)$ and $z = \eta(x,t)$ for the case of two-dimensional potential flow.
The equations of motion can be obtained by determining the critical point(s) of the Lagrangian, using the first variation around Φ and η,

$$\delta\mathcal{L}(\Phi+\varepsilon\delta\Phi,\eta+\varepsilon\delta\eta) = \frac{d}{d\varepsilon}\mathcal{L}(\Phi+\varepsilon\delta\Phi,\eta+\varepsilon\delta\eta)\Big|_{\varepsilon=0} = 0. \tag{2.31}$$

Writing (2.31) out (see Appendix ???? for the complete derivation):

$$\begin{aligned}
\delta\mathcal{L}(\Phi+\varepsilon\delta\Phi,\eta+\varepsilon\delta\eta) = &\int_{t_0}^{t_1}\int_\Omega\Big(-\nabla\cdot(\delta\Phi\nabla\Phi) + \delta\Phi\nabla\cdot(\nabla\Phi)\Big)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\,+\\
&\int_{t_0}^{t_1}\int_{\partial\Omega_s}\Big\{-g\eta\delta\eta + \Phi\partial_t(\delta\eta) + \delta\Phi\partial_t\eta\Big\}\,\mathrm{d}x\,\mathrm{d}t = 0
\end{aligned}$$

Applying Gauss' divergence theorem to the first term and integrating the fourth term by parts with respect to t gives:

$$\begin{aligned}
\delta\mathcal{L}(\Phi+\varepsilon\delta\Phi,\eta+\varepsilon\delta\eta) = &\int_{t_0}^{t_1}\int_{\Omega}(\delta\Phi)(\nabla^2\Phi)\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\,+\\
&\int_{t_0}^{t_1}\int_{\partial\Omega_s}(\delta\eta)(-\partial_t\Phi-g\eta)+(\delta\Phi)(\partial_t\eta-\nabla\Phi\cdot n_s)\ \mathrm{d}x\,\mathrm{d}t\,+\\
&\int_{\partial\Omega_s}\Big[\Phi\delta\eta\Big]_{t_0}^{t_1}\mathrm{d}x+\int_{t_0}^{t_1}\int_{\partial\Omega_b}(\delta\Phi)(-\partial_x\Phi)\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\,+\\
&\int_{t_0}^{t_1}\int_{\partial\Omega_l}(\delta\Phi)(\partial_x\Phi)\ \mathrm{d}z\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\partial\Omega_r}(\delta\Phi)(-\partial_x\Phi)\ \mathrm{d}z\,\mathrm{d}t=0
\end{aligned}$$

Because we are considering only domain I in the paragraph, we take the internal boundary $\partial\Omega_i$ to be a fixed wall. Then because $\delta\eta$ and $\delta\Phi$ are zero at $x=0$, $x=L_p$, $t=t_0$, $t=t_1$ and because of the arbitrariness of $\delta\eta$ and $\delta\Phi$, this results in:

$$\begin{aligned}
\nabla^2\Phi &= 0 \quad \text{in } \Omega,\\
-\partial_t\Phi-g\eta &= 0 \quad \text{on } \partial\Omega_s,\\
\partial_t\eta-\nabla\Phi\cdot n_s &= 0 \quad \text{on } \partial\Omega_s,\\
\partial_z\Phi &= 0 \quad \text{on } \partial\Omega_b,\\
\partial_x\Phi &= 0 \quad \text{on } \partial\Omega_l,\\
-\partial_x\Phi &= 0 \quad \text{on } \partial\Omega_r.
\end{aligned}$$

Analytical solutions for certain boundary conditions and initial conditions, with and without wave maker.

2.3 Shallow water model

2.3.1 Governing equations

The wave motion in the shallow part of the basin (see figure 2 for a sketch of the basin) is modeled by the one-dimensional depth-averaged shallow water equation. These shallow water equations are derived from the conservation of mass and momentum. The domain that is considered is shown in figure 5.
Conservation of mass is guarenteed when:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0 \tag{2.32}$$

Which for incompressible flow reduces to: (see section linear potential flow model!!!!)

$$\nabla\cdot\mathbf{u}=0, \tag{2.33}$$

or

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{2.34}$$

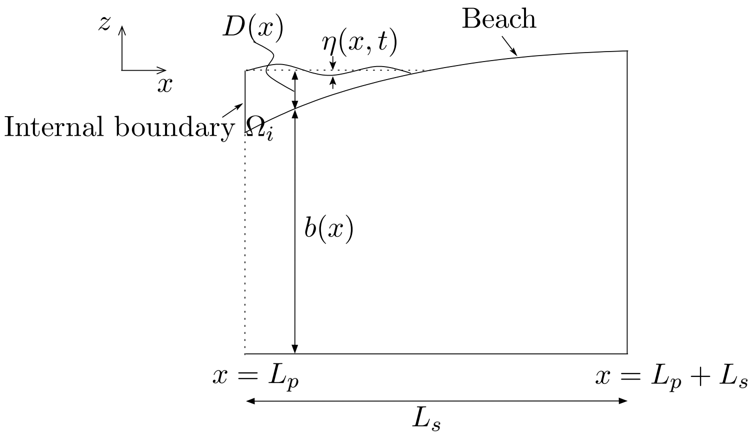

Figure 5: Shallow water domain (domain II)

In the depth averaged shallow water equation, the fluid motion in the depth direction is neglecting. Therefore the velocities u and v are independent of z. Integrating the last equation over the depth (from $z = 0$ to $z = H + \eta$) gives:

$$(H+\eta)\frac{\partial u}{\partial x} + (H+\eta)\frac{\partial v}{\partial y} + w(H+\eta) = 0. \tag{2.35}$$

Particles are at the free surface always stay at the free surface, so we have

$$w(H+\eta) = \frac{D}{Dt}(H+\eta) = \frac{\partial}{\partial t}(H+\eta) + u\frac{\partial}{\partial x}(H+\eta) + v\frac{\partial}{\partial y}(H+\eta) \tag{2.36}$$

Combining these equations gives:

$$(H+\eta)\frac{\partial u}{\partial x} + (H+\eta)\frac{\partial v}{\partial y} + \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = 0, \tag{2.37}$$

or

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(u(H+\eta)) + \frac{\partial}{\partial y}(v(H+\eta)) = 0. \tag{2.38}$$

The second equation of the shallow water equations is derived from the momentum equation. The momentum equation for an incompressible and inviscid fluid is given by Euler's equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{g}. \tag{2.39}$$

From the bottom to the surface the pressure is hydrostatic:

$$p = \rho g(H + \eta - z) \tag{2.40}$$

So we have

$$\frac{\partial p}{\partial x} = \rho g\frac{\partial \eta}{\partial x}, \qquad \frac{\partial p}{\partial y} = \rho g\frac{\partial \eta}{\partial y} \tag{2.41}$$

Then equation (2.39) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial t}. \tag{2.42}$$

The quasi-linear formulation of the shallow water equations in one spatial dimension is given by:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = -g\frac{\partial b}{\partial x}, \tag{2.43}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0, \tag{2.44}$$

where $h(x,t)$ is the depth of the fluid, $u(x,t)$ the velocity, $g = 9.81m/s^2$ the acceleration of gravity and $b(x)$ the topography defined from a certain reference level. The term $-g\frac{\partial b}{\partial x}$ in the first equation is the source term.
The domain of consideration is $x \in (L_p, L_p + L_s)$, see Figure 5 for a sketch of the domain, with $h(x,t) = D(x,t) + b(x)$.
The dimensional equations (2.43)-(2.44) are scaled using the following scalings:

$$u = Uu', \quad x = L_s x', \quad t = \frac{L_s}{U}t', \quad h = Hh', \quad b = Hb' \quad \text{and} \quad g = g'\frac{U^2}{H},$$

where U is the velocity scale, L_s the horizontal scale and H the vertical scale. The dimenionless quasi-linear formulation of the shallow water equations in one spatial dimension then read as:

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + g'\frac{\partial h'}{\partial x'} = -g'\frac{\partial b'}{\partial x'}, \tag{2.45}$$

$$\frac{\partial h'}{\partial t'} + \frac{\partial (h'u')}{\partial x'} = 0, \tag{2.46}$$

The dimensionless quasi-linear formulation (2.45)-(2.46) can be written in a form, conservative for b,

$$\frac{\partial \mathrm{u}}{\partial t} + \frac{\partial f(\mathrm{u})}{\partial x} = S \tag{2.47}$$

with $\mathrm{u} = (hu, h)^T$ and $S = (-gh\frac{\partial b}{\partial x}, 0)^T$, topographic term S, and transpose $(.,.)^T$

2.3.2 Variational formulation

The Lagrangian functional of the non-linear 1D shallow water equations is given as:

$$\mathcal{L}(\phi, h) = \int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s} \left\{ \Big(\frac{1}{2}h(\partial_x\phi)^2 + \frac{1}{2}g((h+b)^2 - b^2)\Big) - \phi\partial_t h \right\} \mathrm{d}x\,\mathrm{d}t, \tag{2.48}$$

with $\phi = \phi(x,t)$, $h = h(x,t)$, $b = b(x)$ and L the length of the domain in the x direction.

