Social Welfare Orderings for Different Subgroup Utility Scales⋆

Anna B. Khmelnitskaya⋆⋆

SPb Institute for Economics and Mathematics Russian Academy of Sciences,
1 Tchaikovsky St., 191187 St.Petersburg, Russia
and Center for Rationality at The Hebrew University of Jerusalem
Givat Ram, Jerusalem 91904, Israel
(e-mail: anna@math.huji.ac.il, anna@AK3141.spb.edu)

Abstract. This paper characterizes social welfare orderings for different scales of individual utility measurement in distinct population subgroups. Different combinations of ordinal, interval, ratio, and translation scales are studied. We consider situations when utility comparisons among subgroups of individuals by unit and/or zeropoint can or cannot be made, that is when subgroup scales are dependent or independent. We show that for combinations of independent subgroups scales, every corresponding social ordering is fully determined by the opinions of only one subgroup of individuals and is in accord with the measurement scales of its members' utilities. We also investigate social orderings admissible given various combinations of interval scales with a common unit and given combinations of arbitrary ratio scales that combine individual utilities from different subgroups.

JEL classification number: D71.

Key words: Arrovian social choice, social welfare ordering, interpersonal utility comparisons.

⋆ The last version of the paper was completed during my stay at the Center for Rationality and Interactive Decision Theory at The Hebrew University of Jerusalem whose hospitality and financial support are gratefully acknowledged. Support from Russian Foundation for Basic Research (grant #96-01-00170) and the European Community (grant #INTAS-96-106) are also appreciated. Earlier versions of the paper were presented at the Symposium on Operation Research in Passau, Germany, September 1995, and at the Third International Meeting of the Society for Social Choice and Welfare in Maastricht, The Netherlands, June 1996.

⋆⋆ I would like to express my sincere gratitude to Elena Yanovskaya, John Weymark, Kevin Roberts and Natalia Naumova for fruitful discussions and helpful suggestions.

1 Introduction

In Arrow's famous impossibility theorem [1], individual preferences are ordinally measurable and not interpersonally comparable. In a number of studies (Sen [14], Roberts [10], [11], d'Aspremont [3], Yanovskaya [16], [17], Bossert and Weymark [2]), more restrictive measurability-comparability assumptions have been proposed and investigated. These studies show that under different measurability-comparability conditions for individual preferences, nondictatorial social choice rules exist. However, in these studies, the measurement scales for individual preferences are assumed to be of the same type across the entire society.

The goal of this paper is to study Arrovian social choice problems when individual preferences in disjoint subgroups of individuals are measured by different scale types, in other words, when separate subgroups of individuals admit different types of information. This situation is common in real decision making. A typical example is the partitioning of a human society into families which in turn consist of individuals. The kind of of information available within and between families will be different in general. If an outsider is making the comparisons based on reports from individuals, it is reasonable to suppose that although no inter-family comparisons may be possible, families can make some comparisons within themselves and these are the comparisons reported to the observer.

We adopt the welfarist viewpoint of social choice and assume that only individual utilities matter for ranking a feasible set of social alternatives. In an Arrovian social choice problem, this is implied by the requirement of Pareto Indifference; the social choice rule can be equivalently described either in terms of a social welfare ordering — a social ordering of the admissible profiles of individual utilities (admissibility is understood as the satisfaction of several *a priori* appealing conditions), or in terms of a social welfare function — a function that represents a social welfare ordering and measures social welfare. Various assumptions concerning the measurability and interpersonal comparability of utility can be formalized by partitioning the set of feasible individual profiles and requiring the social welfare ordering to be constant over a cell of the partition.

In Section 2, we introduce basic definitions and notation and a formal statement of the problem. In Section 3, various combinations of subgroup scale types that lead to existence of a dictatorial subgroup are considered. It is proved that for combinations of independent subgroup scales there exists a decisive coalition equal to one of the subgroups and that the social welfare function is fully determined in accordance with the scale type of this subgroup. For instance, if in any one subgroup, individual preferences are measured by means of independent ordinal or interval scales, then either the social choice rule is dictatorial where the dictator is from this subgroup, or it is completely independent of the individual preferences in this subgroup and is fully determined by the rest of the society in agreement with the measurement scales of the latter's individual preferences. Section 4 studies cases of interrelations among subgroup scales for which the set of admissible social orderings includes ones that combine preferences of individuals from different subgroups. Section 5 provides some concluding remarks.

2 The framework

Consider a society consisting of a finite set $N = \{1, \dots, n\}$ of $n \geq 2$ individuals. Let X be a finite set of at least three alternatives and let $\mathcal{R}$ denote the set of all possible preference orderings over X. The members of $\mathcal{R}$ are assumed to be weak orders, i.e., complete, reflexive and transitive binary relations. A *social choice problem* is a triple $< X, N, \{R_i\}_{i \in N} >$, where $\{R_i\}_{i \in N}$ is a profile of individual preferences $R_i \in \mathcal{R}$, $i \in N$. To introduce measurability/comparability assumptions, we consider individual preferences represented as individual utilities, which may be interpreted as measurements of these preferences. So, let U be the set of all real-valued functions defined on $X \times N$: for any $u \in U$, let $u(x, i)$ denote the ith individual utility at the alternative $x \in X$. By a solution to a social choice problem we understand a *social welfare functional*, which is a mapping $f \colon \mathcal{D} \to \mathcal{R}$ where $\mathcal{D} \subseteq U$ is the domain of f. We assume f satisfies three welfarism axioms:

Unrestricted Domain. $\mathcal{D} = U$, i.e., f is defined for all $u \in U$.

Independence of Irrelevant Alternatives. For any $u, u' \in \mathcal{D}$ and $A \subseteq X$, if $u(x, i) = u'(x, i)$ for all $x \in A$ and $i \in N$, then $R \colon A = R' \colon A$ where $R = f(u)$ and $R' = f(u')$. ($R \colon A$ denotes the restriction of R to $A \subseteq X$.)

Pareto Indifference. For any pair $x, y \in X$ and for all $u \in \mathcal{D}$, if $u(x, i) = u(y, i)$ for all $i \in N$ then xIy, where I denotes the indifference relation corresponding to $R = f(u)$.

According to the welfarism theorem (D'Aspremont and Gevers [4] and Hammond [6]), these three axioms ensure that only individual utilities matter when ranking social alternatives, so any vector $u = (u_1, \dots, u_n)$ in the n-dimensional Euclidian space $\mathbb{R}^n$ can be considered as a profile of individual utilities for the society N; here u_i is the utility of ith individual. From this perspective, a solution to a social choice problem can be regarded as a *social welfare ordering* (SWO), which is a weak order R^* on $\mathbb{R}^n$, the set of possible profiles of utility vectors. We assume that R^* also possesses the property:

Weak Pareto (WP). For all $u, v \in \mathbb{R}^n$, if $u_i > v_i$ for all $i \in N$, then uP^*v, where P^* denotes the strict preference relation corresponding to R^*.

A function $W \colon \mathbb{R}^n \to \mathbb{R}^1$ *represents* the SWO R^* if for all $u, v \in \mathbb{R}^n$

$$uR^*v \Longleftrightarrow W(u) \geq W(v).$$

The representation W is called a *social welfare function* (SWF). By WP, any SWF W increases with an increase in all arguments, i.e., for all $u, v \in \mathbb{R}^n$

$$u \gg v \Longrightarrow W(u) > W(v).$$

We impose one more restriction on an SWO R^*, that R^* be continuous.

Continuity (C). For all $u \in \mathbb{R}^n$, the sets $\{v \in \mathbb{R}^n | vR^*u\}$ and $\{v \in \mathbb{R}^n | uR^*v\}$ are closed in $\mathbb{R}^n$.

Continuity guarantees the existence of a continuous SWF [5].

