Master's thesis

Draft report

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Contents

1 Introduction

The Maritime Research Institute Netherlands (MARIN) facilitates a basin where offshore models are tested in a realistic environment (source: MARIN site). In this Offshore Basin waves are generated using wave makers. In order to prevent reflected waves from interfering with the waves in the basin, passive wave absorbers are installed opposite to the wave makers. The current computational models at MARIN describing the waves in the Offshore Basin do not incorporate the simulation of the fluid at these wave absorbers. The goal of this research is to develop a numerical model of the surface waves in the Offshore Basin included simulating of the behaviour of the fluid at the passive wave absorbers. The approach used in the research is coupling a linear potential flow model with a shallow water model. The linear potential flow model to be used is developed by Ambati/Bokhove (more?) and describes surface waves in a incompressible, irrotional fluid when no wave breaking is present. The shallow water model describing the wave motion at the wave absorbers can handle these breaking waves. In chapter 2 the mathematical model is described. Chapter 3 involves the presentation of the numerical schemes that are used for the simulation of water waves in the Offshore Basin. The numerical results for each of the two schemes are compared with exact solutions. In chapter 4 simulation are compared with real data of the Offshore Basin. This report ends with conclusions and recommendations which can be found in chapter 5.

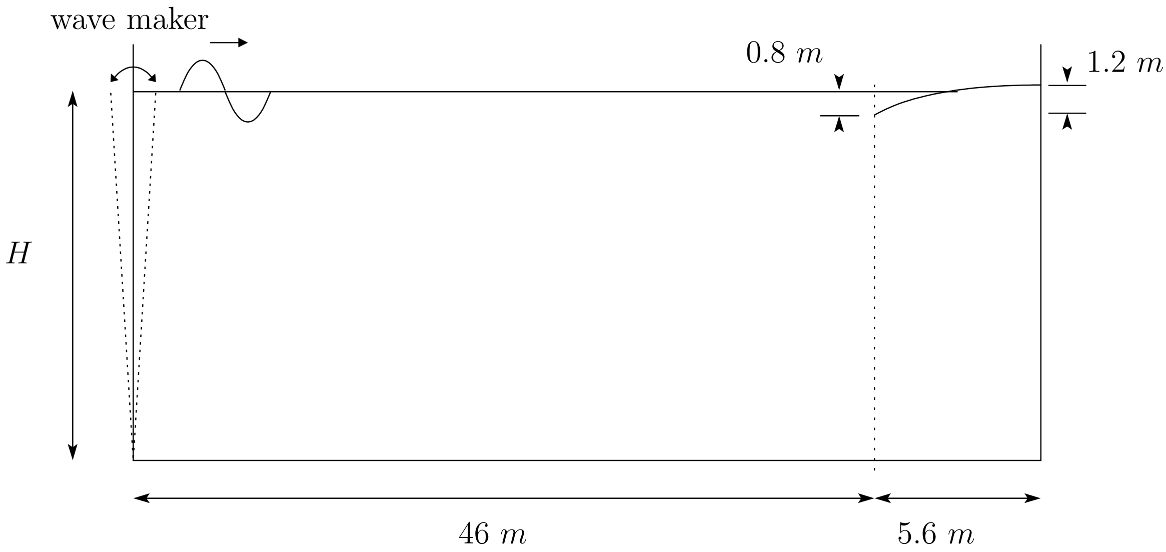

Figure 1: Sketch of the wave tank

2 Mathematical models

2.1 Introduction

A sketch of the offshore basin is depicted in figure 2. Domain I is modeled with a linear potential flow model and domain II with a 1D depth-averaged shallow water model. Each of these models will now be treated.

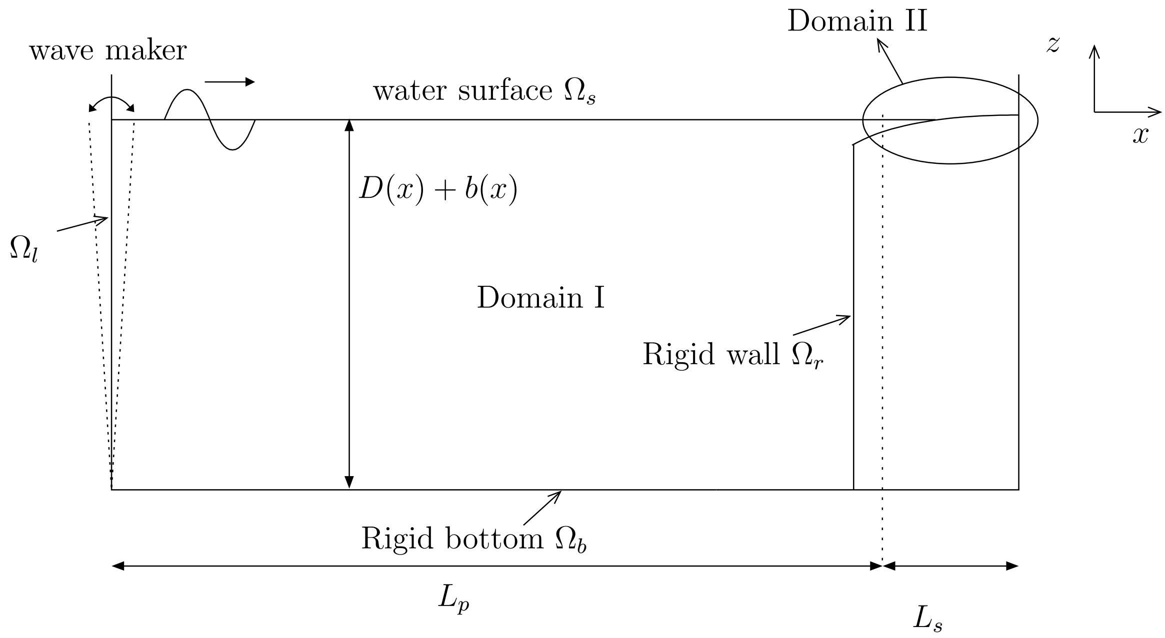

Figure 2: Domain of consideration

2.2 Potential flow model

The waves in the deep water part of the offshore can be described by a linear potential flow model, which is given by Laplace's equation