The equations of motion can be obtained using the first variation around ϕ and h:

$$\delta\mathcal{L}(\phi + \varepsilon\delta\phi, h + \varepsilon\delta h) = \frac{d}{d\varepsilon}\mathcal{L}(\phi + \varepsilon\delta\phi, h + \varepsilon\delta h)\Big|_{\varepsilon=0} = 0, \quad \text{and} \tag{2.49}$$

Writing (2.49) out (see Appendix ????? for complete derivation):

$$
\begin{aligned}
&\int_{t_0}^{t_1} \Big[h\partial_x\phi\delta\phi\Big]_{L_p}^{L_p+L_s} \mathrm{d}t + \int_{L_p}^{L_p+L_s} \Big[-\phi\delta h\Big]_{t_0}^{t_1} \mathrm{d}x + \\
&\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s} \Big\{(\delta h)(\partial_t\phi + \frac{1}{2}(\partial_x\phi)^2 + gh + gb) + (\delta\phi)(-\partial_t h - \partial_x(h\partial_x\phi))\Big\} \,\mathrm{d}x\,\mathrm{d}t = 0
\end{aligned}
\tag{2.50}
$$

Because we are considering only domain II in this paragraph, the internal boundary $\partial\Omega_i$ is taken to be a fixed wall. Then because δh and $\delta\phi$ are zero at $x = L_p$, $x = L_p + L_s$, $t = t_0$, $t = t_1$ and because of the arbitrariness of δh and $\delta\phi$, this results in:

$$
\begin{aligned}
\partial_t\phi + \frac{1}{2}(\partial_x\phi)^2 + g(h+b) &= 0 \\
-\partial_t h - \partial_x(h\partial_x\phi) &= 0.
\end{aligned}
$$

Differentiating the first equation with respect to x and substituting $\partial_x\phi = u$ this becomes:

$$
\partial_t u + \partial_x(\frac{1}{2}u^2 + g(h+b)) = 0 \tag{2.51a}
$$

$$
\partial_t h + \partial_x(hu) = 0. \tag{2.51b}
$$

2.4 Coupling of the models

The shallow water model and the potential flow model have to be coupled in such a way that the information is passed correctly from one domain into the other. The coupling is accomplished by specifying boundary conditions for each model at the internal boundary between the two domains. In order to determine the internal boundary conditions for each model, the functional for the whole domain is considered. This functional can be obtained by adding the functional of the linear potential flow model and the non-linear shallow water model on the corresponding domains:

$$
\begin{aligned}
\mathcal{L}(\Phi,\phi,\eta,h) = &\int_{t_0}^{t_1}\int_{\Omega} -\frac{1}{2}|\nabla\Phi|^2 \,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\partial\Omega_s} \Big\{-\frac{1}{2}g\eta^2 + \Phi\partial_t\eta\Big\} \,\mathrm{d}x\,\mathrm{d}t + \\
&+ \int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s} \Big\{\Big(\frac{1}{2}h(\partial_x\phi)^2 + \frac{1}{2}g((h+b)^2 - b^2)\Big) - \phi\partial_t h\Big\} \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{2.52}
$$

The equations of motion can again be obtained by determining the critical point(s) of the Lagrangian using the first variation around Φ, ϕ, η and h. This results in:

$$
\begin{aligned}
&\delta\mathcal{L}(\Phi+\varepsilon\delta\Phi,\phi+\varepsilon\delta\phi,\eta+\varepsilon\delta\eta,h+\varepsilon\delta h)=\\
&\int_{t_0}^{t_1}\int_{\Omega}(\delta\Phi)(\nabla^2\Phi)\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\partial\Omega_s}(\delta\eta)(-\partial_t\Phi-g\eta)+(\delta\Phi)(\partial_t\eta-\nabla\Phi\cdot n_s)\ \mathrm{d}x\,\mathrm{d}t+\\
&\int_{\partial\Omega_s}\Big[\Phi\delta\eta\Big]_{t_0}^{t_1}\mathrm{d}x+\int_{t_0}^{t_1}\int_{\partial\Omega_b}(\delta\Phi)(-\partial_x\Phi)\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\partial\Omega_l}(\delta\Phi)(\partial_x\Phi)\ \mathrm{d}z\,\mathrm{d}t+\\
&\int_{t_0}^{t_1}\int_{\partial\Omega_r}(\delta\Phi)(-\partial_x\Phi)\ \mathrm{d}z\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\partial\Omega_{b2}}(\delta\Phi)(-\nabla\Phi\cdot n_{b2})\ \mathrm{d}z\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\partial\Omega_i}(\delta\Phi)(-\partial_x\Phi)\ \mathrm{d}z\,\mathrm{d}t+\\
&\int_{t_0}^{t_1}\Big[h\partial_x\phi\delta\phi\Big]_{L_p}^{L_p+L_s}\mathrm{d}t+\int_{L_p}^{L_p+L_s}\Big[-\phi\delta h\Big]_{t_0}^{t_1}\mathrm{d}x+\\
&\int_{t_0}^{t_1}\int_{0}^{L_p}\Big\{(\delta h)(\partial_t\phi+\frac{1}{2}(\partial_x\phi)^2+gh+gb)+(\delta\phi)(-\partial_t h-\partial_x(h\partial_x\phi))\Big\}\ \mathrm{d}x\,\mathrm{d}t=0
\end{aligned}
\tag{2.53}
$$

Because of the arbitrariness of the variations and the fact that the variations are zero at the boundaries excluded $\partial\Omega_i$, this reduces to:

$$
\begin{aligned}
\nabla^2\Phi&=0 \text{ on } \Omega,\\
-\partial_t\Phi-g\eta=0 \text{ and } \partial_t\eta-\nabla\Phi\cdot n_s&=0 \text{ on } \partial\Omega_s,\\
\partial_z\Phi&=0 \text{ on } \partial\Omega_b,\\
\partial_x\Phi&=0 \text{ on } \partial\Omega_l,\\
-\partial_x\Phi&=0 \text{ on } \partial\Omega_r,\\
-\nabla\Phi\cdot n_{b2}&=0 \text{ on } \partial\Omega_{b2},\\
\int_{t_0}^{t_1}\Big[h\partial_x\phi\delta\phi\Big]_{L_p}^{L_p+L_s}\mathrm{d}t&=-\int_{t_0}^{t_1}\int_{\partial\Omega_i}(\delta\Phi)(\partial_x\Phi)\ \mathrm{d}z\,\mathrm{d}t \text{ on } \partial\Omega_i,\\
\partial_t u+\partial_x(\frac{1}{2}u^2+g(h+b))&=0 \text{ on domain II},\\
\partial_t h+\partial_x(hu)&=0 \text{ on domain II}.
\end{aligned}
$$

