Graph-Restricted Games with Coalition Structures*

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Abstract We consider a new model of a TU game endowed with both coalition and two-level cooperation structures that applies to various network situations. The approach to the value is close to that of both Myerson (1977) and Aumann and Drèze (1974): it is based on ideas of component efficiency and of one or another deletion link property, and it treats an a priori union as a self-contained unit; moreover, our approach incorporates also the idea of the Owen's quotient game property (1977). The axiomatically introduced values possess an explicit formula representation and in many cases can be quite simply computed. The results obtained are applied to the problem of sharing an international river among multiple users without international firms.

Keywords: TU game, coalition structure, cooperation structure, Myerson value, Owen value, Aumann-Drèze value, component efficiency, deletion link property

Mathematics Subject Classification (2000): 91A12, 91A40, 91A43

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1. Introduction

The study of TU games with coalition structures was initiated first by Aumann and Dréze (1974), then Owen (1977). Later this approach was extended in Winter (1989) to games with level structures. Another model of a game with limited cooperation presented by means of a communication graph was introduced in Myerson (1977). Various studies in both directions were done during the last three decades but mostly either within one model or another. The generalization of the Owen and the Myerson values, applied to the combination of both models that resulted in a TU game with both independent coalition and cooperation structures, was investigated by Vázquez-Brage et al. (1996).

In the paper we study TU games endowed with both coalition and cooperation structures, the so-called graph games with coalition structures. Different from Vázquez-Brage et al. (1996), in our case a cooperation structure is a two-level cooperation structure that relates fundamentally to the given coalition structure. It is assumed that cooperation (via bilateral agreements between participants) is only

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possible either among the entire coalitions of a coalition structure, in other terms a priori unions, or among single players within a priori unions. No communication and therefore no cooperation is allowed between single players from distinct elements of the coalition structure. This approach allows to model various network situations, in particular, telecommunication problems, distribution of goods among different cities (countries) along highway networks connecting the cities and local road networks within the cities, or sharing an international river with multiple users but without international firms, i.e., when no cooperation is possible among single users located at different levels along the river, and so on. A two-level cooperation structure is introduced by means of graphs of two types, first, presenting links among a priori unions of the coalition structure and second, presenting links among players within each a priori union. We consider cooperation structures presented by combinations of graphs of different types both undirected – general graphs and cycle-free graphs, and directed – line-graphs with linearly ordered players, rooted trees and sink trees. Fig. 1(a) illustrates one of possible situations within the model while Fig. 1(b) provides an example of a possible situation within the model of Vázquez-Brage et al. with the same set of players, the same coalition structure, and even the same links connecting players within a priori unions. In general, the newly introduced model of a game with two-level cooperation structure cannot be reduced to the model of Vázquez-Brage et al.. Consider for example negotiations between two countries held on the level of prime ministers who in turn are citizens of their countries. The communication link between countries can be replaced neither by communication link connecting the prime ministers as single persons and therefore presenting only their personal interests, nor by all communication links connecting citizens of one country with citizens of another country that also present links only on personal level. The two models coincide only if a communication graph between a priori unions in our model is empty and components of a communication graph in the model of Vázquez-Brage et al. are subsets of a priori unions. An example illustrating this situation with the same player set, the same coalition structure, and the same graphs within a priori unions, as on Fig. 1(a) is given on Fig. 1(c).