$$-\nabla^2\Phi = 0 \text{ on } \Omega, \tag{2.1}$$

and boundary conditions at the free surface $\partial\Omega_S$

$$\partial_t\Phi + g\eta = 0 \text{ and} \tag{2.2}$$

$$\partial_t\eta - \partial_z\Phi = 0, \tag{2.3}$$

and a no normal flow boundary condition on the rigid bed $\partial\Omega_b$:

$$\mathbf{n} \cdot \nabla\Phi = 0 \tag{2.4}$$

The wave maker is located at the left boundary, which involves a boundary condition with a prescribed normal velocity. The modeling of the wave maker is explained in detail in the next paragraph.

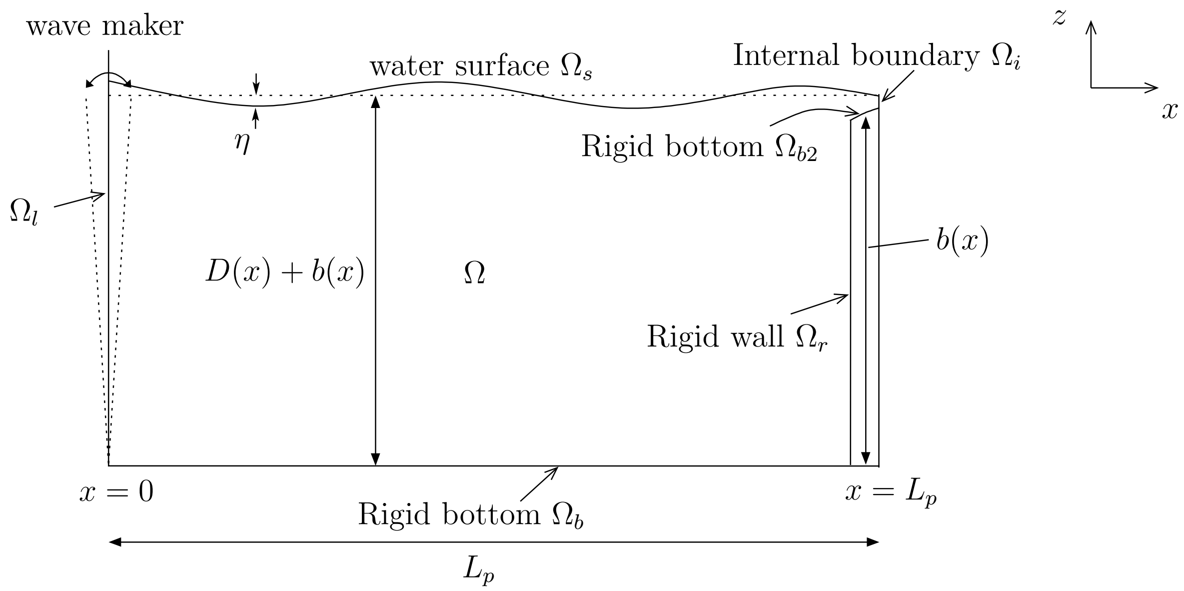

Figure 3: Potential flow domain (domain I)

Depending on the application of interest, the boundary condition at the right of the domain can be a Neumann (fixed wall) or periodic boundary condition.
domain
assumptions: irrotational, incompressible, inviscid flow

2.2.1 Variational formulation

The Lagrangian functional for the modeling of linear potential flow is stated as:

$$\mathcal{L}(\Phi, \eta) = \int_{t_0}^{t_1}\int_{\Omega} -\frac{1}{2}|\nabla\Phi|^2 \,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\Omega_s} \left\{ -\frac{1}{2}g\eta^2 + \Phi\partial_t\eta \right\} \mathrm{d}x\,\mathrm{d}t, \tag{2.5}$$

with $\Phi = \Phi(x, z, t)$ and $z = \eta(x, t)$ for the case of two-dimensional potential flow.

The equations of motion can be obtained by determining the critical point(s) of the Lagrangian, using the first variation around Φ and η,

$$\delta\mathcal{L}(\Phi + \varepsilon\delta\Phi, \eta + \varepsilon\delta\eta) = \frac{d}{d\varepsilon}\mathcal{L}(\Phi + \varepsilon\delta\Phi, \eta + \varepsilon\delta\eta)\Big|_{\varepsilon=0} = 0, \tag{2.6}$$