In the classic case of Arrow utilities were ordinally measurable and interpersonally non-comparable. More generally, within the SWO framework, the degree of measurability and comparability of utility inside the society N can be specified by a class of invariance transforms Φ, where each transform $\phi \in \Phi$ is a list of functions $\phi = \{\phi_i\}_{i \in N}$, $\phi_i : \mathbb{R}^1 \to \mathbb{R}^1$, with the property: for all $u, v \in \mathbb{R}^n$

$$uR^*v \Longleftrightarrow (\phi u)R^*(\phi v), \tag{1}$$

where $\phi u = \{\phi_i u_i\}_{i \in N}$. In what follows if we wish to specify that the transforms of a class Φ apply to a particular society N, we use the notation Φ_N; when there is no ambiguity the index N will be omitted.

For a class Φ to be a scale in the sense of the standard theory of measurement it has to satisfy the stronger condition of being a group. Different scale types for individual utility measurement have been examined in the literature (Roberts [11], d'Aspremont [3], Bossert and Weymark [2]). Next we list the scales to be considered.

Ordinal Measurability (OM). $\phi \in \Phi$ iff ϕ is a list of independent strictly increasing transforms ϕ_i, $i \in N$.

Ordinal Measurability and Full Comparability (OFC). $\phi \in \Phi$ iff ϕ is a list of identical strictly increasing transforms, i.e., for any real t and all $i \in N$, $\phi_i(t) = \phi_0(t)$ where ϕ_0 is a strictly increasing function independent of i.

Cardinal Measurability (CM). $\phi \in \Phi$ iff ϕ is a list of independent strictly positive affine transforms, i.e., for any real t and all $i \in N$, $\phi_i(t) = \alpha_i + \beta_i t$ for some real α_i and real $\beta_i > 0$.

Cardinal Measurability and Origin Comparability (COC). $\phi \in \Phi$ iff ϕ is a list of strictly positive affine transforms with common constant term, i.e., for any real t and all $i \in N$, $\phi_i(t) = \alpha + \beta_i t$ for some real α and $\beta_i > 0$ with α independent of i.

Notice that a class of COC transforms is not a scale since it does not possess the group property. All the others are scales.

Cardinal Measurability and Unit Comparability (CUC). $\phi \in \Phi$ iff ϕ is a list of strictly positive affine transforms with common unit, i.e., for any real t and all $i \in N$, $\phi_i(t) = \alpha_i + \beta t$ for some real α_i and $\beta > 0$ with β independent of i.

Cardinal Measurability and Full Comparability (CFC). $\phi \in \Phi$ iff ϕ is a list of identical strictly positive affine transforms, i.e., for any real t and all $i \in N$ $\phi_i(t) = \alpha + \beta t$ for some real α and $\beta > 0$, both independent of i.

Ratio-Scale Measurability (RSM). $\phi \in \Phi$ iff ϕ is a list of strictly positive affine transforms with zero constant term, i.e., for any real t and all $i \in N$, $\phi_i(t) = \beta_i t$ for some $\beta_i > 0$.

Ratio-Scale Measurability and Full Comparability (RSF). $\phi \in \Phi$ iff ϕ is a list of identical strictly positive affine transforms with zero constant term, i.e., for any real t and all $i \in N$, $\phi_i(t) = \beta t$ for some $\beta > 0$ independent of i.

Translation-Scale Measurability (TSM). $\phi \in \Phi$ iff ϕ is a list of strictly positive affine transforms, for any real t and all $i \in N$ having the form $\phi_i(t) = \alpha_i + t$ for some real α_i.

Translation-Scale Measurability and Full Comparability (TSF). $\phi \in \Phi$ iff ϕ is a list of identical strictly positive affine transforms for any real t and all $i \in N$ having the form $\phi_i(t) = \alpha + t$ for some real α independent of i.

Thus, under conditions imposed, the Arrovian social choice problem in the informational environment introduced by an invariance class Φ can be equivalently described in terms of SWF W which

1) *is a continuous real-valued function defined on* $\mathbb{R}^n$ *such that* $W(c_N) = c$ *for any real* c;

2) *does not decrease when none of its arguments decreases*[3], *i.e., for all* $u, v \in \mathbb{R}^n$,

$$u \geq v \Longrightarrow W(u) \geq W(v);$$

3) *is invariant under invariance transforms of class* Φ, *i.e., for any* $\phi \in \Phi$ *and for all* $u, v \in \mathbb{R}^n$,

$$W(u) \geq W(v) \Longrightarrow W(\phi u) \geq W(\phi v). \tag{2}$$

By D_n we denote the diagonal of $\mathbb{R}^n$. For any real c, c_N is a vector in $\mathbb{R}^n$ with all components equal to c. $\gamma(c) = \{u \in \mathbb{R}^n | W(u) = c\}$ is a c-level surface of the SWF W; obviously, for every $u \in \mathbb{R}^n$, $\gamma(W(u))$ is a level surface of W containing u.

[3] This holds because W is continuous and increasing with an increase in all of its arguments.

Remark. Because of continuity and strict monotonicity of all SWF, every level surface of any SWF intersects a diagonal D_n of $\mathbb{R}^n$ and moreover, this intersection is a singleton. So, a natural scale for the meanings of SWF arises: since every SWF W is defined up to monotonic strictly increasing transforms, then without loss of generality it may be assumed that for any $u \in \mathbb{R}^n$, $W(u) = c$, with c defined by the equality $\gamma(W(u)) \cap D_n = \{c_N\}$.

The main concern of this paper is the situation where the entire society N is partitioned into m disjoint subgroups of individuals, i.e., $N = N_1 \cup N_2 \cup \cdots \cup N_m$ with $N_i \cap N_j = \emptyset$ for $i \neq j$. It is assumed that an SWF W defined on $\mathbb{R}^n$ for different subgroups of variables indexed by N_k, $k \in \overline{1,m}$, may admit invariance transforms of different invariance classes Φ_{N_k}, which amounts to W being invariant under transforms of a class Φ_N such that $\Phi_N = \{\Phi_{N_k}\}_{k=1}^m$, i.e., for every $\phi \in \Phi_N$ for all $k \in \overline{1,m}$, $\phi_{N_k} = \{\phi_i\}_{i \in N_k} \in \Phi_{N_k}$. It is worth noting that the class Φ_N has to satisfy condition (1); however, even if all invariant classes Φ_{N_k} are scales, Φ_N may not be a scale: the condition of being a group may no longer hold.

To complete this section we introduce some additional notation. By n_k we denote the cardinality of N_k. It is obvious that $\Sigma_{k=1}^m n_k = n$. For any $u \in \mathbb{R}^n$ and all $k \in \overline{1,m}$, let u_{N_k} be a subvector of u that belongs to $\mathbb{R}^{n_k}$ and that is composed of components u_i with $i \in N_k$. Suppose $\mathbb{R}^{N_k}$ is a coordinate subspace of $\mathbb{R}^n$ induced by coordinates with indices from N_k, i.e.

$$\mathbb{R}^{N_k} = \{v \in \mathbb{R}^n | v_i = 0, i \notin N_k\}.$$

For all $u \in \mathbb{R}^n$ and $k \in \overline{1,m}$, let

$$\mathbb{R}^{N_k}(u) = \{u' \in \mathbb{R}^n | u'_{N \setminus N_k} = u_{N \setminus N_k}\}$$

be a hyperplane of dimension n_k parallel to coordinate subspace $\mathbb{R}^{N_k}$. Obviously, $\mathbb{R}^{N_k} = \mathbb{R}^{N_k}(\mathbf{0})$ and $\mathbb{R}^{N_k}(u) = u + \mathbb{R}^{N_k}$.

Denote by

$$D_{N_k} = \{u \in \mathbb{R}^{N_k} | u_i = u_j,\ i,j \in N_k,\quad \&\quad u_i = 0,\ i \notin N_k\}$$

a diagonal of a coordinate subspace $\mathbb{R}^{N_k}$, and suppose L^D is a subspace of $\mathbb{R}^n$ spanned by the diagonals D_{N_k}, $k \in \overline{1,m}$. It is easy to see that every $u \in L^D$ has the form $u = \{u_i\}_{i \in N}$ with $u_i = c_k(u)$, where $c_k(u)$ is a real constant depending on u and k, and where k is defined by the relation $i \in N_k$.