The boundary integral on $\partial\Omega_i$ can be written as:

$$
-h\int_{t_0}^{t_1}\partial_x\phi\delta\phi\Big|_{x=L_p}\mathrm{d}t=-\int_{t_0}^{t_1}\int_{\partial\Omega_i}(\delta\Phi)(\partial_x\Phi)\ \mathrm{d}z\,\mathrm{d}t \text{ on } \partial\Omega_i. \tag{2.54}
$$

or

$$
h\partial_x\phi\delta\phi\Big|_{x=L_p}=\int_{\partial\Omega_i}(\partial_x\Phi)(\delta\Phi)\ \mathrm{d}z \text{ on } \partial\Omega_i. \tag{2.55}
$$

Because the depth-averaged shallow water equations are considered, the velocity potential $\phi(x,t)$ represents the depth-averaged velocity potential:

$$
\phi(x,t)=\bar{\Phi}(x,t)=\frac{1}{\eta(x)+D(x)}\int_{-D(x)}^{\eta(x)}\Phi(x,z,t)\ \mathrm{d}z, \tag{2.56}
$$

and also

$$\delta\phi(x,t) = \overline{\delta\Phi}(x,t) = \frac{1}{\eta(x)+D(x)}\int_{-D(x)}^{\eta(x)} \delta\Phi(x,z,t)\ \mathrm{d}z. \tag{2.57}$$

The waterheight h for the shallow water model at the internal boundary equals $\eta(L_p)+D(L_p)$, so the left hand side of (2.55) becomes:

$$(\eta(L_p)+D(L_p))\partial_x\bar{\Phi}(L_p,z,t)\overline{\delta\Phi}(L_p,z,t) = \tag{2.58}$$

$$= (\eta(L_p)+D(L_p))\frac{1}{\eta(L_p)+D(L_p)}\partial_x\bar{\Phi}(L_p,z,t)\int_{-D(L_p)}^{\eta(L_p)} \delta\Phi(L_p,z,t)\ \mathrm{d}z = \tag{2.59}$$

$$= \int_{-D(L_p)}^{\eta(L_p)} \partial_x\bar{\Phi}(L_p,z,t)\delta\Phi(L_p,z,t)\ \mathrm{d}z. \tag{2.60}$$

In the linear potential flow model $\eta \approx 0$, so $D+\eta \approx D$. At the internal boundary $\partial\Omega_i$ we then have:

$$\int_{-D(L_p)}^{0} \partial_x\Phi(L_p,z,t)\delta\Phi(L_p,z,t)\ \mathrm{d}z \tag{2.61}$$

Equation (2.55) then becomes:

$$\int_{-D(L_p)}^{\eta(L_p)} \partial_x\bar{\Phi}(L_p,z,t)\delta\Phi(L_p,z,t)\ \mathrm{d}z = \int_{-D(L_p)}^{0} \partial_x\Phi(L_p,z,t)\delta\Phi(L_p,z,t)\ \mathrm{d}z, \tag{2.62}$$

Splitting the first integral into two integrals:

$$\int_{-D(L_p)}^{0} \partial_x\bar{\Phi}(L_p,z,t)\delta\Phi(L_p,z,t)\ \mathrm{d}z + \int_{0}^{\eta(L_p)} \partial_x\bar{\Phi}(L_p,z,t)\delta\Phi(L_p,z,t)\ \mathrm{d}z = \tag{2.63}$$

$$\int_{-D(L_p)}^{0} \partial_x\Phi(L_p,z,t)\delta\Phi(L_p,z,t)\ \mathrm{d}z, \tag{2.64}$$

which can be written as

$$\int_{0}^{\eta(L_p)} \partial_x\bar{\Phi}(L_p,z,t)\delta\Phi(L_p,z,t)\ \mathrm{d}z + \int_{-D(L_p)}^{0} (\partial_x\bar{\Phi}(L_p,z,t) - \partial_x\Phi(L_p,z,t))\delta\Phi(L_p,z,t)\ \mathrm{d}z = 0. \tag{2.65}$$

or

$$\int_{-D(L_p)}^{0} (\partial_x\bar{\Phi}(L_p,z,t) - \partial_x\Phi(L_p,z,t))\delta\Phi(L_p,z,t)\ \mathrm{d}z = -\int_{0}^{\eta(L_p)} \partial_x\bar{\Phi}(L_p,z,t)\delta\Phi(L_p,z,t)\ \mathrm{d}z. \tag{2.66}$$

If $D+\eta$ is approximated by D for the shallow water model at the internal boundary, the above equation reduces to

$$\int_{-D(L_p)}^{0} \partial_x\bar{\Phi}(L_p,z,t)\delta\Phi(L_p,z,t)\ \mathrm{d}z = \int_{-D(L_p)}^{0} \partial_x\Phi(L_p,z,t)\delta\Phi(L_p,z,t)\ \mathrm{d}z, \tag{2.67}$$

or

$$\int_{-D(L_p)}^{0} (\partial_x \bar{\Phi}(L_p, z, t) - \partial_x \Phi(L_p, z, t))\delta\Phi(L_p, z, t) \, \mathrm{d}z = 0. \tag{2.68}$$

Because of the arbitrariness of the variation, this reduces to the following condition at the internal boundary:

$$\partial_x \bar{\Phi}(L_p, z, t) - \partial_x \Phi(L_p, z, t) = 0 \tag{2.69}$$

$$\partial_x \bar{\Phi}(L_p, z, t) = \partial_x \Phi(L_p, z, t). \tag{2.70}$$

Meaning that the depth-averaged velocity of the shallow water model and the velocity of the potential flow model have to be equal at the internal boundary.

Give criteria for the location of the interface between the two models (wave breaking).
The effect of changing the location of the internal boundary will be analysed in Chapter 3 and 4.

2.5 Conclusion

Conclusion(s) of chapter 2.
- Conditions at the internal boundary are:...
- Take into account the location of the internal boundary.

3 Numerical schemes

3.1 Introduction

In this chapter the numerical methods for solving the potential flow model (section 3.2) and the shallow water equations (section 3.3) are presented. The chapter ends with a comparison between the numerical solutions and the analytical solutions.

3.2 Linear potential flow model

3.2.1 Discretization

The potential flow model is solved using a space discontinuous Galerkin method. The time integration is handled by an implicit time integration method (which one exactly, details?). In the next subsections numerical solutions of the linear potential flow model are compared with exact solutions for the case of harmonic waves without a wave maker and linear waves generated by a wave maker.

3.2.2 Meshing of the domain

3.2.3 Stability, dispersion and dissipation analysis

Stability analysis

Dispersion analysis

Dissipation analysis

3.2.4 Numerical verification

The numerical results fort the linear potential flow model are compared with exact solutions for two cases: - Linear harmonic waves - Linear waves generated by a wave maker
For the case of linear harmonic waves a domain $\Omega = [0, L] \times [-H, 0]$ is considered, with a flat bottom and periodic boundary conditions at the left ($x = 0$) and right boundary ($x = L$). The governing equations are:

$$\nabla^2\Phi = 0 \quad \text{in } \Omega, \tag{3.1}$$
$$-\partial_t\Phi - g\eta = 0 \quad \text{at } z = 0, \tag{3.2}$$
$$\partial_t\eta - \nabla\Phi \cdot n_s = 0 \quad \text{at } z = 0, \tag{3.3}$$
$$\partial_z\Phi = 0 \quad \text{at } z = -H, \tag{3.4}$$
$$\partial_x\Phi = 0 \quad \text{at } x = 0, \tag{3.5}$$
$$-\partial_x\Phi = 0 \quad \text{at } x = L. \tag{3.6}$$

The equation (3.1)-(3.6) are satisfied with the following exact solution for the velocity potential Φ (see Appendix ??? for the derivation of the exact solutions):

$$\Phi(x, z, t) = \tag{3.7}$$

- Comparison between analytical and numerical solutions.
- dispersion relation
- wave numbers
- grid resolution
- $\eta(x,t)$
- Dispersion, dissipation error (numerical).
- initial conditions for η and Φ
- Give table of errors.
- length and depth domain, amplitude etc.