Writing (2.6) out:

$$
\begin{aligned}
&\delta\mathcal{L}(\Phi+\varepsilon\delta\Phi,\eta+\varepsilon\delta\eta)=\frac{d}{d\varepsilon}\mathcal{L}(\Phi+\varepsilon\delta\Phi,\eta+\varepsilon\delta\eta)\Big|_{\varepsilon=0}=\\
&=\frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_{\Omega}-\frac{1}{2}|\nabla(\Phi+\varepsilon\delta\Phi)|^2\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t+\\
&\quad+\int_{t_0}^{t_1}\int_{\Omega_s}\Big\{-\frac{1}{2}g(\eta+\varepsilon\delta\eta)^2+(\Phi+\varepsilon\delta\Phi)\partial_t(\eta+\varepsilon\delta\eta)\Big\}\ \mathrm{d}x\,\mathrm{d}t\Big|_{\varepsilon=0}=\\
&=\frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_{\Omega}-\frac{1}{2}|\nabla(\Phi+\varepsilon\delta\Phi)|^2\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t+\\
&\quad+\int_{t_0}^{t_1}\int_{\Omega_s}\Big\{-\frac{1}{2}g(\eta^2+2\varepsilon\eta\delta\eta+\varepsilon^2(\delta\eta)^2)+\Phi\partial_t\eta+\Phi\varepsilon\partial_t(\delta\eta)+\varepsilon\delta\Phi\partial_t\eta+\varepsilon^2\delta\Phi\partial_t(\delta\eta)\Big\}\ \mathrm{d}x\,\mathrm{d}t\Big|_{\varepsilon=0}=\\
&=\int_{t_0}^{t_1}\int_{\Omega}-|\nabla\Phi|\cdot|\nabla(\delta\Phi)|\ \mathrm{d}x\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\Omega_s}\Big\{-g\eta\delta\eta+\Phi\partial_t(\delta\eta)+\delta\Phi\partial_t\eta\Big\}\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t=\\
&=\int_{t_0}^{t_1}\int_{\Omega}\Big(-\nabla\cdot(\delta\Phi\nabla\Phi)+\delta\Phi\nabla\cdot(\nabla\Phi)\Big)\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\ +\int_{t_0}^{t_1}\int_{\Omega_s}\Big\{-g\eta\delta\eta+\Phi\partial_t(\delta\eta)+\delta\Phi\partial_t\eta\Big\}\ \mathrm{d}x\,\mathrm{d}t=0
\end{aligned}
$$

Applying Gauss' divergence theorem to the first term and integrating the fourth term by parts with respect to t gives:

$$
\begin{aligned}
&=\int_{t_0}^{t_1}\int_{\Omega}\delta\Phi\nabla\cdot(\nabla\Phi)\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\Omega_s}(-\partial_t\Phi(\delta\eta)+(\delta\Phi)\partial_t\eta-g\eta(\delta\eta))\ \mathrm{d}x\,\mathrm{d}t+\int_{\Omega_s}\Big[\Phi\delta\eta\Big]_{t_0}^{t_1}\mathrm{d}x+\\
&\quad+\int_{t_0}^{t_1}\int_{\Omega_s}-(\delta\Phi)\nabla\Phi\cdot n_s\ \mathrm{d}x\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\Omega_b}-(\delta\Phi)\nabla\Phi\cdot n_b\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\Omega_l}-(\delta\Phi)\nabla\Phi\cdot n_l\ \mathrm{d}z\,\mathrm{d}t+\\
&\quad+\int_{t_0}^{t_1}\int_{\Omega_r}-(\delta\Phi)\nabla\Phi\cdot n_r\ \mathrm{d}z\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\Omega_{b2}}-(\delta\Phi)\nabla\Phi\cdot n_{b2}\ \mathrm{d}z\,\mathrm{d}t++\int_{t_0}^{t_1}\int_{\Omega_i}-(\delta\Phi)\nabla\Phi\cdot n_I\ \mathrm{d}z\,\mathrm{d}t=\\
&=\int_{t_0}^{t_1}\int_{\Omega}(\delta\Phi)(\nabla^2\Phi)\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\Omega_s}(\delta\eta)(-\partial_t\Phi-g\eta)+(\delta\Phi)(\partial_t\eta-\nabla\Phi\cdot n_s)\ \mathrm{d}x\,\mathrm{d}t+\\
&\quad+\int_{\Omega_s}\Big[\Phi\delta\eta\Big]_{t_0}^{t_1}\mathrm{d}x+\int_{t_0}^{t_1}\int_{\Omega_b}(\delta\Phi)(-\partial_x\Phi)\ \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\Omega_l}(\delta\Phi)(\partial_x\Phi)\ \mathrm{d}z\,\mathrm{d}t+\\
&\quad+\int_{t_0}^{t_1}\int_{\Omega_r}(\delta\Phi)(-\partial_x\Phi)\ \mathrm{d}z\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\Omega_{b2}}(\delta\Phi)(-\nabla\Phi\cdot n_{b2})\ \mathrm{d}z\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{\Omega_i}(\delta\Phi)(-\partial_x\Phi)\ \mathrm{d}z\,\mathrm{d}t=0
\end{aligned}
$$

Because we are considering only domain I in the paragraph, we take the internal boundary Ω_i to be a fixed wall. Then because $\delta\eta$ and $\delta\Phi$ are zero at $x=0$, $x=L_p$, $t=t_0$, $t=t_1$ and because of the arbitrariness of $\delta\eta$ and $\delta\Phi$, this results in:

$$
\begin{aligned}
\nabla^2\Phi &= 0 \text{ on } \Omega,\\
-\partial_t\Phi - g\eta &= 0 \text{ and } \partial_t\eta - \nabla\Phi\cdot n_s = 0 \,\text{on}\, \Omega_s,\\
\partial_z\Phi &= 0 \text{ on } \Omega_b,\\
\partial_x\Phi &= 0 \text{ on } \Omega_l,\\
-\partial_x\Phi &= 0 \text{ on } \Omega_r,\\
-\nabla\Phi\cdot n_{b2} &= 0 \text{ on } \Omega_{b2},\\
-\partial_x\Phi &= 0 \text{ on } \Omega_i.
\end{aligned} \tag{2.7}
$$

2.2.2 Modeling of the wave maker

How to model the wave maker.