For every vector $u \in \mathbb{R}^n$ and for any real c, denote by $u \| c_{N_k}$ a vector in $\mathbb{R}^n$ with components

$$(u \| c_{N_k})_i = \begin{cases} u_i, & i \in N \setminus N_k; \\ c, & i \in N_k. \end{cases}$$

We can see that $u \| c_{N_k}$ is an orthogonal projection of u on a hyperplane $\mathbb{R}^{N \setminus N_k}(c_{N_k})$. For any real c, let $(c_{N_k}, \mathbf{0}_{N \setminus N_k})$ denote the vector in $\mathbb{R}^n$ with components

$$(c_{N_k}, \mathbf{0}_{N \setminus N_k})_i = \begin{cases} c, & i \in N_k; \\ 0, & i \in N \setminus N_k. \end{cases}$$

We denote an orthogonal projection of the level surface $\gamma(c)$ to the hyperplane $\mathbb{R}^{N_k}(c_N)$ by $\gamma_{N_k}(c)$. For any two points $u, u' \in \mathbb{R}^n$, $u \neq u'$, let $l(u, u')$, $r[u, u')$ and $r(u, u')$ be, respectively, a straight line passing through these two points, a ray starting from u in a direction to u' and a ray $r[u, u')$ without its origin u.

As usual we designate a mean value of a vector $u \in \mathbb{R}^n$ by $\bar{u}$, i.e. $\bar{u} = (\sum_{i=1}^n u_i)/n$. A nonnegative orthant in $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$, i.e.,

$$\mathbb{R}^n_+ = \{u \in \mathbb{R}^n | u_i \geq 0,\ i \in N, \quad \& \quad u \neq \mathbf{0}\}.$$

In what follows we use also the definition of a fan from [2]. For a vector $u \in \mathbb{R}^n$, the *fan generated by* u is

$$Y(u) = \{u \in \mathbb{R}^n | u = \theta 1_n + \lambda u, \theta \in \mathbb{R}, \lambda \in \mathbb{R}_+\}.$$

A subset Y of $\mathbb{R}^n$ is a *fan* if it is a fan generated by some $u \in \mathbb{R}^n$.

If it is desirable to stress what class of invariant transforms is admissible for an SWF, we use the designation SWF(Φ), and an SWF(Φ) characterization we call an Φ-characterization.

3 Independent subroup scales

Clearly, every continuous nondecreasing n-dimensional function that is only determined by variables with indices from one of the subgroups N_k and that is invariant under invariance transforms proper to this subgroup of variables is an SWF. Now we examine situations with mutually independent subgroup scales for which such a form of an SWF is the only possible one, or equivalently, for which a dictatorial subgroup, or a decisive coalition equal to one of the subgroups of individuals, must exist. The social ordering is then determined in accordance with the scale type proper to this subgroup.

The subgroup scales Φ_{N_k}, $k = 1, \ldots, m$, are *mutually independent* if, for any distinct $k_1, k_2 \in \{1, \ldots, m\}$, for every $i \in N_{k_1}$, and for every $j \in N_{k_2}$, there exist $\phi_i \in \Phi_{N_{k_1}}$ and $\phi_j \in \Phi_{N_{k_2}}$ such that $\phi_i(t) = \alpha_i + \beta_i t$ with $\beta_i > 0$ and $\phi_j(t) = \alpha_j + \beta_j t$ with $\beta_j > 0$, where $\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$. Note that because OM and CM include the positive affine transforms, these classes are covered by this definition as well. It should also be stressed that Mutual Independence is a property of the set of subgroup classes of transforms $\{\Phi_{N_1}, \ldots, \Phi_{N_m}\}$, not of individual transforms within these classes. Mutual Independence preserves the group property and therefore guarantees that Φ_N is a group if each of the Φ_{N_k} is a group.

The assumption of non-comparability (mutual independence) across subgroups imposes restrictions on what comparability is possible within subgroups; hence not every combination of subgroup scales is admissible under this hypothesis. There may be a combination of independent CUC and CFC or CFC and RSM scales but no combination of independent RSM and RSM scales since all transforms presenting both of the last scales have the same constant term equal to zero. Each combination of OM or CM scales with any one from the listed above provides an example of independent subgroup scales. However, in any admissible set of more than two independent subgroup scales there can be at most one RSM (or RSF): RSM for a number of subgroups is equivalent to RSM for the merged subgroup, and therefore a combination of RSM scales constitutes exactly a combination of dependent subgroup scales. The same is true for combinations with TSM (or TSF) scales: under the condition of mutual independence there cannot be more than one TSM (or TSF) subgroup scales.

Theorem 1. *Let $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$, and let a continuous real-valued function W defined on $\mathbb{R}^n$, nondecreasing with a nondecrease in all arguments, be invariant with respect to the variables indexed by N_1 under OM or CM transforms and with respect to the rest of the variables indexed by N_2 under any one of the invariance transform classes listed above. Then either*

$$W(u) = u_i \quad \textit{for some} \quad i \in N_1,$$

or $W(u)$ is independent of all variables u_i, $i \in N_1$, i.e.,

$$W(u) = W(u_{N_2}),$$

and so, function W is fully characterized by the invariance class admissible for the variables indexed by N_2.

It follows directly from Theorem 1

Corollary. *Let W be a continuous real-valued function that is strictly nondecreasing with a nondecrease in all arguments, and assume W is invariant with respect to variables indexed by one subset of indices under OM transforms and with respect to the rest variables under CM transforms. Then*

$$W(u) = u_i \quad \textit{for some} \quad i \in N,$$

and therefore for such a society a dictator must exist.

Proof of Theorem 1. Let $u' \in \gamma(W(u))$ and $u' \neq u$. Then there exists $i_0 \in N$ such that $u'_{i_0} \neq u_{i_0}$. Assume $i_0 \in N_1$ and $u'_{i_0} \neq 0$. Consider an admissible transform $\phi = \{\phi_i\}_{i \in N}$,

$$\phi_i(t) = \begin{cases} (1-\alpha)u_{i_0} + \alpha t, & \alpha > 0, \quad i = i_0, \\ t, & \qquad\quad\;\; i \neq i_0. \end{cases}$$

Due to (2) for all $\alpha > 0$, $W(\phi u') = W(\phi u)$. But for all $\alpha > 0$, $\phi u = u$, and so for all $\alpha > 0$, $\phi u' \in \gamma(W(u))$. If $\alpha = 1$ then $\phi u' = u'$. Therefore, the ray starting from the point $\hat{u} = \begin{cases} u_{i_0}, \; i = i_0 \\ u'_i, \; i \neq i_0 \end{cases}$ and proceeding parallel to i_0-axis passes through the point u' and belongs to $\gamma(W(u))$. Using another admissible transform $\tilde{\phi} = \{\tilde{\phi}_i\}_{i \in N}$,

$$\tilde{\phi}_i(t) = \begin{cases} (1-\alpha)u'_{i_0} + \alpha t, & \alpha > 0, \quad i = i_0, \\ t, & \qquad\quad\;\; i \neq i_0, \end{cases}$$

we can check that the ray starting from u' and running parallel to i_0-axis contains the point $\hat{u}$ and belongs to $\gamma(W(u))$. Hence, a whole straight line parallel to i_0-axis and containing u' lies completely in the level surface $\gamma(W(u))$. The case $u' \in N(W(u))$, $u'_{i_0} \neq u_{i_0}$, when $i_0 \in N_1$ and $u'_{i_0} = 0$, due to the equality $\gamma(W(u)) = \gamma(W(u'))$, may be considered similarly by replacing u by u'.