The equations for the case with a piston wave maker located at $x = 0$ and solid walls at the bottom, $z = -H$, and right boundary, $x = L$ are stated as:

$$\begin{aligned}
\nabla^2\Phi &= 0 && \text{in } \Omega, && (3.8)\\
-\partial_t\Phi - g\eta &= 0 && \text{at } z = 0, && (3.9)\\
\partial_t\eta - \nabla\Phi \cdot n_s &= 0 && \text{at } z = 0, && (3.10)\\
\partial_z\Phi &= 0 && \text{at } z = -H, && (3.11)\\
\partial_x\Phi &= g_N(t) && \text{at } x = 0, && (3.12)\\
-\partial_x\Phi &= 0 && \text{at } x = L. && (3.13)
\end{aligned}$$

The wave maker is modeled by a prescribed normal velocity $g_N(t) = ...$ at the left boundary. Taking $g_N(t) = ...$, exact solutions are be constructed for the velocity potential Φ and free surface height η (see Appendix ??? for the derivation):

$$\Phi(x,z,t) = , \tag{3.14}$$

$$\eta(x,t) = . \tag{3.15}$$

- Comparison between analytical and numerical solutions.
- dispersion relation
- wave numbers
- grid resolution
- Dispersion, dissipation error (numerical).
- initial conditions for η and Φ
- Give table of errors.
- length and depth domain, amplitude etc.

3.3 Shallow water model

3.3.1 Discretization

The waves in the shallow water domain are described by the one dimensional depth-averaged shallow water equations (see section ???). These equations in conservative form are stated as (see (2.47)):

$$\frac{\partial \mathsf{u}}{\partial t} + \frac{\partial \mathsf{f}(\mathsf{u})}{\partial x} = S, \tag{3.16}$$

with

$$\mathsf{u} = \begin{pmatrix} hu \\ h \end{pmatrix} \qquad \mathsf{f}(\mathsf{u}) = \begin{pmatrix} hu^2 + \frac{1}{2}gh^2 \\ hu \end{pmatrix},$$

and the topographic term S being:

$$S = \begin{pmatrix} -gh\frac{\partial b}{\partial x} \\ 0 \end{pmatrix}.$$

The above equations are discretized using a Godunov finite volume scheme. In order to handle flooding and drying correctly, the discretization of the shallow water equations according to Audusse et al. (2004) is applied, which is as follows:

$$\mathsf{U}_k^{n+1} = \mathsf{U}_k^n - \frac{\Delta t}{h_k}\Big(\mathsf{F}_{k+\frac{1}{2}}(\mathsf{U}^n_{(k+\frac{1}{2})-}, \mathsf{U}^n_{(k+\frac{1}{2})+}) - \mathsf{F}_{k-\frac{1}{2}}(\mathsf{U}^n_{(k-\frac{1}{2})-}, \mathsf{U}^n_{(k-\frac{1}{2})+})\Big) + \frac{\Delta t}{h_k} S_k, \tag{3.17}$$

with the topographic term

$$S_k = \frac{1}{2} g h^2_{(k+\frac{1}{2})-} - \frac{1}{2} g h^2_{(k-\frac{1}{2})+}, \tag{3.18}$$

$$\mathsf{U}^n_{(k-\frac{1}{2})-} = \begin{pmatrix} h_{(k+\frac{1}{2})-} u_k \\ h_{(k+\frac{1}{2})-} \end{pmatrix} \quad \text{and} \quad \mathsf{U}^n_{(k-\frac{1}{2})+} = \begin{pmatrix} h_{(k+\frac{1}{2})+} u_{k+1} \\ h_{(k+\frac{1}{2})+} \end{pmatrix}. \tag{3.19}$$

To ensure that the waterdepths $h_{(k+\frac{1}{2})-}$ and $h_{(k+\frac{1}{2})+}$ are non-negative, they are chosen as follows:

$$h_{(k+\frac{1}{2})-} = \max(h_k + b_k - b_{k+\frac{1}{2}}, 0), \quad h_{(k+\frac{1}{2})+} = \max(h_{k+1} + b_{k+1} - b_{k+\frac{1}{2}}, 0), \text{ with} \tag{3.20}$$

$$b_{k+\frac{1}{2}} = \max(b_k, b_{k+1}).$$

The HLL flux is used as a numerical flux, which is stated as:

$$\mathsf{F}^{hll}_{k+\frac{1}{2}} = \begin{cases} \mathsf{F}_L & \text{if } 0 < S_L, \\ \frac{S_R \mathsf{F}_L - S_L \mathsf{F}_R + S_L S_R (U_R - U_L)}{S_R - S_L} & \text{if } S_L \leq 0 \leq S_R, \\ \mathsf{F}_R & \text{if } 0 > S_R. \end{cases} \tag{3.21}$$

The wave speeds S_L and S_R are approximated as the smallest respectively the largest eigenvalue at the corresponding node.
To ensure stability of this explicit scheme, a CFL stability condition per cell is used for all eigenvalues λ_p at each U_j^n:

$$\left|\frac{\Delta t}{h_k} \lambda_p(\mathsf{U}_k^n)\right| \leq 1, \tag{3.22}$$

where h_k is the cell width of cell k.

3.3.2 Stability, dispersion and dissipation analysis

Stability, dispersion and dissipation analysis is done for the linearized one dimensional depth-averaged shallow water equations. The linearized equations are obtained by linearizing around ...... These equations are given by:

Stability analysis

Dispersion analysis

Dissipation analysis

3.3.3 Numerical verification

The finite volume discretization of the shallow water equations is verifying by comparing the numerical results with exact solutions of the Riemann problem and with exact solutions to the Burgers' equation for the case that the shallow water equations reduce to the Burgers' equation.
The error between the exact and numerical solution are calculated using the $L2$ norm, which is calculated as follows:
The Riemann problem consists of a conservation law, in this case hyperbolic partial differential equations, together with piecewise constant data. It is therefore defined as

$$\mathsf{u}_t + \mathsf{f}_x = 0 \tag{3.23}$$

with a flux $\mathsf{f} = \mathsf{f}(\mathsf{u})$ and initial conditions

$$\mathsf{u}(x, t_0) = \begin{cases} \mathsf{u}_l & \text{if } x < x_0 \\ \mathsf{u}_r & \text{if } x > x_0 \end{cases} . \tag{3.24}$$

With

$$\mathsf{u} = \begin{pmatrix} hu \\ h \end{pmatrix}$$

for the case of the shallow water equations. Depending on the initial conditions, a shock wave or rarefaction wave can occur for each of the variables hu and u. This allows for four types of solutions: - left moving shock wave, right moving shock wave
- left moving rarefaction wave, right moving shock wave
- left moving shock wave, right moving rarefaction wave
- left moving rarefaction wave, right moving rarefaction wave. The characteristics of each of these cases are depicted in figure ???.

For these cases exact solutions can be obtained (see Appendix ??? for the derivations), which makes comparison to the numerical solutions possible. In figures ??? the exact solutions are plotted against the numerical solutions on different time levels, for all of the four cases.

- grid resolution
- Dispersion, dissipation error (numerical).
- initial conditions for h and hu
- Give table of errors.
- boundary conditions.
- length and depth domain
- no bottom topography

The shallow water equations with the topography term S equal to zero (see equations ???? (3.3)) take the form of the Burger' equation $\partial_t q + q\partial_x q = 0$, when one of its Riemann invariants is taken to be constant as $u + 2\sqrt{gh} = c$,with $q(t,x) = c - 3\sqrt{gh}$ (see Appendix ??? for a

detailed derivation). The solutions of the shallow water equations are then $h(x, t) =$ and $u(x, t) = ...$
- grid resolution
- Dispersion, dissipation error (numerical)?
- initial conditions for h and hu
- Give table of errors.
- boundary conditions.
- length and depth domain
- no bottom topography

3.4 Coupling of the numerical schemes

The potential flow code and the shallow water code are executed sequentially. The information between the codes is exchanged through boundary conditions at the internal boundary. The shallow water code needs a velocity u and a waterheight h from the potential flow code. The potential flow code needs only a velocity u from the shallow water code. The algorithm of the combined codes is described in the flow chart in figure 6. At first the mesh of the whole domain is generated. After this both codes are initialized. In the next step the average velocity $u = \Phi_x$ and the waterheight h at the internal boundary of the potential flow part are computed. These values are necessary to check if the CFL condition is satisfied. If not, the time step is adjusted. Subsequently solutions of the shallow water are calculated at the new time step $n + 1$, with the water height h and $u = \Phi_x$ the average velocity at time step n as input. Next, solutions of the potential flow code are calculated at time step $n + 1$, with only the velocity u from the shallow water code at time step n as an input.