2.2.3 Analytical solutions

For the potential flow model (2.1)-(2.4) analytical solutions can be obtained. Suppose the solution is of the form

$$\Phi(x,z,t) = \hat{\Phi}(x,z)e^{iwt} \tag{2.8}$$

Then because of (2.1) we have on Ω:

$$\nabla^2(\hat{\Phi}(x,z)e^{iwt}) = 0,$$

or

$$\hat{\Phi}_{xx}e^{iwt} + \hat{\Phi}_{zz}e^{iwt} = 0,$$

so

$$\nabla^2\hat{\Phi}(x,z) = 0. \tag{2.9}$$

Boundary conditions (2.2) and (2.3) on $\partial\Omega_S$ can be written as a single boundary condition, by differentiating (2.2) with respect to t and substituting (2.3) into (2.2), resulting in:

$$\partial_{tt}\Phi - g\partial_z\Phi = 0 \text{ on } \partial\Omega_S. \tag{2.10}$$

Putting (2.8) into the free surface boundary condition (2.10) gives

$$i^2\omega^2\hat{\Phi}e^{iwt} + g\partial_z\hat{\Phi}e^{iwt} = 0 \text{ on } \partial\Omega_S,$$

or

$$e^{iwt}(g\partial_z\hat{\Phi} - \omega^2\hat{\Phi}) = 0 \text{ on } \partial\Omega_S,$$

resulting in

$$g\partial_z\hat{\Phi} - \omega^2\hat{\Phi} = 0 \text{ on } \partial\Omega_S. \tag{2.11}$$

Substituting (2.8) in (2.4) leads to:

$$\partial_z \hat{\Phi} e^{i\omega t} = 0 \text{ on } \partial\Omega_b,$$

or

$$\partial_z \hat{\Phi} = 0 \text{ on } \partial\Omega_b, \tag{2.12}$$

Now we apply the method of separation of variables for (2.9), (2.11) and (2.12). Suppose

$$\hat{\Phi}(x,z) = f(x)h(z),$$

Then because of (2.9) we have on Ω:

$$\nabla^2(\hat{\Phi}(x,z)) = \nabla^2(f(x)h(z)) = 0,$$

or

$$\frac{\partial^2 f}{\partial x^2} h + \frac{\partial^2 h}{\partial z^2} f = 0$$

Deviding by fh gives:

$$\frac{\partial^2 h}{\partial z^2}\frac{1}{h} + \frac{\partial^2 f}{\partial x^2}\frac{1}{f}.$$

Because $\frac{\partial^2 h}{\partial z^2}\frac{1}{h}$ depends only on z and $\frac{\partial^2 f}{\partial x^2}\frac{1}{f}$ only on x, we have:

$$\frac{\partial^2 h}{\partial z^2}\frac{1}{h} = k^2 \text{ and } \frac{\partial^2 f}{\partial x^2}\frac{1}{f} = -k^2 \text{ , with } k \text{ a constant.}$$

Shorter notated as:

$$f'' = -k^2 f \text{ and } h'' = k^2 h \text{ , with } k \text{ a constant.} \tag{2.13}$$

Boundary condition (2.11) (at the free surface $\partial\Omega_S$) then becomes:

$$gf\partial_z h - \omega^2 fh = 0 \text{ on } \partial\Omega_S,$$

or

$$f(g\partial_z h - \omega^2 h) = 0 \text{ on } \partial\Omega_S.$$

Because $f \neq 0$ (trivial solutions not allowed):

$$g\partial_z h - \omega^2 h = 0 \text{ on } \partial\Omega_S. \tag{2.14}$$

Boundary condition (2.12) then becomes:

$$f\partial_z h = 0 \text{ on } \partial\Omega_b,$$

because $f \neq 0$:

$$\partial_z h = 0 \text{ on } \partial\Omega_b. \tag{2.15}$$

Take $f(x) = e^{-ikx}$ and $h(z) = e^{kz}$, then (2.13) holds. Substituting these in (2.14) gives:

$$gke^{kz} - \omega^2 e^{kz} = 0 \text{ on } \partial\Omega_S \text{ or at } z = 0,$$

or

$$gke^0 - \omega^2 e^0 = 0,$$

or

$$gk - \omega^2 = 0,$$

so

$$\omega = \sqrt{gk}.$$

Substitution of $f(x)$ and $g(z)$ in (2.15) gives:

$$ke^{kz} = 0 \text{ on } \partial\Omega_b \text{ or at } z = -H,$$

or

$$ke^{-kH} = 0,$$

and because $k = 0$ results in $\hat{\Phi} = 1$, we take $h(z) = e^{k(z+H)} + e^{-k(z+H)}$, then (2.15) becomes

$$ke^{k(z+H)} - ke^{-k(z+H)} = 0 \text{ at } z = -H,$$

or

$$ke^0 - ke^0 = 0,$$

so (2.15) holds. With the new f and h boundary condition (2.14) gives:

$$gke^{k(z+H)} - gke^{-k(z+H)} - \omega^2\big(e^{k(z+H)} + e^{-k(z+H)}\big) = 0 \text{ at } z = 0,$$

or

$$gke^{kH} - gke^{-kH} - \omega^2\big(e^{kH} + e^{-kH}\big) = 0,$$

or

$$\omega^2\big(e^{kH} + e^{-kH}\big) = gk\big(e^{kH} - e^{-kH}\big),$$

so

$$\omega^2 = \frac{gk(e^{kH} - e^{-kH})}{(e^{kH} + e^{-kH})} = gk\frac{2\sinh kH}{2\cosh kH} = gk\tanh kH \quad \text{or} \quad \omega = \sqrt{gk\tanh kH}.$$

Summerizing, we have $f(x) = e^{-ikx}$ and $h(z) = e^{k(z+H)} + e^{-k(z+H)} = 2\cosh k(z+H)$, resulting in

$$\Phi(x,z,t) = \hat{\Phi}(x,z)e^{i\omega t} = f(x)h(z)e^{i\omega t} = 2\cosh\big(k(z+H)\big)e^{-ikx}e^{i\omega t} = \tag{2.16}$$

$$= 2\cosh\big(k(z+H)\big)e^{i(\omega t - kx)} = A\cosh\big(k(z+H)\big)e^{i(\omega t - kx)}. \tag{2.17}$$

Analytical solutions for certain boundary conditions and initial conditions, with and without wave maker.