Thus, if for any $i_0 \in N_1$ on some fixed level surface γ of W there exist $u', u'' \in \gamma$, such that $u'_{i_0} \neq u''_{i_0}$, then for every $u \in \gamma$ the whole straight line containing u and parallel to i_0-axis belongs to γ. For any function W invariant under TSF transforms admissible within the framework of the theorem, with the exception of the case of RSM and RSF scales, every level surface can be obtained from any other one by a shift along the diagonal of $\mathbb{R}^n$. In case of RSF transforms, W is a homothetic function, and under homothetic transforms any straight line transforms into a parallel one. Hence, if on any level surface of function $W(u)$) there exist two points with different i_0-components for some $i_0 \in N_1$ then the function $W(u)$ is independent of i_0-variable. It follows immediately that either for all $u' \in \gamma(W(u))$ for some fixed $i_0 \in N_1$ $u'_{i_0} = u_{i_0} = \text{Const}$, or $W(u)$ is independent of all variables u_i, $i \in N_1$. This completes the proof. ∎

Theorem 2. *Let $N = N_1 \cup N_2 \cup \cdots \cup N_m$ and $N_i \cap N_j = \emptyset$ for $i \neq j$, and let a continuous real-valued function W defined on $\mathbb{R}^n$, nondecreasing with a nondecrease in all arguments, be invariant with respect to variables indexed by N_k under any one of invariant transform classes OM, OFM, CM, CUC and CFC. Invariant transforms proper to distinct subgroups of indices N_k are assumed to be mutually independent. Then there exists a unique integer $k \in \overline{1,m}$, such that the function W for all $u \in \mathbb{R}^n$ has the form*

$$W(u) = W(u_{N_k}),$$

i.e., W is determined only by variables indexed by N_k, and so is fully characterized by the invariance class admissible to this subset of variables.[4]

Remark. Note that every CFC transform is at the same time a transform of any one of the OM, OFM, CM and CUC invariant classes. Therefore, it is possible to simplify the statement of Theorem 2 by requiring only that the function $W(u)$ be invariant with respect to variables indexed by N_k for all $k \in \overline{1,m}$ under mutually independent CFC transforms.

Theorem 2 allows us to construct an SWF characterization for various combinations of OM, OFM, CM, CUC and CFC independent subgroup utility scales on the basis of well-known results for social choice problems with the same measurement scales of individual utilities for the entire society.

Theorem 2 has close affinities to a result of Roberts [12] where different people form different interpersonal comparisons but these different views are not

[4] A particular case for combinations of independent among disjoint subgroups CUC scales was investigated by Plata-Pérez [9]. This theorem with proof omitted was stated in Khmelnitskaya [8].

comparable. Dictatorship by one view was still obtained. The situation considered in [12] amounts in our framework to a combination of mutually independent CFC scales for each subgroup being the same size where the interpersonal comparisons within a subgroup are equivalent to one person's view. However, the two results cannot be nested. Theorem 2, unlike Roberts, works on an unrestricted domain (which makes it easier to get a negative result) but it shows that the characterization extends to a much wider class of situations than that covered by [12].

In terms of level surfaces, the statement of Theorem 2 means that for any function $W(u)$, there exists a unique $k \in \overline{1,m}$ such that every level surface $\gamma(c)$ is parallel to the coordinate subspace $\mathbb{R}^{N\setminus N_k}$. This says exactly that for all $u \in \mathbb{R}^n$, $\mathbb{R}^{N\setminus N_k}(u) \subset \gamma(W(u))$. It is easy to see that in order to prove the last inclusion, it is sufficient to show that all sections $\gamma(W(u)) \cap \mathbb{R}^{N_k}(u)$, $k \in \overline{1,m}$, except one are hyperplanes of dimension n_k. For different combinations of mutually independent OM, CM and CUC scales, the result may be easily obtained based on the admissibility of the transform $\phi = \{\phi_i\}_{i \in N}$:

$$\phi_i(t) = \begin{cases} t, & i \in N_k; \\ (1-\alpha)a_i + \alpha t, \quad \alpha > 0, & i \in N\setminus N_k. \end{cases}$$

For all combinations of OM, CM and CUC scales, for all $k \in \overline{1,m}$, every section $\gamma(W(u)) \cap \mathbb{R}^{N_k}(u)$ together with any two points contains the whole straight line passing through these points, and therefore has to be a hyperplane. So, for this case the proof of Theorem 1 is rather simple. However, if we append OFC and CFC scales, then the upper defined transform ϕ is inadmissible for all combinations of scales, and not all sections $\gamma(W(u)) \cap \mathbb{R}^{N_k}(u)$ are hyperplanes.

The proof of Theorem 2 is based on the following plan. First we show that every level surface $\gamma(c)$ contains its own orthogonal projection on each hyperplane $\mathbb{R}^{N_k}(c_N)$, $k \in \overline{1,m}$, which in turn is the same as the section $\gamma(c) \cap \mathbb{R}^{N_k}(c_N)$ (Lemma 1). Next, we derive a necessary and sufficient condition in terms of this projection for a function $W(u)$ to be fully determined only by variables indexed by some fixed subgroup N_k (Lemma 2). And finally, we prove that this condition holds under the hypothesis of the theorem (Lemma 3 and Lemma 4).

Until the end of the section we work within the framework of Theorem 2.

Lemma 1. *Any level surface $\gamma(c)$ for all $k \in \overline{1,m}$ contains its own orthogonal projection on the hyperplane $\mathbb{R}^{N_k}(c_N)$, i.e.,*

$$\gamma_{N_k}(c) \subset \gamma(c), \tag{3}$$

moreover, either $\dim \gamma_{N_k}(c) = n_k$ *or* $\dim \gamma_{N_k}(c) = n_k - 1$ *and*

$$\gamma_{N_k}(c) = \gamma(c) \cap \mathbb{R}^{N_k}(c_N). \tag{4}$$

Proof of Lemma 1. Fix some $k \in \overline{1,m}$. To prove (3) it suffices to show that for every $u \in \gamma(c)$, $u \| c_{N \setminus N_k} \in \gamma(c)$. Take $u \in \gamma(c)$. If $u \in \mathbb{R}^{N_k}(c_N)$, then $u \| c_{N \setminus N_k} = u$ and obviously, $u \| c_{N \setminus N_k} \in \gamma(c)$. Assume $u \notin \mathbb{R}^{N_k}(c_N)$. In accordance with Remark in Section 2, $c_N \in \gamma(c)$. Take an admissible transform $\phi = \{\phi_i\}_{i \in N}$:

$$\phi_i(t) = \begin{cases} t, & i \in N_k; \\ (1-\alpha)c + \alpha t, \quad \alpha > 0, & i \in N \setminus N_k. \end{cases}$$

By (2) for all $\alpha > 0$, $W(\phi u) = W(\phi c_N)$. But for any $\alpha > 0$, $\phi c_N = c_N \in \gamma(c)$. Hence, for all $\alpha > 0$, $\phi u \in \gamma(c)$, and moreover, since $u \notin \mathbb{R}^{N_k}(c_N)$, ϕu corresponding to different α are different. If $\alpha = 1$, $\phi u = u$, whence $r(u \| c_{N \setminus N_k}, u) \subset \gamma(c)$. Therefore, every neighborhood of a point $u \| c_{N \setminus N_k}$ has a nonempty intersection with $\gamma(c)$. Then by continuity of $W(u)$, $u \| c_{N \setminus N_k} \in \gamma(c)$. The function $W(u)$ is defined on $\mathbb{R}^n$ and so we may consider $W(u \| c_{N \setminus N_k})$. Let $W(u \| c_{N \setminus N_k}) = a \neq c$. Because of continuity of W, a neighborhood S of $u \| c_{N \setminus N_k}$ exists such that for any $u' \in S$, $|W(u') - a| < |c - a|/2$. Hence, $|W(u') - c| > |c - a|/2$, i.e., for all $u' \in S$, $W(u') \neq c$. The contradiction obtained proves (3).