3.4.1 Numerical verification

The coupled discontinuous Galerkin potential flow and finite volume shallow water model is verified by comparing numerical results with exact solutions. Exact solutions are obtained for the situation where there is a flat bottom for the potential flow and shallo water domain (see figure 7).

- Specify BC!
- Formulate exact solutions.
- Give plots with exact solutions and numerical solutions.
- Give table with errors for different grid sizes.
- Give results with more elements at internal boundary.
- Give results with changing the location of the internal boundary.

3.5 Conclusion

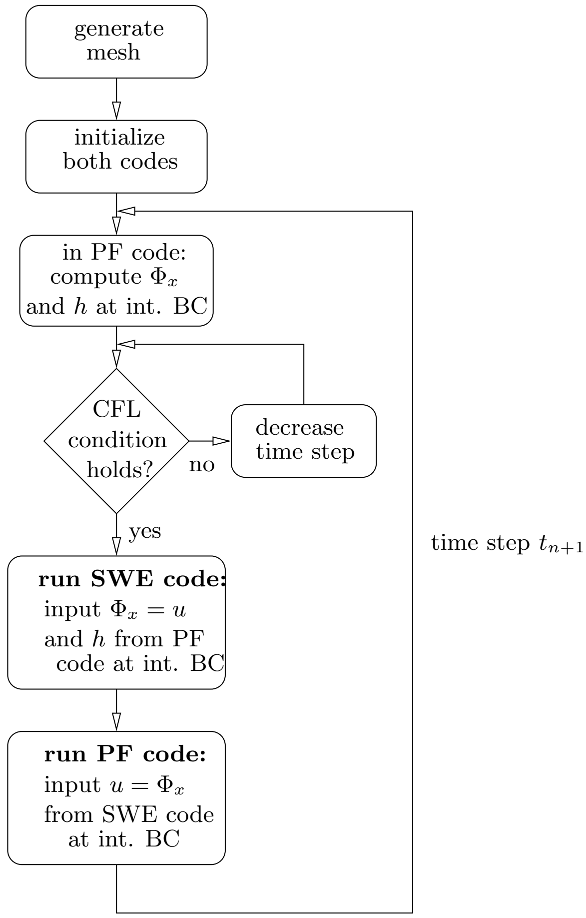

Figure 6: Flow chart of the coupled codes

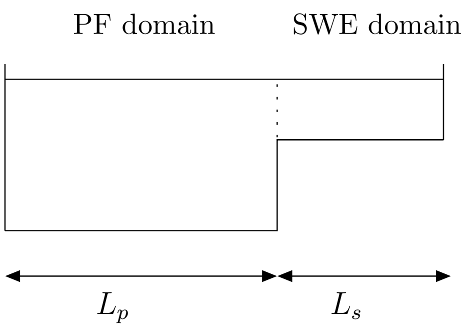

Figure 7: Domain for comparison with exact solutions.

4 Experimental validation

4.1 Introduction

In this chapter the coupled linear potential and shallow water equations model is validated with experimental data from a wave tank.

4.2 Experimental data

The experimental data - Describe setup of the Offshore Basin.
- Describe format of available data.
- Compare measurements with target wave patterns.

4.3 Numerical simulation

An exponential distribution of the elements in depth is applied for the potential flow model. For this, a transformation is used that transforms a uniform grid into a grid that decreases exponentially with decreasing depth. This transformation holds for a grid pictured in figure 8 and is stated as:

$$\bar{z} = \frac{log(\frac{z-\beta}{h-1})}{log(\frac{0-1}{h-1})}, \tag{4.1}$$

with $h \leq 0$. Increasing β results in more clustering near the top of the domain.

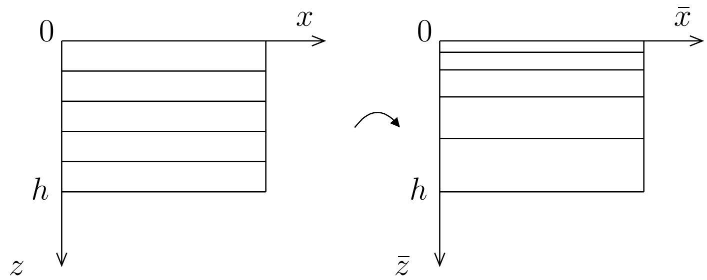

Figure 8: Transformation

- Describe scaling.
- Effect of location of internal boundary.
- Grid refinement

4.4 Conclusion

Conclusion(s) of chapter 4.

5 Conclusions and recommendations

5.1 Introduction

Introduction of chapter 6.

5.2 Conclusions

Conclusion(s) of the report.

5.2.1 Recommendations for MARIN

Recommendations for wave dampening in the offshore basin at MARIN.

5.2.2 Future research

Recommendations for future research with respect to the research done in this report.

6 Appendix

6.1 Derivation exact solutions Riemann problem for the shallow water equations with no bottom topography

The Riemann problem consists of a conservation law, in this case hyperbolic partial differential equations, together with piecewise constant data. It is therefore defined as:

$$\frac{\partial \mathsf{u}}{\partial t} + \frac{\partial \mathsf{f}(\mathsf{u})}{\partial x} = 0 \tag{6.1}$$

with a flux $\mathsf{f} = \mathsf{f}(\mathsf{u})$ and initial conditions

$$\mathsf{u}(x, t_0) = \begin{cases} \mathsf{u}_l & \text{if } x < x_0 \\ \mathsf{u}_r & \text{if } x > x_0 \end{cases} . \tag{6.2}$$

For the depth averaged one dimensional shallow water equations with a constant bottom topography u and $\mathsf{f}(\mathsf{u})$ are stated as:

$$\mathsf{u} = \begin{pmatrix} hu \\ h \end{pmatrix} \qquad \mathsf{f}(\mathsf{u}) = \begin{pmatrix} hu^2 + \frac{1}{2}gh^2 \\ hu \end{pmatrix}.$$

Due to the discontinuous initial conditions of the Riemann problem, shock waves and rarefaction waves can occur. The shallow water equations consist of a system of two equation, therefor two shock or rarefaction waves will occur. This results in four possible states:
- left moving shock wave, right moving shock wave
- left moving shock wave, right moving rarefaction wave
- left moving rarefaction wave, right moving shock wave
- left moving rarefaction wave, right moving rarefaction wave.

In figure ??? these states are pictured, together with the characteristics (see page 71-74 of Toro).
Solutions are constant along the characteristics.
Riemann invariants.
Riemann invariants are constant along the characteristics.
Eigenvalues.
Speed of the shock/discontinuity.