2.3 Shallow water model

The modeling of the offshore basin at MARIN considered in this report consists of a shallow water part and a deep water part. In the previous section a potential flow model for the deep water part was treated. In this section the shallow water part is presented. This section begins with a description of the mathematical model and gives analytical solutions for some cases. Subsequently ....
The wave run-up in the shallow water part of the domain is modeled by the shallow water equations. The quasi-linear formulation of the shallow water equations in one spatial dimension is given by:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = -g\frac{\partial b}{\partial x}, \tag{2.18}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0, \tag{2.19}$$

where $h(x,t)$ is the depth of the fluid, $u(x,t)$ the velocity, $g = 9.81 m/s^2$ the acceleration of gravity and $b(x)$ the topography defined from a certain reference level. The term $-g\frac{\partial b}{\partial x}$ in the first equation is the source term.
The domain of consideration is $x \in (L_p, L_p + L_s)$, see Figure 4 for a sketch of the domain, with $h(x,t) = D(x,t) + b(x)$.

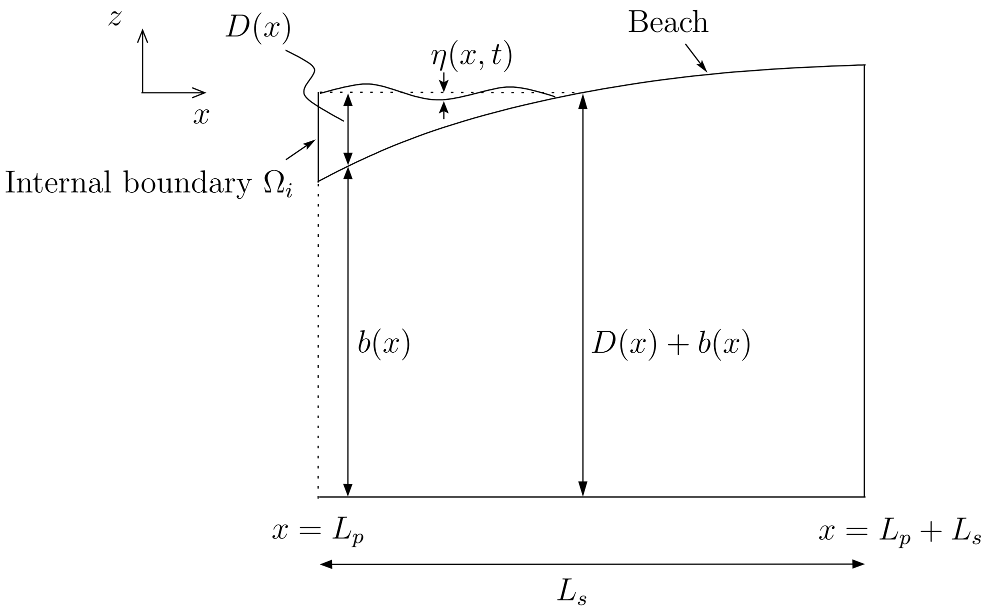

Figure 4: Shallow water domain (domain II)

The dimensional equations (2.18)-(2.19) are scaled using the following scalings:

$$u = Uu', \quad x = L_s x', \quad t = \frac{L_s}{U}t', \quad h = Hh', \quad b = Hb' \quad \text{and} \quad g = g'\frac{U^2}{H},$$

where U is the velocity scale, L_s the horizontal scale and H the vertical scale. The dimenionless quasi-linear formulation of the shallow water equations in one spatial dimension then read as:

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + g'\frac{\partial h'}{\partial x'} = -g'\frac{\partial b'}{\partial x'}, \tag{2.20}$$

$$\frac{\partial h'}{\partial t'} + \frac{\partial (h'u')}{\partial x'} = 0, \tag{2.21}$$

The dimensionless quasi-linear formulation (2.20)-(2.21) can be written in a form, conservative for b,

$$\frac{\partial \mathrm{u}}{\partial t} + u\frac{\partial f(\mathrm{u})}{\partial x} = S \tag{2.22}$$

with $\mathrm{u} = (hu, h)^T$ and $S = (-gh\frac{\partial b}{\partial x}, 0)^T$, topographic term S, and transpose $(.,.)^T$

2.3.1 Variational formulation

The Lagrangian functional of the linear 1D shallow water equations is given as:

$$\mathcal{L}(\phi, \eta) = \int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s} \left\{ \left(\frac{1}{2}D(\partial_x\phi)^2 + \frac{1}{2}g\eta^2\right) - \phi\partial_t\eta \right\} \mathrm{d}x\,\mathrm{d}t, \tag{2.23}$$

with $\phi = \phi(x,t)$, $\eta = \eta(x,t)$ and L_s the length of the domain in the x direction.