The validity of

$$\gamma_{N_k}(c) \subset \mathbb{R}^{N_k}(c_N) \tag{5}$$

and

$$\gamma(c) \subset \gamma_{N_k}(c) + \mathbb{R}^{N \setminus N_k}$$

comes directly from the definition of orthogonal projection. It follows immediately that

$$\dim \gamma_{N_k}(c) \leq n_k$$

and

$$\dim \gamma(c) \leq \dim \gamma_{N_k}(c) + (n - n_k).$$

Combining these inequalities and taking into account the equality $\dim \gamma(c) = n - 1$, we arrive at

$$n_k - 1 \leq \dim \gamma_{N_k}(c) \leq n_k.$$

From the definition of orthogonal projection it also follows that

$$\gamma(c) \cap \mathbb{R}^{N_k}(c_N) \subset \gamma_{N_k}(c).$$

Then, (4) immediately comes from (3) because of (5) and the last inclusion. ∎

Remark 1. Lemma 1 remains true under a coarser partition of N into disjoint subgroups when a few subgroups N_k, $k \in \overline{1,m}$, may unite into one. It is worth noting that this statement is valid for all next propositions of the section.

Remark 2. Due to the admissibility of the transform $\{\phi_i(t) = \alpha + t\}_{i \in N}$ for any real α, all level surfaces $\gamma(c)$ corresponding to different c can be obtained from each other by parallel shifts along the diagonal D_n. (This property was mentioned earlier in [11]). From this and (4), it follows that for all real c and c',

$$\gamma_{N_k}(c') = \gamma_{N_k}(c) + (c' - c)_N, \tag{6}$$

to wit, all projections $\gamma_{N_k}(c)$ relevant to the same k and different c can be obtained from each other by parallel shifts along D_n.

Remark 3. Note that $\gamma_{N_k}(c)$ is a cone in $\mathbb{R}^{N_k}(c_N)$ with a top in c_N. Indeed, if $u' \in \gamma_{N_k}(c)$ and $u' \neq c_N$, then there exists $u \in \gamma(c)$, $u \neq c_N$, such that $u' = u \| c_{N \setminus N_k}$. Since $c_N \in \gamma(c)$ and because of the admissibility of the transform $\{\psi_i(t) = (1-\alpha)c + \alpha t\}_{i \in N}$ for all $\alpha > 0$, $r[c_N, u) \subset \gamma(c)$. But the ray $r[c_N, u')$ is a projection of the ray $r[c_N, u)$ onto the hyperplane $\mathbb{R}^{N_k}(c_N)$. Thus, for every $u' \in \gamma_{N_k}(c)$ such that $u' \neq c_N$, $r[c_N, u') \subset \gamma_{N_k}(c)$, which proves that $\gamma_{N_k}(c)$ is a cone. In particular, a cone $\gamma_{N_k}(c)$ with $\dim \gamma_{N_k}(c) = n_k$ may coincide with $\mathbb{R}^{N_k}(c_N)$. If $\dim \gamma_{N_k}(c) = n_k - 1$, it may be a hyperplane in $\mathbb{R}^{N_k}(c_N)$ which passes through c_N.

Denote by $H_{N_k}(c)$ the cylinder $\gamma_{N_k}(c) + \mathbb{R}^{N \setminus N_k}$.

Remark 4. As it was already noted in the proof of Lemma 1, for any real c and all $k \in \overline{1, m}$,

$$\gamma(c) \subset H_{N_k}(c). \tag{7}$$

Lemma 2. *A function W for any $u \in \mathbb{R}^n$ has the form*

$$W(u) = W(u_{N_k}) \qquad \text{for some} \qquad k \in \overline{1, m},$$

i.e. depends only on the variables u_i with indices $i \in N_k$, if and only if there is a real c for which $\dim \gamma_{N_k}(c) = n_k - 1$.

Proof of Lemma 2. First we prove the necessity. Clearly, for every real c

$$\gamma(c) \cap \mathbb{R}^{N_k}(c_N) = \{u \in \mathbb{R}^{N_k}(c_N) | W(u) = c\}.$$

By the hypothesis, for all $u \in \mathbb{R}^n$, and in particular for all $u \in \mathbb{R}^{N_k}(c_N)$, $W(u) = W(u_{N_k})$. But the variables u_{N_k} are intrinsic coordinates of the hyperplane $\mathbb{R}^{N_k}(c_N)$. Therefore and because of (4), the projection $\gamma_{N_k}(c)$, being a subset of the n_k-dimensional hyperplane $\mathbb{R}^{N_k}(c_N)$, is characterized by the unique equality $W(u_{N_k}) = c$ in the intrinsic coordinates of $\mathbb{R}^{N_k}(c_N)$, whence it follows that for every c, $\dim \gamma_{N_k}(c) = n_k - 1$.

Turning to the proof of sufficiency, from (6) for all real c and c',

$$H_{N_k}(c') \cap \mathbb{R}^{N_k}(c_N) = \gamma_{N_k}(c) + ((c' - c)_{N_k}, \mathbf{0}_{N \setminus N_k}),$$

i.e., for every c and different c', all sections $H_{N_k}(c') \cap \mathbb{R}^{N_k}(c_N)$ are obtained from $\gamma_{N_k}(c)$ by parallel shifts along the diagonal $D_{N_k}(c)$ of the hyperplane $\mathbb{R}^{N_k}(c_N)$,

$$D_{N_k}(c) = \{u \in \mathbb{R}^n | u_i = u_j,\ i, j \in N_k \ \ \& \ \ u_i = c,\ i \notin N_k\},$$

Next, if we show that for any $k \in \overline{1, m}$ such that $\dim \gamma_{N_k}(c) = n_k - 1$, all parallel shifts of $\gamma_{N_k}(c)$ along $D_{N_k}(c)$ in $\mathbb{R}^{N_k}(c_N)$ do not mutually intersect and cover the whole $\mathbb{R}^{N_k}(c_N)$, then it will follow that cylinders $H_{N_k}(c)$ for different c do not intersect and cover $\mathbb{R}^n$. On the other hand, since a function W is defined on $\mathbb{R}^n$ for every $u \in \mathbb{R}^n$, then there is a level surface of W that contains any

u. Hence, because of (7) for every real c, $\gamma(c) = H_{N_k}(c)$, which is the same as for all $u \in \mathbb{R}^n$, $W(u) = W(u_{N_k})$. Thus, to complete the proof of sufficiency, it is enough to show that for every $k \in \overline{1,m}$ for which $\dim \gamma_{N_k}(c) = n_k - 1$, the parallel shifts of $\gamma_{N_k}(c)$ along $D_{N_k}(c)$ in $\mathbb{R}^{N_k}(c_N)$ do not mutually intersect and cover $\mathbb{R}^{N_k}(c_N)$.