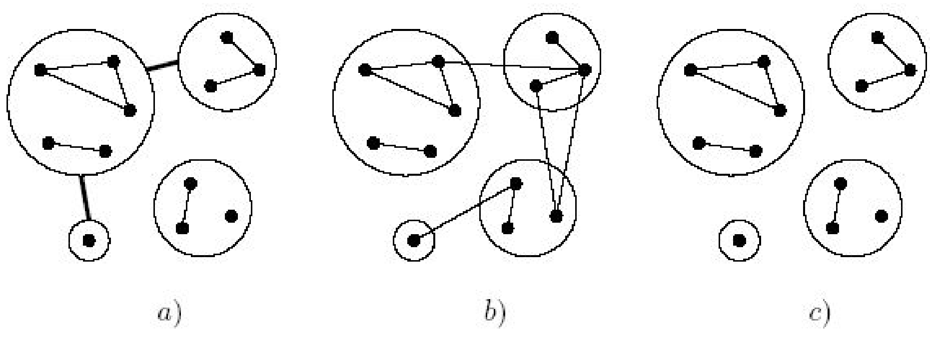

Figure1. *a*) model of the paper; *b*) model of Vázquez-Brage et al.; *c*) case of the coincidence

Our main concern is the theoretical justification of solution concepts reflecting the two-stage distribution procedure. It is assumed that at first, a priori unions through upper level bargaining based only on cumulative interests of all members of every involved entire a priori union, when nobody's personal interests are taken into account, collect their total shares. Thereafter, via bargaining within a priori unions based now on personal interests of participants, the collected shares are distributed to single players. As a bargaining output on both levels one or another value for games with cooperation structures, in other terms graph games, can be reasonably applied. Following Myerson (1977) we assume that cooperation possible only among connected players or connected groups of players and, therefore, we concentrate on component efficient values. Different component efficient values for graph games with graphs of various types, both undirected and directed, are known in the literature. We introduce a unified approach to a number of component efficient values for graph games that allows application of various combinations of known solutions concepts, first at the level of entire a priori unions and then at the level within a priori unions, within the unique framework. Our approach to values for graph games with coalition structures is close to that of both Myerson (1977) and Aumann and Drèze (1974): it is based on ideas of component efficiency and one or another deletion link property, and it treats an a priori union as a self-contained unit. Moreover, to link both communication levels between and within a priori unions we incorporate the idea of the Owen's quotient game property (Owen, 1977). This approach generates two-stage solution concepts that provide consistent application of values for graph games on both levels. The incorporation of different solutions aims not only to enrich the solution concept for graph games with coalition structures but, because there exists no universal solution for graph games applicable to full variety of possible undirected and directed graph structures, it also opens the broad diversity of applications impossible otherwise. Moreover, it also allows to chose, depending on types of graph structures under scrutiny, the most preferable, in particular, the most computationally efficient combination of values among others suitable. The idea of the two-stage construction of solutions is not new. The well known example is the Owen value (Owen, 1977) for games with coalition structures that is defined by applying the Shapley value (Shapley, 1953) twice, first, the Shapley value is employed at the level of a priori unions to define a new game on each one of them, and then the Shapley value is applied to these new games. Other applications of the two-stage construction of solutions can be found in Albizuri and Zarzuelo (2004) and in Kamijo (2009). As a practical application of the new model we consider the problem of sharing of an international river among multiple users.

The structure of the paper is as follows. Basic definitions and notation along with the formal definition of a graph game with coalition structure and its core are introduced in Sect. 2.. Sect. 3. provides the uniform approach to several known component efficient values for games with cooperation structures. In Sect. 4. we introduce values for graph games with coalition structures axiomatically and present the explicit formula representation, we also investigate stability and distribution of Harsanyi dividends. Sect. 5. deals with the generalization on graph games with level structures. Sect. 6. discusses application to the water distribution problem of an international river among multiple users without international firms.