The equations of motion can be obtained by determining the critical point(s) of the Lagrangian using the first variation around ϕ and η:

$$\delta\mathcal{L}(\phi + \varepsilon\delta\phi, \eta + \varepsilon\delta\eta) = \frac{d}{d\varepsilon}\mathcal{L}(\phi + \varepsilon\delta\phi, \eta + \varepsilon\delta\eta)\Big|_{\varepsilon=0} = 0, \quad \text{and} \tag{2.24}$$

Writing (2.24) out:

$$
\begin{aligned}
&\delta\mathcal{L}(\phi+\varepsilon\delta\phi,\eta+\varepsilon\delta\eta) = \frac{d}{d\varepsilon}\mathcal{L}(\phi+\varepsilon\delta\phi,\eta+\varepsilon\delta\eta)\Big|_{\varepsilon=0} = \\
&= \frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\bigg\{\frac{1}{2}D\Big(\partial_x(\phi+\varepsilon\delta\phi)\Big)^2+\frac{1}{2}g(\eta+\varepsilon\delta\eta)^2+ \\
&\qquad\qquad -(\phi+\varepsilon\delta\phi)\partial_t(\eta+\varepsilon\delta\eta)\bigg\}\,\mathrm{d}x\,\mathrm{d}t\Big|_{\varepsilon=0} = \\
&= \frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\bigg\{\frac{1}{2}D\Big((\partial_x\phi)^2+2\varepsilon\partial_x\phi\partial_x(\delta\phi)+\varepsilon^2(\partial_x(\delta\phi))^2\Big)+ \\
&\qquad\qquad +\frac{1}{2}g\Big(\eta^2+2\varepsilon\eta\delta\eta+\varepsilon^2(\delta\eta)^2\Big)+ \\
&\qquad\qquad -\phi\partial_t\eta-\varepsilon\phi\partial_t(\delta\eta)-\varepsilon\delta\phi\partial_t\eta-\varepsilon^2\delta\phi\partial_t(\delta\eta)\bigg\}\,\mathrm{d}x\,\mathrm{d}t\Big|_{\varepsilon=0} = \\
&= \int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\Big\{D\partial_x\phi\partial_x(\delta\phi)+g\eta\delta\eta-\phi\partial_t(\delta\eta)-\delta\phi\partial_t\eta\Big\}\,\mathrm{d}x\,\mathrm{d}t = \\
&= \int_{t_0}^{t_1}\Big[D\partial_x\phi\delta\phi\Big]_0^L\,\mathrm{d}t+\int_{L_p}^{L_p+L_s}\Big[-\phi\delta\eta\Big]_{t_0}^{t_1}\,\mathrm{d}x+ \\
&\qquad +\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\Big\{\delta\eta\partial_t\phi-\delta\phi\partial_t\eta-D\partial_{xx}\phi(\delta\phi)+g\eta\delta\eta\Big\}\,\mathrm{d}x\,\mathrm{d}t = \\
&= \int_{t_0}^{t_1}\Big[D\partial_x\phi\delta\phi\Big]_{L_p}^{L_p+L_s}\,\mathrm{d}t+\int_0^L\Big[-\phi\delta\eta\Big]_{t_0}^{t_1}\,\mathrm{d}x+ \\
&\qquad +\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\Big\{(\delta\eta)(\partial_t\phi+g\eta)+(\delta\phi)(-\partial_t\eta-D\partial_{xx}\phi)\Big\}\,\mathrm{d}x\,\mathrm{d}t = 0
\end{aligned}
\tag{2.25}
$$

Because we are considering only domain II in the paragraph, the internal boundary Ω_i is taken to be a fixed wall. Then because δeta and $\delta\phi$ are zero at $x = L_p$, $x = L_p + L_s$, $t = t_0$, $t = t_1$ and because of the arbitrariness of δeta and $\delta\phi$, this results in:

$$
\begin{aligned}
\partial_t\phi + g\eta &= 0 \\
-\partial_t\eta - D\partial_{xx}\phi &= 0.
\end{aligned}
$$

Diffentiating the first equation with respect to x and substituting $\partial_x\phi = u$ this becomes:

$$\partial_t u + \partial_x(g\eta) = 0 \tag{2.26a}$$

$$\partial_t\eta + D\partial_x u = 0. \tag{2.26b}$$

The Lagrangian functional of the non-linear 1D shallow water equations is given as:

$$\mathcal{L}(\phi,h) = \int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\bigg\{\Big(\frac{1}{2}h(\partial_x\phi)^2+\frac{1}{2}g((h+b)^2-b^2)\Big)-\phi\partial_t h\bigg\}\,\mathrm{d}x\,\mathrm{d}t, \tag{2.27}$$

with $\phi = \phi(x,t)$, $h = h(x,t)$, $b = b(x)$ and L the length of the domain in the x direction.

The equations of motion can be obtained using the first variation around ϕ and h:

$$\delta\mathcal{L}(\phi+\varepsilon\delta\phi,h+\varepsilon\delta h)=\frac{d}{d\varepsilon}\mathcal{L}(\phi+\varepsilon\delta\phi,h+\varepsilon\delta h)\Big|_{\varepsilon=0}=0,\quad\text{and}\tag{2.28}$$