First, we show that for every $k \in \overline{1,m}$, the parallel shifts of $\gamma_{N_k}(c)$ cover $\mathbb{R}^{N_k}(c_N)$. Because a certain level surface $\gamma(W(u))$ passes through any $u \in \mathbb{R}^n$, every $\gamma(W(u))$ is a cone with a top in $\{W(u)\}_N \in D_n$ and all level surfaces may be obtained from each other by parallel shifts along D_n. Therefore, through every point in any two-dimensional half-plane with a boundary D_n, denoted in the sequel by $\mathbb{R}^2_\pm(D_n)$, a ray starts with a top in some $c_N \in D_n$ and completely belonging to $\gamma(c)$ there passes. Moreover, since different level surfaces do not intersect, from every point $c_N \in D_n$ in any half-plane $\mathbb{R}^2_\pm(D_n)$ a unique ray emanates that belongs to $\gamma(c)$ and that does not intersect the other level surfaces $\gamma(c')$, $c' \neq c$.[5] Parallel rays starting from different $c_N \in D_n$ and belonging to a half-plane $\mathbb{R}^2_\pm(D_n)$ cover the entire $\mathbb{R}^2_\pm(D_n)$. Hence, in any two-dimensional plane $\mathbb{R}^2(D_n)$ passing through D_n, for every $c_N \in D_n$ there are exactly two rays starting from c_N. Furthermore, they are located in distinct half-planes of $\mathbb{R}^2(D_n)$ separated by D_n, i.e., in $\mathbb{R}^2_+(D_n)$ and $\mathbb{R}^2_-(D_n)$ respectively. A collection of mutually non-intersecting pairs of rays relevant to different level surfaces $\gamma(c)$ covers $\mathbb{R}^2(D_n)$ (in particular, rays r_1 and r_2 may form a straight line which intersects D_n in c_N). A set of all two-dimensional planes $\mathbb{R}^2(D_{N_k}(c)) \subset \mathbb{R}^{N_k}(c_N)$ containing the diagonal $D_{N_k}(c)$ of the hyperplane $\mathbb{R}^{N_k}(c_N)$ cover $\mathbb{R}^{N_k}(c_N)$ (each point $u \in \mathbb{R}^{N_k}(c_N)\backslash D_{N_k}(c)$ and a straight line $D_{N_k}(c)$ unambiguously determine a two-dimensional plane). A plane $\mathbb{R}^2(D_{N_k}(c))$ may be considered as a projection of a cylinder $\mathbb{R}^2(D_{N_k}(c)) + \mathbb{R}^{N\backslash N_k}$ on $\mathbb{R}^{N_k}(c_N)$. Since $D_n \subset \mathbb{R}^2(D_{N_k}(c)) + \mathbb{R}^{N\backslash N_k}$, every cylinder $\mathbb{R}^2(D_{N_k}(c)) + \mathbb{R}^{N\backslash N_k}$ is covered by a set of all two-dimensional planes $\mathbb{R}^2(D_n) \subset \mathbb{R}^2(D_{N_k}(c)) + \mathbb{R}^{N\backslash N_k}$. Note that $D_n \| c_{N\backslash N_k} = D_{N_k}(c)$. Therefore, for each plane $\mathbb{R}^2(D_n) \subset \mathbb{R}^2(D_{N_k}(c)) + \mathbb{R}^{N\backslash N_k}$ that is not orthogonal to $\mathbb{R}^2(D_{N_k}(c))$, a projection of $\gamma(c) \cap \mathbb{R}^2(D_n)$ on $\mathbb{R}^{N_k}(c_N)$ consists of exactly two rays $\tilde{r}_1, \tilde{r}_2 \subset \gamma_{N_k}(c)$ starting from $c_N \in D_{N_k}(c)$ and belonging to the different half-planes $\mathbb{R}^2_+(D_{N_k}(c))$ and $\mathbb{R}^2_-(D_{N_k}(c))$ of the plane $\mathbb{R}^2(D_{N_k}(c))$ and that are separated by the diagonal $D_{N_k}(c)$: $\tilde{r}_1 \subset \mathbb{R}^2_+(D_{N_k}(c))$, $\tilde{r}_2 \subset \mathbb{R}^2_-(D_{N_k}(c))$. (Note that any plane orthogonal to $\mathbb{R}^2(D_{N_k}(c))$ maps completely on $D_{N_k}(c)$). It is also worth noting here, that under parallel shifts along $D_{N_k}(c)$, rays $\tilde{r}_1$ and $\tilde{r}_2$ cover the entire plane $\mathbb{R}^2(D_{N_k}(c))$, whence the collection of all shifts of $\gamma_{N_k}(c)$ along $D_{N_k}(c)$ covers the hyperplane $\mathbb{R}^{N_k}(c_N)$.

Now we show that for any $k \in \overline{1,m}$ for which $\dim \gamma_{N_k}(c) = n_k - 1$, parallel shifts of $\gamma_{N_k}(c)$ along $D_{N_k}(c)$ in $\mathbb{R}^{N_k}(c_N)$ do not intersect. For this, it suffices to show that every half-plane $\mathbb{R}^2_\pm(D_{N_k}(c))$ contains a unique ray from

[5] A point $c_N \in D_n$ completely determines a level surface $\gamma(c)$ and so, even if from any one $c_N \in D_n$ in any one half-plane $\mathbb{R}^2_\pm(D_n)$ two rays $r_1, r_2 \subset \gamma(c)$ started, then since all level surfaces are obtained from each other by parallel shifts along D_n, it would follow that different level surfaces intersect, which is impossible.

the projection $\gamma_{N_k}(c)$. Suppose the opposite, and let at least two rays $r_1, r_2 \subset \gamma_{N_k}(c) \cap \mathbb{R}^2_{\pm}(D_{N_k}(c))$. Then because of continuity of the level surface $\gamma(c)$ and continuity of a projection mapping $\mathrm{Pr}: \mathbb{R}^n \to \mathbb{R}^{N_k}(c_N)$, the part of a half-plane $\mathbb{R}^2_{\pm}(D_{N_k}(c))$ in between rays r_1 and r_2 also belongs to $\gamma_{N_k}(c)$. But this is impossible since by hypothesis, $\dim \gamma_{N_k}(c) = n_k - 1$. ∎

Remark. The necessary and sufficient condition in Lemma 2 may be restated equivalently in terms of cylinders $H_{N_k}(c)$. Indeed, the equality

$$\dim \gamma_{N_k}(c) = n_k - 1$$

for some $k \in \overline{1, m}$ is equivalent to the equality

$$\gamma(c) = H_{N_k}(c). \tag{8}$$

for the same k.

Lemma 3. *For every level surface $\gamma(c)$, if for some $k \in \overline{1, m}$*

1) $\dim \gamma_{N_k}(c) = n_k - 1$, *then* $\gamma_{N \setminus N_k}(c) = \mathbb{R}^{N \setminus N_k}(c_N)$ *and for all* $k' \in \overline{1, m}$, $k' \neq k$, $\gamma_{N_{k'}}(c) = \mathbb{R}^{N_{k'}}(c_N)$;

2) $\dim \gamma_{N_k}(c) = n_k$, *then* $\gamma_{N \setminus N_k}(c) \neq \mathbb{R}^{N \setminus N_k}(c_N)$,
and furthermore, if $\gamma_{N_k}(c) = \mathbb{R}^{N_k}(c_N)$, *then* $\dim \gamma_{N \setminus N_k}(c) = n - n_k - 1$,
while if $\gamma_{N_k}(c) \neq \mathbb{R}^{N_k}(c_N)$, *then* $\dim \gamma_{N \setminus N_k}(c) = n - n_k$.

Proof of Lemma 3. To prove the first statement, let $\dim \gamma_{N_k}(c) = n_k - 1$ for some $k \in \overline{1, m}$. Then by (4) and (8), for all $k \in \overline{1, m}$

$$\gamma_{N \setminus N_k}(c) = \gamma(c) \cap \mathbb{R}^{N \setminus N_k}(c_N) = H_{N_k}(c) \cap \mathbb{R}^{N \setminus N_k}(c_N) = \mathbb{R}^{N \setminus N_k}(c_N).$$

Similarly we may show that for all $k' \in \overline{1, m}$, $k' \neq k$, $\gamma_{N_{k'}}(c) = \mathbb{R}^{N_{k'}}(c_N)$. Indeed,

$$\gamma_{N_{k'}}(c) = \gamma(c) \cap \mathbb{R}^{N_{k'}}(c_N) = H_{N_k}(c) \cap \mathbb{R}^{N_{k'}}(c_N) = \mathbb{R}^{N_{k'}}(c_N).$$

Thus, the first statement is proved.

Turning to the second one, suppose the opposite, i.e., $\gamma_{N \setminus N_k}(c) = \mathbb{R}^{N \setminus N_k}(c_N)$. Then because of (3), $\mathbb{R}^{N \setminus N_k}(c_N) \subset \gamma(c)$, which is equivalent to $W(u) = W(u_{N_k})$. Therefore, by Lemma 2, $\dim \gamma_{N_k}(c) = n_k - 1$, which contradicts to the hypothesis.