The left moving rarefaction wave is related to the eigenvalue $\lambda = u - \sqrt{gh}$

6.2 Derivation exact solutions linear harmonic waves

Consider a domain $\Omega = [0, L] \times [-H, 0]$ with a non sloping bottom and solid walls at the bottom, $z = -H$, left boundary, right $x = 0$ and boundary, $x = L$. The equations representing this case are:

$$\begin{aligned} \nabla^2\Phi &= 0 \quad \text{in } \Omega, && (6.3)\\ -\partial_t\Phi - g\eta &= 0 \quad \text{at } z=0, && (6.4)\\ \partial_t\eta - \nabla\Phi\cdot n_s &= 0 \quad \text{at } z=0, && (6.5)\\ \partial_z\Phi &= 0 \quad \text{at } z=-H, && (6.6)\\ \partial_x\Phi &= 0 \quad \text{at } x=0, && (6.7)\\ -\partial_x\Phi &= 0 \quad \text{at } x=L. && (6.8) \end{aligned}$$

For the potential flow model (6.3)-(6.8) analytical solutions can be obtained. Suppose the solution is of the form

$$\Phi(x,z,t) = \hat{\Phi}(x,z)e^{iwt} \tag{6.9}$$

Then because of (6.3) we have on Ω:

$$\nabla^2(\hat{\Phi}(x,z)e^{iwt}) = 0,$$

or

$$\hat{\Phi}_{xx}e^{iwt} + \hat{\Phi}_{zz}e^{iwt} = 0,$$

so

$$\nabla^2\hat{\Phi}(x,z) = 0. \tag{6.10}$$

Boundary conditions (6.4) and (6.5) on $\partial\Omega_S$ can be written as a single boundary condition, by differentiating (6.4) with respect to t and substituting (6.5) into (6.4), resulting in:

$$\partial_{tt}\Phi - g\partial_z\Phi = 0 \text{ on } \partial\Omega_S. \tag{6.11}$$

Putting (6.9) into the free surface boundary condition (6.11) gives

$$i^2\omega^2\hat{\Phi}e^{iwt} + g\partial_z\hat{\Phi}e^{iwt} = 0 \text{ on } \partial\Omega_S,$$

or

$$e^{iwt}(g\partial_z\hat{\Phi} - \omega^2\hat{\Phi}) = 0 \text{ on } \partial\Omega_S,$$

resulting in

$$g\partial_z\hat{\Phi} - \omega^2\hat{\Phi} = 0 \text{ on } \partial\Omega_S. \tag{6.12}$$

Substituting (6.9) in (6.6) leads to:

$$\partial_z\hat{\Phi}e^{i\omega t} = 0 \text{ on } \partial\Omega_b,$$

or

$$\partial_z\hat{\Phi} = 0 \text{ on } \partial\Omega_b, \tag{6.13}$$

Now we apply the method of separation of variables for (6.10), (6.12) and (6.13).
Suppose

$$\hat{\Phi}(x,z) = f(x)h(z),$$

Then because of (6.10) we have on Ω:

$$\nabla^2(\hat{\Phi}(x,z)) = \nabla^2(f(x)h(z)) = 0,$$

or

$$\frac{\partial^2 f}{\partial x^2}h + \frac{\partial^2 h}{\partial z^2}f = 0$$

Deviding by fh gives:

$$\frac{\partial^2 h}{\partial z^2}\frac{1}{h} + \frac{\partial^2 f}{\partial x^2}\frac{1}{f}.$$

Because $\frac{\partial^2 h}{\partial z^2}\frac{1}{h}$ depends only on z and $\frac{\partial^2 f}{\partial x^2}\frac{1}{f}$ only on x, we have:

$$\frac{\partial^2 h}{\partial z^2}\frac{1}{h} = k^2 \text{ and } \frac{\partial^2 f}{\partial x^2}\frac{1}{f} = -k^2 \text{ , with } k \text{ a constant.}$$

Shorter notated as:

$$f'' = -k^2 f \text{ and } h'' = k^2 h \text{ , with } k \text{ a constant.} \tag{6.14}$$

Boundary condition (6.12) (at the free surface $\partial\Omega_S$) then becomes:

$$gf\partial_z h - \omega^2 fh = 0 \text{ on } \partial\Omega_S,$$

or

$$f(g\partial_z h - \omega^2 h) = 0 \text{ on } \partial\Omega_S.$$

Because $f \neq 0$ (trivial solutions not allowed):

$$g\partial_z h - \omega^2 h = 0 \text{ on } \partial\Omega_S. \tag{6.15}$$

Boundary condition (6.13) then becomes:

$$f\partial_z h = 0 \text{ on } \partial\Omega_b,$$

because $f \neq 0$:

$$\partial_z h = 0 \text{ on } \partial\Omega_b. \tag{6.16}$$

Take $f(x) = e^{-ikx}$ and $h(z) = e^{kz}$, then (6.14) holds. Substituting these in (6.16) gives:

$$gke^{kz} - \omega^2 e^{kz} = 0 \text{ on } \partial\Omega_S \text{ or at } z = 0,$$

or

$$gke^0 - \omega^2 e^0 = 0,$$

or

$$gk - \omega^2 = 0,$$

so

$$\omega = \sqrt{gk}.$$

Substitution of $f(x)$ and $g(z)$ in (6.16) gives:

$$ke^{kz} = 0 \text{ on } \partial\Omega_b \text{ or at } z = -H,$$

or

$$ke^{-kH} = 0,$$

and because $k = 0$ results in $\hat{\Phi} = 1$, we take $h(z) = e^{k(z+H)} + e^{-k(z+H)}$, then (6.16) becomes

$$ke^{k(z+H)} - ke^{-k(z+H)} = 0 \text{ at } z = -H,$$

or

$$ke^0 - ke^0 = 0,$$

so (6.16) holds. With the new f and h boundary condition (6.16) gives:

$$gke^{k(z+H)} - gke^{-k(z+H)} - \omega^2\left(e^{k(z+H)} + e^{-k(z+H)}\right) = 0 \text{ at } z = 0,$$

or

$$gke^{kH} - gke^{-kH} - \omega^2\left(e^{kH} + e^{-kH}\right) = 0,$$

or

$$\omega^2\left(e^{kH} + e^{-kH}\right) = gk\left(e^{kH} - e^{-kH}\right),$$

so

$$\omega^2 = \frac{gk(e^{kH} - e^{-kH})}{(e^{kH} + e^{-kH})} = gk\frac{2\sinh kH}{2\cosh kH} = gk\tanh kH \quad \text{or} \quad \omega = \sqrt{gk\tanh kH}.$$

Summerizing, we have $f(x) = e^{-ikx}$ and $h(z) = e^{k(z+H)} + e^{-k(z+H)} = 2\cosh k(z+H)$, resulting in

$$\Phi(x,z,t) = \hat{\Phi}(x,z)e^{i\omega t} = f(x)h(z)e^{i\omega t} = 2\cosh\left(k(z+H)\right)e^{-ikx}e^{i\omega t} = \tag{6.17}$$

$$= 2\cosh\left(k(z+H)\right)e^{i(\omega t - kx)} = A\cosh\left(k(z+H)\right)e^{i(\omega t - kx)}. \tag{6.18}$$

6.3 Derivation exact solutions piston type wave maker

Consider a domain $\Omega = [0, L] \times [-H, 0]$ with a non sloping bottom, a piston type wave maker located at $x = 0$ and solid walls at the bottom, $z = -H$, and right boundary, $x = L$. The equations representing this case are:

$$\nabla^2\Phi = 0 \quad \text{in } \Omega, \tag{6.19}$$

$$-\partial_t\Phi - g\eta = 0 \quad \text{at } z = 0, \tag{6.20}$$

$$\partial_t\eta - \nabla\Phi \cdot n_s = 0 \quad \text{at } z = 0, \tag{6.21}$$

$$\partial_z\Phi = 0 \quad \text{at } z = -H, \tag{6.22}$$

$$\partial_x\Phi = g_N(t) \quad \text{at } x = 0, \tag{6.23}$$

$$-\partial_x\Phi = 0 \quad \text{at } x = L. \tag{6.24}$$

The prescribed normal velocity at the left boundary is chosen as $g_N(t) =$. The goal is to derive an exact solution for the above equations with the initial water at rest, so $\eta(x, 0) = 0$. Suppose the velocity potential Φ is chosen as the exact solution for the case with a solid wall at the left boundary in stead of a wave maker (see ???):

$$\Phi(x,z,t) = \frac{A}{\omega}\frac{\cosh(k(z+H))}{\cosh(kH)}\cos(\omega t - kx) \tag{6.25}$$