Writing (2.28) out:

$$\begin{aligned}
&\delta\mathcal{L}(\phi+\varepsilon\delta\phi,h+\varepsilon\delta h)=\frac{d}{d\varepsilon}\mathcal{L}(\phi+\varepsilon\delta\phi,h+\varepsilon\delta h)\Big|_{\varepsilon=0}=\\
&=\frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\bigg\{\frac{1}{2}(h+\varepsilon\delta h)(\partial_x(\phi+\varepsilon\delta\phi))^2+\frac{1}{2}g\Big((h+\varepsilon\delta h+b)^2-b^2\Big)+\\
&\qquad-(\phi+\varepsilon\delta\phi)\partial_t(h+\varepsilon\delta h)\bigg\}\,\mathrm{d}x\,\mathrm{d}t\Big|_{\varepsilon=0}=\\
&=\frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\bigg\{\frac{1}{2}(h+\varepsilon\delta h)\Big((\partial_x\phi)^2+2\varepsilon\partial_x\phi\partial_x(\delta\phi)+\varepsilon^2(\partial_x(\delta\phi))^2\Big)+\\
&\qquad+\frac{1}{2}g\Big(h^2+2\varepsilon h\delta h+2hb+2\varepsilon b\delta h+\varepsilon^2(\delta h)^2+b^2-b^2\Big)+\\
&\qquad-\phi\partial_t h-\varepsilon\phi\partial_t(\delta h)-\varepsilon\delta\phi\partial_t h-\varepsilon^2\delta\phi\partial_t(\delta h)\bigg\}\,\mathrm{d}x\,\mathrm{d}t\Big|_{\varepsilon=0}=\\
&=\frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\bigg\{\frac{1}{2}\Big(h(\partial_x\phi)^2+2\varepsilon h\partial_x\phi\partial_x(\delta\phi)+h\varepsilon^2(\partial_x(\delta\phi))^2+\varepsilon\delta h(\partial_x\phi)^2+2\varepsilon^2\delta h\partial_x\phi\partial_x(\delta\phi)+\\
&\qquad+\varepsilon^3\delta h(\partial_x(\delta\phi))^2\Big)+\frac{1}{2}g\Big(h^2+2\varepsilon h\delta h+2hb+2\varepsilon b\delta h+\varepsilon^2(\delta h)^2\Big)+\\
&\qquad-(\phi\partial_t h+\varepsilon\phi\partial_t(\delta h)+\varepsilon\delta\phi\partial_t h+\varepsilon^2\delta\phi\partial_t(\delta h))\bigg\}\,\mathrm{d}x\,\mathrm{d}t\Big|_{\varepsilon=0}=\\
&=\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\bigg\{\frac{1}{2}\Big(2h\partial_x\phi\partial_x(\delta\phi)+\delta h(\partial_x\phi)^2\Big)+\frac{1}{2}\Big(2h\delta h+2b\delta h\Big)-\phi\partial_t(\delta h)-\delta\phi\partial_t h\bigg\}\,\mathrm{d}x\,\mathrm{d}t=\\
&=\int_{t_0}^{t_1}\Big[h\partial_x\phi\delta\phi\Big]_{L_p}^{L_p+L_s}\mathrm{d}t+\int_{L_p}^{L_p+L_s}\Big[-\phi\delta h\Big]_{t_0}^{t_1}\mathrm{d}x+\\
&\quad+\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}ss\bigg\{(\delta h)\partial_t\phi-\delta\phi\partial_t h+\frac{1}{2}\delta h(\partial_x\phi)^2+gh\delta h+gb\delta h-\partial_x(h\partial_x\phi)(\delta\phi)\bigg\}\,\mathrm{d}x\,\mathrm{d}t=\\
&=\int_{t_0}^{t_1}\Big[h\partial_x\phi\delta\phi\Big]_{L_p}^{L_p+L_s}\mathrm{d}t+\int_{L_p}^{L_p+L_s}\Big[-\phi\delta h\Big]_{t_0}^{t_1}\mathrm{d}x+\\
&\quad+\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\bigg\{(\delta h)(\partial_t\phi+\frac{1}{2}(\partial_x\phi)^2+gh+gb)+(\delta\phi)(-\partial_t h-\partial_x(h\partial_x\phi))\bigg\}\,\mathrm{d}x\,\mathrm{d}t=0
\end{aligned}\tag{2.29}$$

Because we are considering only domain II in the paragraph, the internal boundary Ω_i is taken to be a fixed wall. Then because δh and $\delta\phi$ are zero at $x=L_p$, $x=L_p+L_s$, $t=t_0$, $t=t_1$ and because of the arbitrariness of δh and $\delta\phi$, this results in:

$$
\begin{aligned}
\partial_t\phi + \frac{1}{2}(\partial_x\phi)^2 + g(h+b) &= 0\\
-\partial_t h - \partial_x(h\partial_x\phi) &= 0.
\end{aligned}
$$

Integrating the first equation with respect to x and substituting $\partial_x\phi = u$ this becomes:

$$
\partial_t u + \partial_x(\frac{1}{2}u^2 + g(h+b)) = 0 \tag{2.30a}
$$

$$
\partial_t h + \partial_x(hu) = 0. \tag{2.30b}
$$

2.4 Coupling of the models

The functional for the two coupled domains can be obtained by adding the functional of the linear potential flow model and the non-linear shallow water model on the corresponding domain:

$$
\begin{aligned}
\mathcal{L}(\Phi,\phi,\eta,h) = &\int_{t_0}^{t_1}\int_{\Omega} -\frac{1}{2}|\nabla\Phi|^2 \,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\Omega_s}\Big\{-\frac{1}{2}g\eta^2 + \Phi\partial_t\eta\Big\}\,\mathrm{d}x\,\mathrm{d}t +\\
&+\int_{t_0}^{t_1}\int_{L_p}^{L_p+L_s}\Big\{\Big(\frac{1}{2}h(\partial_x\phi)^2 + \frac{1}{2}g((h+b)^2-b^2)\Big) - \phi\partial_t h\Big\}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.31}
$$