Next, from Lemma 1 and Remark 1 on Lemma 1 it follows that either $\dim \gamma_{N \setminus N_k}(c) = n - n_k$, or $\dim \gamma_{N \setminus N_k}(c) = n - n_k - 1$. If $\dim \gamma_{N \setminus N_k}(c) = n - n_k - 1$, then by the Remark on Lemma 2, $\gamma(c) = H_{N \setminus N_k}(c)$. Clearly,

$$H_{N \setminus N_k}(c) \cap \mathbb{R}^{N_k}(c_N) = \mathbb{R}^{N_k}(c_N),$$

i.e.,

$$\gamma(c) \cap \mathbb{R}^{N_k}(c_N) = \mathbb{R}^{N_k}(c_N),$$

whence by (4), $\gamma_{N_k}(c) = \mathbb{R}^{N_k}(c_N)$.

Further, if we suppose $\gamma_{N_k}(c) = \mathbb{R}^{N_k}(c_N)$ and then repeat the previous arguments, because of (3) and Lemma 2 we arrive at $\dim \gamma_{N \setminus N_k}(c) = n - n_k - 1$. ∎

Remark. From Remark 1 on Lemma 1, we can prove the second statement of the second part of Lemma 3 from the first statement as well.

From the first statement of Lemma 3 we may derive the next corollary by induction with respect to the number m of subgroups N_k in the partition of N.

Corollary 1. *For every level surface $\gamma(c)$ not all projections $\gamma_{N_k}(c)$, $k \in \overline{1,m}$, coincide with the corresponding hyperplanes $\mathbb{R}^{N_k}(c_N)$.*

Lemma 4. *For every level surface $\gamma(c)$ for any $k \in \overline{1,m}$, a projection $\gamma_{N_k}(c)$ either coincides with a hyperplane $\mathbb{R}^{N_k}(c_N)$ or $\dim \gamma_{N_k}(c) = n_k - 1$.*

Proof of Lemma 4. From the first statement of Lemma 3, if $\dim \gamma_{N_k}(c) = n_k$ and $\gamma_{N_k}(c) \neq \mathbb{R}^{N_k}(c_N)$, then $\dim \gamma_{N \backslash N_k}(c) = n - n_k$. It follows that there exist a real $\epsilon > 0$ and two points $u_1 \in \gamma_{N_k}(c)$, $u_2 \in \gamma_{N \backslash N_k}(c)$, such that $u_1^\epsilon = u_1 + (\epsilon_{N_k}, \mathbf{0}_{N \backslash N_k}) \in \gamma_{N_k}(c)$ and $u_2^\epsilon = u_2 + (\mathbf{0}_{N_k}, \epsilon_{N \backslash N_k}) \in \gamma_{N \backslash N_k}(c)$. Moreover, by Lemma 1 and Remark 1 on Lemma 1 , $u_1, u_1^\epsilon, u_2, u_2^\epsilon \in \gamma(c)$. Consider the admissible transform $\phi = \{\phi_i\}_{i \in N}$:

$$\phi_i(t) = \begin{cases} t + \epsilon, & i \in N_k; \\ t, & i \in N \backslash N_k. \end{cases}$$

By (2), $W(\phi u_1) = W(\phi u_2^\epsilon)$. Then note that $\phi u_1 = u_1^\epsilon \in \gamma(c)$. Hence, $\phi u_2^\epsilon \in \gamma(c)$. But $\phi u_2^\epsilon = u_2 + \epsilon_N$, whence, since $u_2 \in \gamma(c)$ and since all level surfaces $\gamma(c')$ for different c' can be obtained from each other by parallel shifts along D_n, $\phi u_2^\epsilon \in \gamma(c + \epsilon)$. But if $\phi u_2^\epsilon \in \gamma(c)$, this is impossible. ∎

Remark on Lemma 3 and Lemma 4. From Remark 2 on Lemma 1 it follows that if for some c and some $k \in \overline{1,m}$, Lemma 3 or Lemma 4 is true, then it still holds for every other $c' \neq c$ under the same k.

Proof of Theorem 2. Due to Corollary to Lemma 3 and Lemma 4, for some $k \in \overline{1,m}$, $\dim \gamma_{N_k}(c) = n_k - 1$. Moreover, by the first statement of Lemma 3, this k is unique one. From this and Lemma 2, Theorem 2 follows directly. ∎

4 Interrelations among subgroup scales

Now we consider some other combinations of subgroup scales that do not give rise to the dictatorial subgroup characterization for an SWF. We start with the following general remark.

Remark. Suppose we examine a social choice problem for the society N in the informational environment prescribed by invariance class Φ (which in particular may be a certain combination of subgroup scales). Assume also that there are two other scales Φ' and Φ'' for which we know SWF characterizations. Suppose moreover, that $\Phi' \subset \Phi \subset \Phi''$, and assume there exists a subset $L^\Phi \subset \mathbb{R}^n$, $\dim L^\Phi = m$, such that for any $W \in \text{SWF}(\Phi)$ the restriction $W|_{L^\Phi}$ to L^Φ can be identified with a real-valued function W^* on $\mathbb{R}^m$ that admits invariance transforms from the class Φ''. Then we can state that an SWF(Φ) possesses a Φ'-characterization that narrows to a Φ''-characterization on L^Φ. Furthermore, if Φ is a certain combination of subgroup scales, i.e., $N = N_1 \cup N_2 \cup \cdots \cup N_m$ with $N_i \cap N_j = \emptyset$ for $i \neq j$ and $\Phi = \Phi_N = \{\Phi_{N_k}\}_{k=1}^m$, then it is not difficult to see that for any SWF(Φ_N) its Φ'-characterization, for every $k \in \overline{1,m}$ and any fixed $\hat{u}_{N\setminus N_k} \in \mathbb{R}^{N\setminus N_k}$ considered as a function on $\mathbb{R}^{n_k}$, must correspond to Φ_{N_k}-characterization.

The last observation allows us to obtain SWF characterizations for various combinations of interval scales with common for the entire society unit and for different combinations of RSF scales.

I. Interval scales with common unit.

Let Φ be a combination of CFC subgroup scales, possibly different for different k but with the same unit for all k. Then CFC $\subset \Phi \subset$ CUC and for any SWF(Φ) W, its restriction $W|_{L^D}$ to L^D can be identified with a real-valued function W^* defined on $\mathbb{R}^m$ that is an SWF(CUC). Therefore, utilizing the SWF characterization under CUC and CFC invariance [11], [2], we may state:

Theorem 3. *Let $N = N_1 \cup N_2 \cup \cdots \cup N_m$ and $N_i \cap N_j = \emptyset$ for $i \neq j$, and let W be a continuous real-valued function defined on $\mathbb{R}^n$, nondecreasing with a nondecrease in all arguments, and assume that for all $k \in \overline{1,m}$, W is invariant with respect to variables indexed by N_k under CFC transforms that may be different for different k but that have the same common unit. Then W has the form*

$$W(u) = \bar{u} + \theta(u - \bar{u})$$

for some $\theta\colon \mathbb{R}^n \to \mathbb{R}^1$ continuous and homogeneous of the first degree function, such that

1) on any fan Y in $\mathbb{R}^n$ and on a diagonal subspace L^D, the function W is linear with semipositive coefficients which in general depend on Y and L^D respectively;

2) for every $k \in \overline{1,m}$, on any fan Y in $\mathbb{R}^{n_k}$, the function $W(u_{N_k}, \hat{u}_{N\setminus N_k})$ considered for any fixed $\hat{u}_{N\setminus N_k} \in \mathbb{R}^{N\setminus N_k}$ as a function on $\mathbb{R}^{n_k}$, is linear with semipositive coefficients depending in general on Y.