The equation for the free surface height is then, using $\partial.... =$

$$\eta(x,t) = -A\frac{\cosh(k(z+H))}{\cosh(kH)}\sin(\omega t - kx) \tag{6.26}$$

However $\eta(x,0)$ is then non zero. By choosing

$$\eta(x,t) = -A\frac{\cosh(k(z+H))}{\cosh(kH)}\sin(\omega t - kx) \tag{6.27}$$

, exact solutions can be constructed for the velocity potential Φ and free surface height η:

6.4 Derivation exact solutions Burgers' equation

The Burgers' equation is stated as:

$$\partial_t q + q\partial_x q = 0, \tag{6.28}$$

with $q = q(x,t)$ the velocity of the fluid in the x-direction.
The Riemann invariants of the one dimensional shallow water equations as stated in equation ????(refer to equations in report) are given by $u - c$ and $u + c$, with $c = 2\sqrt{gh}$, u the velocity of the fluid in x-direction and h the waterheight. For the Riemann invariants the following relation holds:

$$\frac{\partial(u \pm c)}{\partial t} + (u \pm a)\frac{\partial(u \pm c)}{\partial x} = 0, \tag{6.29}$$

with $a = \sqrt{gh}$.
Suppose one of the two Riemann invariants is taken to be constant, so $u + c = k$, with k a constant. Then equations (6.29) reduce to:

$$\frac{\partial(u - c)}{\partial t} + (u - a)\frac{\partial(u - c)}{\partial x} = 0. \tag{6.30}$$

Because $u - c = u + c - c - a = k - c - a = k - 2a - a = k - 3a$ and $u - c = u + c - 2c = k - 2c$, (6.30) becomes

$$\frac{\partial(k - 4a)}{\partial t} + (k - 3a)\frac{\partial(k - 4a)}{\partial x} = 0, \tag{6.31}$$

with the derivatives of k being equal to zero,

$$\frac{\partial(-4a)}{\partial t} + (k - 3a)\frac{\partial(-4a)}{\partial x} = 0, \tag{6.32}$$

multiplying by $\frac{3}{4}$ gives

$$\frac{\partial(-3a)}{\partial t} + (k - 3a)\frac{\partial(-3a)}{\partial x} = 0. \tag{6.33}$$

The constant k can be added within the partial derivatives

$$\frac{\partial(k-3a)}{\partial t} + (k-3a)\frac{\partial(k-3a)}{\partial x} = 0. \tag{6.34}$$

Thus we have

$$\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial x} = 0, \tag{6.35}$$

with $q(x,t) = k - 3a = k - 3\sqrt{gh}$, which corresponds to the Burgers' equation. Furthermore we have $q = k - 3a$ or $a = \frac{k-q}{3}$. Substituting this into $u + c = u + 2a = k$ or $u = k - 2a$ results in $u = k - 2a = k - \frac{2(k-q)}{3} = \frac{k-2q}{3}$.
An expression for h is obtained by rearranging the terms of $q = k - 3\sqrt{gh}$, taking the square and deviding by $9g$, resulting in $h = \frac{(q-k)^2}{9g}$.
So the solution $u(x,t) = \frac{k-2q(x,t)}{3}$ and $h(x,t) = \frac{(q(x,t)-k)^2}{9g}$ of the shallow water equations corresponds to the solution $q(x,t) = k - 3a = k - 3\sqrt{gh(x,t)}$ of the Burgers' equation, with $q(x,t)$ a chosen function and k a chosen constant.

6.5 Derivation equations shallow water model from variational principle

The Lagrangian functional of the non-linear 1D shallow water equations is given as:

$$\mathcal{L}(\phi, h) = \int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s} \left\{ \left(\frac{1}{2}h(\partial_x\phi)^2 + \frac{1}{2}g((h+b)^2 - b^2)\right) - \phi\partial_t h \right\} \mathrm{d}x\,\mathrm{d}t, \tag{6.36}$$

with $\phi = \phi(x,t)$, $h = h(x,t)$, $b = b(x)$ and L the length of the domain in the x direction.

The equations of motion can be obtained using the first variation around ϕ and h:

$$\delta\mathcal{L}(\phi + \varepsilon\delta\phi, h + \varepsilon\delta h) = \frac{d}{d\varepsilon}\mathcal{L}(\phi + \varepsilon\delta\phi, h + \varepsilon\delta h)\Big|_{\varepsilon=0} = 0, \quad \text{and} \tag{6.37}$$

Writing (6.37) out:

$$
\begin{aligned}
\delta\mathcal{L}(\phi+\varepsilon\delta\phi,h+\varepsilon\delta h) &= \frac{d}{d\varepsilon}\mathcal{L}(\phi+\varepsilon\delta\phi,h+\varepsilon\delta h)\Big|_{\varepsilon=0} = \\
&= \frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\Big\{\frac{1}{2}(h+\varepsilon\delta h)(\partial_x(\phi+\varepsilon\delta\phi))^2 + \\
&\frac{1}{2}g\Big((h+\varepsilon\delta h+b)^2-b^2\Big) + -(\phi+\varepsilon\delta\phi)\partial_t(h+\varepsilon\delta h)\Big\}\,\mathrm{d}x\,\mathrm{d}t\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\Big\{\frac{1}{2}(h+\varepsilon\delta h)\Big((\partial_x\phi)^2+2\varepsilon\partial_x\phi\partial_x(\delta\phi)+ \\
&\varepsilon^2(\partial_x(\delta\phi))^2\Big)+\frac{1}{2}g\Big(h^2+2\varepsilon h\delta h+2hb+2\varepsilon b\delta h+\varepsilon^2(\delta h)^2+b^2-b^2\Big)+ \\
&-\phi\partial_t h-\varepsilon\phi\partial_t(\delta h)-\varepsilon\delta\phi\partial_t h-\varepsilon^2\delta\phi\partial_t(\delta h)\Big\}\,\mathrm{d}x\,\mathrm{d}t\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\Big\{\frac{1}{2}\Big(h(\partial_x\phi)^2+2\varepsilon h\partial_x\phi\partial_x(\delta\phi)+h\varepsilon^2(\partial_x(\delta\phi))^2+ \\
&\varepsilon\delta h(\partial_x\phi)^2+2\varepsilon^2\delta h\partial_x\phi\partial_x(\delta\phi)+\varepsilon^3\delta h(\partial_x(\delta\phi))^2\Big)+ \\
&\frac{1}{2}g\Big(h^2+2\varepsilon h\delta h+2hb+2\varepsilon b\delta h+\varepsilon^2(\delta h)^2\Big)+ \\
&-(\phi\partial_t h+\varepsilon\phi\partial_t(\delta h)+\varepsilon\delta\phi\partial_t h+\varepsilon^2\delta\phi\partial_t(\delta h))\Big\}\,\mathrm{d}x\,\mathrm{d}t\Big|_{\varepsilon=0} \\
&= \int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\Big\{\frac{1}{2}\Big(2h\partial_x\phi\partial_x(\delta\phi)+\delta h(\partial_x\phi)^2\Big)+\frac{1}{2}\Big(2h\delta h+2b\delta h\Big)+ \\
&-\phi\partial_t(\delta h)\;-\delta\phi\partial_t h\Big\}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{t_0}^{t_1}\Big[h\partial_x\phi\delta\phi\Big]_{L_p}^{L_p+L_s}\mathrm{d}t+\int_{L_p}^{L_p+L_s}\Big[-\phi\delta h\Big]_{t_0}^{t_1}\mathrm{d}x+ \\
&+\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\Big\{(\delta h)\partial_t\phi-\delta\phi\partial_t h+\frac{1}{2}\delta h(\partial_x\phi)^2+gh\delta h+gb\delta h+ \\
&-\partial_x(h\partial_x\phi)(\delta\phi)\Big\}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{t_0}^{t_1}\Big[h\partial_x\phi\delta\phi\Big]_{L_p}^{L_p+L_s}\mathrm{d}t+\int_{L_p}^{L_p+L_s}\Big[-\phi\delta h\Big]_{t_0}^{t_1}\mathrm{d}x+ \\
&+\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\Big\{(\delta h)(\partial_t\phi+\frac{1}{2}(\partial_x\phi)^2+gh+gb)+ \\
&(\delta\phi)(-\partial_t h-\partial_x(h\partial_x\phi))\Big\}\,\mathrm{d}x\,\mathrm{d}t=0
\end{aligned}
\tag{6.38}
$$