The equations of motion can again be obtained by determining the critical point(s) of the Lagrangian using the first variation around Φ, ϕ, η and h. This results in:

$$
\begin{aligned}
&\int_{t_0}^{t_1}\int_{\Omega}(\delta\Phi)(\nabla^2\Phi)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\Omega_s}(\delta\eta)(-\partial_t\Phi - g\eta) + (\delta\Phi)(\partial_t\eta - \nabla\Phi\cdot n_s)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_s}\Big[\Phi\delta\eta\Big]_{t_0}^{t_1}\mathrm{d}x +\\
&+\int_{t_0}^{t_1}\int_{\Omega_b}(\delta\Phi)(-\partial_x\Phi)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\Omega_l}(\delta\Phi)(\partial_x\Phi)\,\mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\Omega_r}(\delta\Phi)(-\partial_x\Phi)\,\mathrm{d}z\,\mathrm{d}t +\\
&+\int_{t_0}^{t_1}\int_{\Omega_{b2}}(\delta\Phi)(-\nabla\Phi\cdot n_{b2})\,\mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{\Omega_i}(\delta\Phi)(-\partial_x\Phi)\,\mathrm{d}z\,\mathrm{d}t + \int_{t_0}^{t_1}\Big[h\partial_x\phi\delta\phi\Big]_{L_p}^{L_p+L_s}\mathrm{d}t +\\
&+\int_{L_p}^{L_p+L_s}\Big[-\phi\delta h\Big]_{t_0}^{t_1}\mathrm{d}x + \int_{t_0}^{t_1}\int_{0}^{L_p}\Big\{(\delta h)(\partial_t\phi + \frac{1}{2}(\partial_x\phi)^2 + gh + gb) + (\delta\phi)(-\partial_t h - \partial_x(h\partial_x\phi))\Big\}\,\mathrm{d}x\,\mathrm{d}t = 0
\end{aligned} \tag{2.32}
$$

Because of the arbitrariness of the variations and the fact that the variations are zero at the boundaries excluded Ω_i, this reduces to:

$$
\begin{aligned}
\nabla^2\Phi &= 0 \text{ on } \Omega,\\
-\partial_t\Phi - g\eta = 0 \text{ and } \partial_t\eta - \nabla\Phi\cdot n_s &= 0 \text{ on } \Omega_s,\\
\partial_z\Phi &= 0 \text{ on } \Omega_b,\\
\partial_x\Phi &= 0 \text{ on } \Omega_l,\\
-\partial_x\Phi &= 0 \text{ on } \Omega_r,\\
-\nabla\Phi\cdot n_{b2} &= 0 \text{ on } \Omega_{b2},\\
\int_{t_0}^{t_1}\Big[h\partial_x\phi\delta\phi\Big]_{L_p}^{L_p+L_s}\,\mathrm{d}t &= -\int_{t_0}^{t_1}\int_{\Omega_i}(\delta\Phi)(\partial_x\Phi)\;\mathrm{d}z\,\mathrm{d}t \;\text{ on }\; \Omega_i,\\
\partial_t u + \partial_x(\frac{1}{2}u^2 + g(h+b)) &= 0 \;\text{ on domain II},\\
\partial_t h + \partial_x(hu) &= 0 \;\text{ on domain II.}
\end{aligned}
$$

The boundary integral on Ω_i can be written as:

$$
-h\int_{t_0}^{t_1}\partial_x\phi\delta\phi\Big|_{x=L_p}\mathrm{d}t = -\int_{t_0}^{t_1}\int_{\Omega_i}(\delta\Phi)(\partial_x\Phi)\;\mathrm{d}z\,\mathrm{d}t \;\text{ on }\; \Omega_i. \tag{2.33}
$$

$\Phi_2 = \Phi(x, z = 0, t) = \phi(x, t)$

2.5 Conclusion

Conclusion(s) of chapter 2.

3 Numerical schemes

3.1 Introduction

the numerical scheme for the potential flow model is presented. This chapter ends with a comparison between the numerical solutions and the analytical solutions

3.2 Potential flow model

Description of the numerical scheme used in the potential flow code.

3.2.1 Numerical results

Comparison between analytical and numerical solutions for certain (test) cases.

3.3 Shallow water model

Description of the numerical scheme used in the shallow water code.

3.3.1 Finite volume scheme

Describe the finite volume scheme.
- Godunov method
- Method of Audusse et al for bottom topography
- CFL condition
- Accuracy

3.3.2 Discontinuous Galerkin scheme

Describe the discontinuous Galerkin scheme.

3.3.3 Numerical results

Numerical results of the FV Shallow Water code with the implemented topography of the Offshore Basin at MARIN.
Refer to appendix for geometry beach.

Comparison between analytical and numerical solutions for certain (test) cases.
Show results of the FV discretisation by comparing exact and numerical solutions for the Riemann problem, for several cases.
- derivation exact solution Riemann problem
- describe cases
- numerical solutions for cases
- exact solutions for cases; Burgers' equation

Show results of the DG discretisation by comparing exact and numerical solutions for the Riemann problem, for several cases, with the implemented topography of the Offshore Basin at MARIN. Refer to appendix. Comparison between analytical and numerical solutions for certain (test) cases.

3.4 Conclusion

Or: name this section results and put the numerical results of the potential flow model and shallow water model in it.

4 Results offshore basin model

4.1 Introduction

Introduction of chapter 4.

4.2 Situation(s) of interest

Describe setup of the offshore basin.

4.3 Meshing of the domains

Describe type of meshing used for the two parts of the domain.

4.4 Numerical results

Numerical results for differents (test) cases.
Compare numerical results with experimental data.

4.5 Conclusion

Conclusion(s) of chapter 4.

5 Conclusions and recommendations

5.1 Introduction

Introduction of chapter 6.

5.2 Conclusions

Conclusion(s) of the report.

5.2.1 Recommendations for MARIN

Recommendations for wave dampening in the offshore basin at MARIN.

5.2.2 Future research

Recommendations for future research with respect to the research done in this report.