2. Preliminaries

2.1. TU Games and Values

Recall some definitions and notation. A *cooperative game with transferable utility* (*TU game*) is a pair $\langle N, v\rangle$, where $N = \{1, \ldots, n\}$ is a finite set of $n \geq 2$ players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ (or $S \in 2^N$) of s players is called a *coalition*, and the associated real number $v(S)$ presents the *worth* of S. The set of all games with fixed N we denote by $\mathcal{G}_N$. For simplicity of notation and if no ambiguity appears, we write v instead of $\langle N, v\rangle$ when refer to a game. A *value* is a mapping $\xi\colon \mathcal{G}_N \to \mathbb{R}^N$ that assigns to every $v \in \mathcal{G}_N$ a vector $\xi(v) \in \mathbb{R}^N$; the real number $\xi_i(v)$ represents the *payoff* to player i in v. A *subgame* of v with a player set $T \subseteq N$, $T \neq \emptyset$, is a game $v|_T$ defined as $v|_T(S) = v(S)$, for all $S \subseteq T$. A game v is *superadditive*, if $v(S \cup T) \geq v(S) + v(T)$, for all $S, T \subseteq N$, such that $S \cap T = \emptyset$. A game v is *convex*, if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$, for all $S, T \subseteq N$. In what follows for all $x \in \mathbb{R}^N$ and $S \subseteq N$, we use standard notation $x(S) = \sum_{i\in S} x_i$ and $x_S = \{x_i\}_{i\in S}$. The cardinality of a given set A we denote by $|A|$ along with lower case letters like $n = |N|$, $m = |M|$, $n_k = |N_k|$, and so on.

It is well known (Shapley, 1953) that *unanimity games* $\{u_T\}_{\substack{T\subseteq N\\ T\neq\emptyset}}$, defined as $u_T(S) = 1$, if $T \subseteq S$, and $u_T(S) = 0$ otherwise, create a basis in $\mathcal{G}_N$, i.e., every $v \in \mathcal{G}_N$ can be uniquely presented in the linear form $v = \sum\limits_{T\subseteq N, T\neq\emptyset} \lambda_T^v\, u_T$, where $\lambda_T^v = \sum\limits_{S\subseteq T} (-1)^{t-s}\, v(S)$, for all $T \subseteq N$, $T \neq \emptyset$. Following Harsanyi (1959) the coefficient λ_T^v is referred to as a *dividend* of coalition T in game v.

For a permutation $\pi\colon N \to N$, assigning rank number $\pi(i) \in N$ to a player $i \in N$, let $\pi^i = \{j \in N \mid \pi(j) \leq \pi(i)\}$ be the set of all players with rank number smaller or equal to the rank number of i, including i itself. The *marginal contribution vector* $m^\pi(v) \in \mathbb{R}^n$ of a game v and a permutation π is given by $m_i^\pi(v) = v(\pi^i) - v(\pi^i\backslash\{i\})$, $i \in N$. By u we denote the permutation on N relevant to the natural ordering from 1 to n, i.e., $u(i) = i$, $i \in N$, and by l the permutation relevant to the reverse ordering $n, n-1, \ldots, 1$, i.e., $l(i) = n + 1 - i$, $i \in N$.

The *Shapley value* (Shapley, 1953) of a game $v \in \mathcal{G}_N$ can be given by

$$Sh_i(v) = \sum_{T\subseteq N, T\ni i} \frac{\lambda_T^v}{t}, \qquad \text{for all } i \in N.$$

The *core* (Gillies,1953) of $v \in \mathcal{G}_N$ is defined as

$$C(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(S) \geq v(S), \text{ for all } S \subseteq N\}.$$

A value ξ is *stable*, if for any $v \in \mathcal{G}_N$ with nonempty core $C(v)$, $\xi(v) \in C(v)$.

2.2. Games with Coalition Structures

A *coalition structure* or, in other terms, a *system of a priori unions* on a player set N is given by a partition $\mathcal{P} = \{N_1, ..., N_m\}$ of the player set N, i.e., $N_1 \cup ... \cup N_m = N$ and $N_k \cap N_l = \emptyset$ for $k \neq l$. A pair $\langle v, \mathcal{P}\rangle$ of a game $v \in \mathcal{G}_N$ and a coalition structure $\mathcal{P}$ on the player set N constitutes a *game with coalition structure* or, in other terms, a *game with a priori unions* or simply *P-game*. The set of all P-games with a fixed player set N we denote $\mathcal{G}_N^{\mathcal{P}}$. A *P-value* is a mapping $\xi\colon \mathcal{G}_N^{\mathcal{P}} \to \mathbb{R}^N$ that associates with every $\langle v,\mathcal{P}\rangle \in \mathcal{G}_N^{\mathcal{P}}$ a vector $\xi(v,\mathcal{P}) \in \mathbb{R}^N$. Given $\langle v,\mathcal{P}\rangle \in \mathcal{G}_N^{\mathcal{P}}$, Owen (1977) defines a game $v_{\mathcal{P}}$, called a *quotient game*, on $M = \{1,\dots,m\}$ in which each a priori union N_k acts as a player:

$$v_{\mathcal{P}}(Q) = v(\bigcup_{k\in Q} N_k), \qquad \text{for all } Q \subseteq M.$$

Note that $\langle v,\{N\}\rangle$ represents the same situation as v itself. Later on by $\langle N\rangle$ denote the coalition structure composed by singletons, i.e., $\langle N\rangle = \{\{1\},\dots,\{n\}\}$. Furthermore, for every $i \in N$, let $k(i)$ be defined by the relation $i \in N_{k(i)}$, and for any $x\in\mathbb{R}^N$, let $x^{\mathcal{P}} = \big(x(N_k)\big)_{k\in M} \in \mathbb{R}^M$ be the corresponding vector of total payoffs to a priori unions.

2.3. Games with Cooperation Structures

A *cooperation structure* on N is specified by a graph Γ, undirected or directed. An *undirected/directed graph* is a collection of unordered/ordered pairs of nodes (players) $\Gamma \subseteq \Gamma_N^c = \{\,\{i,j\} \mid i,j \in N,\ i \neq j\}$ or $\Gamma \subseteq \bar{\Gamma}_N^c = \{(i,j) \mid i,j \in N, i \neq j\}$ respectively, where an unordered/ordered pair $\{i,j\}$ or correspondingly (i,j) presents a *undirected/directed link* between $i,j \in N$. A pair $\langle v,\Gamma\rangle$ of a game $v \in \mathcal{G}_N$ and a communication graph Γ on N constitutes a *game with graph (cooperation) structure* or simply *Γ-game*. The set of all Γ-games with a fixed player set N we denote $\mathcal{G}_N^{\Gamma}$. A *Γ-value* is a mapping $\xi\colon \mathcal{G}_N^{\Gamma} \to \mathbb{R}^N$ that assigns to every $\langle v,\Gamma\rangle \in \mathcal{G}_N^{\Gamma}$ a vector $\xi(v,\Gamma) \in \mathbb{R}^N$.

For any graph Γ on N and any $S \subseteq N$, the *subgraph* of Γ on S is the graph $\Gamma|_S = \{\{i,j\} \in \Gamma \mid i,j \in S\}$. In an undirected graph Γ on N a sequence of different nodes $(i_1,\dots,i_k)$, $k \geq 2$, is a *path* from i_1 to i_k, if for all $h = 1,\dots,k-1$, $\{i_h,i_{h+1}\} \in \Gamma$. In a digraph Γ on N a sequence of different nodes $(i_1,\dots,i_k)$, $k \geq 2$, is an *undirected path* from i_1 to i_k, if for all $h = 1,\dots,k-1$, $(i_h,i_{h+1}) \in \Gamma$ and/or $(i_{h+1},i_h) \in \Gamma$, and is a *directed path* from i_1 to i_k, if for all $h = 1,\dots,k-1$, $(i_h,i_{h+1}) \in \Gamma$. We consider connectedness with respect to (undirected) paths and say that two nodes are *connected*, if there exists an (undirected) path from one node to another. A graph is *connected*, if any two nodes are connected. Given a graph Γ, $S \subseteq N$ is *connected*, if $\Gamma|_S$ is connected. Denote by $C^{\Gamma}(S)$ the set of all connected subcoalitions of S, by S/Γ the set of maximally connected subcoalitions, called *components*, and let $(S/\Gamma)_i$ be the component of S containing $i \in S$. Notice that S/Γ is a partition of S. Besides, for any coalition structure $\mathcal{P}$, the graph $\Gamma^c(\mathcal{P}) = \bigcup_{P\in\mathcal{P}} \Gamma_P^c$, splits into completely connected components $P \in \mathcal{P}$, and $N/\Gamma^c(\mathcal{P}) = \mathcal{P}$. For any $\langle v,\Gamma\rangle \in \mathcal{G}_N^{\Gamma}$, a payoff vector $x \in \mathbb{R}^N$ is *component efficient*, if $x(C) = v(C)$, for every $C \in N/\Gamma$. Later on, when for avoiding confusion it is necessary to specify the set of nodes N, we write Γ_N instead of Γ.