Because δh and $\delta\phi$ are zero at $x = L_p$, $x = L_p + L_s$, $t = t_0$, $t = t_1$ and because of the arbitrariness of δh and $\delta\phi$, this results in:

$$
\begin{aligned}
\partial_t\phi + \frac{1}{2}(\partial_x\phi)^2 + g(h+b) &= 0 \\
-\partial_t h - \partial_x(h\partial_x\phi) &= 0.
\end{aligned}
$$

Integrating the first equation with respect to x and substituting $\partial_x\phi = u$ this becomes:

$$
\partial_t u + \partial_x(\frac{1}{2}u^2 + g(h+b)) = 0 \tag{6.39a}
$$

$$
\partial_t h + \partial_x(hu) = 0. \tag{6.39b}
$$

6.6 Derivation equations linear potential flow model from variational principle

The Lagrangian functional for the modeling of linear potential flow is stated as:

$$
\mathcal{L}(\Phi,\eta) = \int_{t_0}^{t_1}\int_\Omega -\frac{1}{2}|\nabla\Phi|^2 \,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\partial\Omega_s}\Big\{-\frac{1}{2}g\eta^2 + \Phi\partial_t\eta\Big\}\,\mathrm{d}x\,\mathrm{d}t, \tag{6.40}
$$

with $\Phi = \Phi(x,z,t)$ and $z = \eta(x,t)$ for the case of two-dimensional potential flow. The equations of motion can be obtained by determining the critical point(s) of the Lagrangian, using the first variation around Φ and η,

$$
\delta\mathcal{L}(\Phi+\varepsilon\delta\Phi,\eta+\varepsilon\delta\eta) = \frac{d}{d\varepsilon}\mathcal{L}(\Phi+\varepsilon\delta\Phi,\eta+\varepsilon\delta\eta)\Big|_{\varepsilon=0} = 0, \tag{6.41}
$$

Writing (6.41) out:

$$
\begin{aligned}
\delta\mathcal{L}(\Phi+\varepsilon\delta\Phi,\eta+\varepsilon\delta\eta) &= \frac{d}{d\varepsilon}\mathcal{L}(\Phi+\varepsilon\delta\Phi,\eta+\varepsilon\delta\eta)\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_\Omega -\frac{1}{2}|\nabla(\Phi+\varepsilon\delta\Phi)|^2 \,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\, + \\
&\quad \int_{t_0}^{t_1}\int_{\partial\Omega_s}\Big\{-\frac{1}{2}g(\eta+\varepsilon\delta\eta)^2 + (\Phi+\varepsilon\delta\Phi)\partial_t(\eta+\varepsilon\delta\eta)\Big\}\,\mathrm{d}x\,\mathrm{d}t\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_\Omega -\frac{1}{2}|\nabla(\Phi+\varepsilon\delta\Phi)|^2 \,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\, + \\
&\quad \int_{t_0}^{t_1}\int_{\partial\Omega_s}\Big\{-\frac{1}{2}g(\eta^2 + 2\varepsilon\eta\delta\eta + \varepsilon^2(\delta\eta)^2) + \Phi\partial_t\eta + \Phi\varepsilon\partial_t(\delta\eta)\, + \\
&\quad \varepsilon\delta\Phi\partial_t\eta + \varepsilon^2\delta\Phi\partial_t(\delta\eta)\Big\}\,\mathrm{d}x\,\mathrm{d}t\Big|_{\varepsilon=0} \\
&= \int_{t_0}^{t_1}\int_\Omega -|\nabla\Phi|\cdot|\nabla(\delta\Phi)|\,\mathrm{d}x\,\mathrm{d}t + \\
&\quad \int_{t_0}^{t_1}\int_{\partial\Omega_s}\Big\{-g\eta\delta\eta + \Phi\partial_t(\delta\eta) + \delta\Phi\partial_t\eta\Big\}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t
\end{aligned}
$$

Using $\nabla u\cdot v = \nabla u\cdot v + u\nabla\cdot v$ for scalar $u = \delta\Phi$ and vector $v = \nabla\Phi$, so $\nabla\delta\Phi\cdot\nabla\Phi = \nabla\delta\Phi\cdot\nabla\Phi + \delta\Phi\nabla\cdot\nabla\Phi$ results in

$$
\int_{t_0}^{t_1}\int_\Omega\Big(-\nabla\cdot(\delta\Phi\nabla\Phi) + \delta\Phi\nabla\cdot(\nabla\Phi)\Big)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\partial\Omega_s}\Big\{-g\eta\delta\eta + \Phi\partial_t(\delta\eta) + \delta\Phi\partial_t\eta\Big\}\,\mathrm{d}x\,\mathrm{d}t = 0
$$

Applying Gauss' divergence theorem to the first term and integrating the fourth term by parts with respect to t gives:

$$\begin{aligned}
\delta\mathcal{L}(\Phi+\varepsilon\delta\Phi,\eta+\varepsilon\delta\eta) = &\int_{t_0}^{t_1}\int_{\Omega} \delta\Phi\nabla\cdot(\nabla\Phi)\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\,+ \\
&\int_{t_0}^{t_1}\int_{\partial\Omega_s} (-\partial_t\Phi(\delta\eta)+(\delta\Phi)\partial_t\eta-g\eta(\delta\eta))\ \mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega_s}\Big[\Phi\delta\eta\Big]_{t_0}^{t_1}\mathrm{d}x\,+ \\
&\int_{t_0}^{t_1}\int_{\partial\Omega_s} -(\delta\Phi)\nabla\Phi\cdot n_s\ \mathrm{d}x\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\partial\Omega_b} -(\delta\Phi)\nabla\Phi\cdot n_b\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,+ \\
&\int_{t_0}^{t_1}\int_{\partial\Omega_l} -(\delta\Phi)\nabla\Phi\cdot n_l\ \mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\partial\Omega_r} -(\delta\Phi)\nabla\Phi\cdot n_r\ \mathrm{d}z\,\mathrm{d}t \\
= &\int_{t_0}^{t_1}\int_{\Omega}(\delta\Phi)(\nabla^2\Phi)\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\partial\Omega_s}(\delta\eta)(-\partial_t\Phi-g\eta)\,+ \\
&(\delta\Phi)(\partial_t\eta-\nabla\Phi\cdot n_s)\ \mathrm{d}x\,\mathrm{d}t + \int_{\partial\Omega_s}\Big[\Phi\delta\eta\Big]_{t_0}^{t_1}\mathrm{d}x\,+ \\
&\int_{t_0}^{t_1}\int_{\partial\Omega_b}(\delta\Phi)(-\partial_x\Phi)\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\partial\Omega_l}(\delta\Phi)(\partial_x\Phi)\ \mathrm{d}z\,\mathrm{d}t\,+ \\
&\int_{t_0}^{t_1}\int_{\partial\Omega_r}(\delta\Phi)(-\partial_x\Phi)\ \mathrm{d}z\,\mathrm{d}t = 0
\end{aligned}$$

Because $\delta\eta$ and $\delta\Phi$ are zero at $x=0$, $x=L_p$, $t=t_0$, $t=t_1$ and because of the arbitrariness of $\delta\eta$ and $\delta\Phi$, this results in:

$$\begin{aligned}
\nabla^2\Phi &= 0 \quad \text{in } \Omega, \\
-\partial_t\Phi-g\eta &= 0 \quad \text{on } \partial\Omega_s, \\
\partial_t\eta-\nabla\Phi\cdot n_s &= 0 \quad \text{on } \partial\Omega_s, \\
\partial_z\Phi &= 0 \quad \text{on } \partial\Omega_b, \\
\partial_x\Phi &= 0 \quad \text{on } \partial\Omega_l, \\
-\partial_x\Phi &= 0 \quad \text{on } \partial\Omega_r.
\end{aligned}$$