Theorem 3 is tantamount to every level surface of W flattening on each fan Y in $\mathbb{R}^n$, on each cylinder $Y' + u_{N\setminus N_k}$ where Y' is a fan in $\mathbb{R}^{N_k}$ and $u_{N\setminus N_k} \in \mathbb{R}^{N\setminus N_k}$, and on a diagonal subspace L^D. In particular, the latter expresses the fact that under the stated conditions, an SWF is always a weighted utilitarian SWF if for all $k \in \overline{1,m}$, all individual utilities have the same value within every subgroup of individuals N_k.

It should be mentioned that if all subgroups N_k, $k \in \overline{1,m}$, are singletons, the hypothesis of Theorem 3 is equivalent to the condition of CUC-invariance for all $i \in N$. Also, combinations of different CUC scales with a common unit do not require separate study, since this amounts to supposing CUC-invariance for the entire society.

For combinations of CUC and CFC scales, without loss of generality it suffices to consider only the situation $N = N_1(\text{CUC}) \cup N_2(\text{CFC})$, which in turn is equivalent to the consideration of the combination of $n_1 + 1$ CFC scales with n_1 singleton subgroups.

Remark. As it was noted by Roberts [11], for the characterization of weighted utilitarianism, TSM invariance is enough. Thus, the conclusion of Theorem 3 concerning the linear form of the function W on a diagonal subspace L^D still holds if we consider a combination of TSF subgroup scales instead of that of CFC ones.

It is worth mentioning that if we consider the full continuum of different combinations of scales for the entire society between CM and CFC for all individuals, at "point" CM we have Sen's dictatorial result [14], between "points" CM and CUC there exists a dictatorial subgroup with a utilitarian rule (the result of Plata-Pérez [9] which is a particular case of Theorem 2 for various combinations of independent among distinct subgroups CUC scales), at "point" CUC we arrive at a utilitarian rule for the entire society, between CUC and CFC we arrive at Theorem 3, and at CFC we have the result of Roberts [11] and Yanovskaya [16].

II. RSF subgroup scales.

Similarly, applying the results of Kaneko and Nakamura [7] and of Tsui and Weymark [15] for RSM-characterization, we can show that, for arbitrary combination of RSF subgroup scales, for each relevant SWF W defined on $\mathbb{R}^n_{++}$ (resp. $\mathbb{R}^n_+$), its restriction $W|_{L^D}$ to L^D_{++} (resp. L^D_+)[6] may be identified with a Cobb-Douglas function on $\mathbb{R}^m_{++}$ (resp. $\mathbb{R}^m_+$). The last in turn combined with the SWF characterization under RSF invariance [11] implies

Theorem 4. *Let $N = N_1 \cup N_2 \cup \dots \cup N_m$ and $N_i \cap N_j = \emptyset$ for $i \neq j$, and let a continuous real-valued function W defined on $\mathbb{R}^n_{++}$ (resp. $\mathbb{R}^n_+$), nondecreasing with a nondecrease in all arguments, for all $k \in \overline{1,m}$ be invariant with respect to variables indexed by N_k under RSF transforms defined independently with k different units. Then the function W is a homothetic one on $\mathbb{R}^n$ such that*

1) *for every $k \in \overline{1,m}$ the function $W(u_{N_k}, \hat{u}_{N \setminus N_k})$, considered for any fixed $\hat{u}_{N \setminus N_k} \in \mathbb{R}^{N \setminus N_k}$ as a function on $\mathbb{R}^{n_k}$, is homothetic on $\mathbb{R}^{n_k}$;*

2) *the function W on the subspace L^D_{++} (resp. L^D_+) has the Cobb-Douglas form, i.e., for all $u \in L^D$*

$$W(u) = \prod_{k=1}^{m} [c_k(u)]^{\alpha_k},$$

[6] L^D_{++} (resp. L^D_+) is defined analogously to L^D replacing $\mathbb{R}^n$ by $\mathbb{R}^n_{++}$ (resp. $\mathbb{R}^n_+$).

with $\alpha = \{\alpha_k\}_{k=1}^m \in \mathbb{R}_+^m$.

Roberts [13] obtains an analogous result under an alternative domain assumption.

5 Concluding remarks

To conclude, it is worth noting that social orderings for problems with different utility scales for separate subgroups of individuals may be obtained by aggregating the individual utilities in two ways: either simultaneously over the entire society, or first within subgroups with identical information and then integrating the partially aggregated information. It is clear that in the second case the second stage of the aggregation is equivalent to aggregation with a smaller number of individuals. However, there is no information available about the comparability of utilities for this smaller number of individuals, and so the situation is similar to that captured by the Arrovian conditions. Hence, every social ordering obtained by this way of aggregation is fully determined by the opinions of only one subgroup of individuals and it is in accord with the measurement scales of its members' utilities. We have shown that, for combinations of scales independent among subgroups, all corresponding social orderings obtained by aggregation over the entire society are the same. But full independence across subgroups is not the only case. Thus with respect to various combinations of CUC and CFC scales with a common unit and to combinations of arbitrary RSF subgroup scales, the set of admissible social orderings includes ones that combine utilities of individuals from different subgroups.

References

1. Arrow KJ (1963) Social choice and individual values (2nd ed). Wiley, New York
2. Bossert W, Weymark JA (1997) Utility in social choice. In Barberá S, Hammond PJ, Seidl C (eds) Handbook of utility theory. Kluwer Academic Publishers. Forthcoming.
3. d'Aspremont C (1985) Axioms for social welfare orderings. In Hurwicz L, Schmeidler D, Sonnenschein H (eds) Social goals and social organizations: essays in memory of Elisha Pazner. Cambridge University Press, Cambridge: 19–76
4. d'Aspremont C and Gevers L (1977) Equity and the informational basis of collective choice: Review of Economic Studies 44: 199–209
5. Debreu G (1954) Representation of a preference ordering by a numerical function. In Thrall RM, Coombs CH, Davis RL (eds) Decision processes. Wiley, New York: 159–165
6. Hammond PJ (1979) Equity in two person situations: some consequences: Econometrika 47: 1127–1135
7. Kaneko M, Nakamura K (1979) The Nash social welfare function: Econometrica 47: 423–435
8. Khmelnitskaya AB (1996) Social choice problems with different scales of individual welfares measurement for different subgroups of individuals. In Kleinschmidt P,

Bachem A, Derigs U, Fischer D, Leopold-Wildburger U, Möring R (eds) Operations research proceedings 1995. Springer-Verlag, Berlin: 252–257

9. Plata-Pérez L (1994) Aggregation and Sen's partial comparability. Departamento de Economía, Instituto Technológico Autónomo de México, unpublished manuscript
10. Roberts KWS (1980) Possibility theorems with interpersonally comparable welfare levels: Rev Econ Studies 47: 409–420
11. Roberts KWS (1980) Interpersonal comparability and social choice theory: Rev Econ Studies 47: 421–439
12. Roberts KWS (1995) Valued opinions or opinionated values: the double aggregation problem. In Basu K, Pattanaik P, Suzumura K (eds) Choice welfare and development: essays in honour of Amartya K. Sen. Oxford University Press. Oxford: 141–165
13. Roberts KWS (1997) Objective interpersonal comparisons of utility: Soc Choice Welfare 14: 79–96
14. Sen AK (1970) Collective choice and social welfare. Holden-Day, San Francisco
15. Tsui K-Y and Weymark JA (1997) Social welfare orderings for ratio-scale measurable utilities: Economic Theory 10: 241–256
16. Yanovskaya EB (1988) Social choice functions for different scales of individual preference measurement (in Russian): Annals of VNIISI (All-Union Research Institute for System Studies, Moscow) 6: 64–76
17. Yanovskaya EB (1989) Group choice rules in problems with comparisons of individual preferences: Automation and Remote Control 50: 822–830 (translated from the 1989 Russian original that appeared in Avtomatika i Telemekhanika 6: 129–138)

This article was processed using the LaTeX macro package with LLNCS style