Following Myerson (1977), we assume that for $\langle v,\Gamma\rangle \in \mathcal{G}_N^{\Gamma}$ cooperation is possible only among connected players and consider a *restricted game* $v^{\Gamma} \in \mathcal{G}_N$ defined as

$$v^{\Gamma}(S) = \sum_{C\in S/\Gamma} v(C), \qquad \text{for all } S \subseteq N.$$

The *core* $C(v,\Gamma)$ of $\langle v,\Gamma\rangle \in \mathcal{G}_N^{\Gamma}$ is defined as a set of component efficient payoff vectors that are not dominated by any connected coalition, i.e.,

$$C(v,\Gamma) = \{x \in \mathbb{R}^N \mid x(C) = v(C),\ \forall C \in N/\Gamma, \text{ and } x(T) \geq v(T),\ \forall T \in C^{\Gamma}(N)\}. \quad (1)$$

It is easy to see that $C(v, \Gamma) = C(v^{\Gamma})$.

Below along with cooperation structures given by general undirected graphs we consider also those given by cycle-free undirected graphs and by directed graphs – line-graphs with linearly ordered players, rooted and sink forests. In an undirected graph a path $(i_1, \dots, i_k)$, $k \geq 3$, is a *cycle*, if $i_1 = i_k$. An undirected graph is *cycle-free*, if it contains no cycles. In a directed link (i, j), j is a *subordinate* of i and i is a *superior* of j. In a digraph Γ, $j \neq i$ is a *successor* of i and i is a *predecessor* of j, if there exists a path $(i_1, \dots, i_k)$ with $i_1 = i$ and $i_k = j$. A digraph Γ is a *rooted tree*, if there is one node in N, called a *root*, having no predecessors in Γ and there is a unique directed path in Γ from this node to any other node in N. A digraph Γ is a *sink tree*, if the directed graph, composed by the same set of links as Γ but with the opposite orientation, is a rooted tree; in this case the root of a tree changes its meaning to the absorbing sink. A digraph is a *rooted/sink forest*, if it is composed by a number of nonoverlapping rooted)/sink trees. A *line-graph* is a digraph that contains links only between subsequent nodes. Without loss of generality we may assume that in a line-graph L nodes are ordered according to the natural order from 1 to n, i.e., line-graph $\Gamma \subseteq \{(i, i+1) \mid i = 1, \dots, n-1\}$.

2.4. Graph Games with Coalition Structures

A triple $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}} \rangle$ presenting a combination of a TU game $v \in \mathcal{G}_N$ with a coalition structure $\mathcal{P}$ and with limited cooperation possibilities presented via a two-level graph structure $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_{N_k}\}_{k \in M} \rangle$ constitutes a *graph game with coalition structure* or simply *$P\Gamma$-game*. The set of all $P\Gamma$-games with a fixed player set N we denote $\mathcal{G}_N^{\mathcal{P}\Gamma}$. A *$P\Gamma$-value* is defined as a mapping $\xi \colon \mathcal{G}_N^{\mathcal{P}\Gamma} \to \mathbb{R}^N$ that associates with every $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$ a vector $\xi(v, \mathcal{P}, \Gamma_{\mathcal{P}}) \in \mathbb{R}^N$.

It is worth to emphasize that in the model under scrutiny the primary is a coalition structure and a cooperation structure is introduced above the given coalition structure. The graph structure $\Gamma_{\mathcal{P}}$ is specified by means of graphs of two types – a graph Γ_M connecting a priori unions as single elements, and graphs Γ_{N_k} within a priori unions N_k, $k \in M$, connecting single players. Moreover, observe that $P\Gamma$-games $\langle v, \langle N \rangle, \Gamma_{\langle N \rangle} \rangle$ and $\langle v, \{N\}, \Gamma_{\{N\}} \rangle$ with trivial coalition structures reduce to a Γ-game $\langle v, \Gamma_N \rangle$. Later on for simplicity of notation, when it causes no ambiguity, we denote graphs Γ_{N_k} within a priori unions N_k, $k \in M$, by Γ_k.

Given $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$, one can consider *graph games within a priori unions* $\langle v_k, \Gamma_k \rangle \in \mathcal{G}_{N_k}^{\Gamma}$, with $v_k = v|_{N_k}$, $k \in M$. Moreover, owning a coalition structure one can consider a quotient game. However, a quotient game relating to a $P\Gamma$-game should take into account the limited cooperation within a priori unions, and hence, it must differ from the classical one of Owen. For any $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$, we define the *quotient game* $v_{\mathcal{P}\Gamma} \in \mathcal{G}_M$ as

$$v_{\mathcal{P}\Gamma}(Q) = \begin{cases} v_k^{\Gamma_k}(N_k), & Q = \{k\}, \\ v(\bigcup\limits_{k \in Q} N_k), & |Q| > 1, \end{cases} \qquad \text{for all } Q \subseteq M. \tag{2}$$

Next, it is natural to consider a *quotient Γ-game* $\langle v_{\mathcal{P}\Gamma}, \Gamma_M \rangle \in \mathcal{G}_M^{\Gamma}$.

Furthermore, given a Γ-value ϕ, for any $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}} \rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$ with a graph structure Γ_M on the level of a priori unions suitable for application of ϕ to the corresponding quotient Γ-game $\langle v_{\mathcal{P}\Gamma}, \Gamma_M\rangle$[1], along with a subgame v_k within a priori union N_k, $k \in M$, one can also consider a ϕ_k*-game* v_k^ϕ defined as

$$v_k^\phi(S) = \begin{cases} \phi_k(v_{\mathcal{P}\Gamma}, \Gamma_M), & S = N_k, \\ v(S), & S \neq N_k, \end{cases} \qquad \text{for all } S \subseteq N_k,$$

where $\phi_k(v_{\mathcal{P}\Gamma}, \Gamma_M)$ is the payoff to N_k given by ϕ in $\langle v_{\mathcal{P}\Gamma}, \Gamma_M\rangle$. In particular, for any $x \in \mathbb{R}^M$, a x_k-game v_k^x within N_k, $k \in M$, is defined by

$$v_k^x(S) = \begin{cases} x_k, & S = N_k, \\ v(S), & S \neq N_k, \end{cases} \qquad \text{for all } S \subseteq N_k.$$

In this context it is natural to consider Γ-games $\langle v_k^\xi, \Gamma_k\rangle$, $k \in M$, as well.

Following the similar approach as for games with cooperation structure, the *core* $C(v, \mathcal{P}, \Gamma_{\mathcal{P}})$ of $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$ is the set of payoff vectors that are

(i) component efficient both in the quotient Γ-game $\langle v_{\mathcal{P}\Gamma}, \Gamma_M\rangle$ and in all graph games within a priori unions $\langle v_k, \Gamma_k\rangle$, $k \in M$, containing more than one player,
(ii) not dominated by any connected coalition:

$$C(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = \Big\{ x \in \mathbb{R}^N \mid \Big[\big[x^{\mathcal{P}}(K) = v_{\mathcal{P}\Gamma}(K),\ \forall K \in M/\Gamma_M \big] \ \& \\ \big[x^{\mathcal{P}}(Q) \geq v_{\mathcal{P}\Gamma}(Q),\ \forall Q \in C^{\Gamma_M}(M) \big] \Big] \ \& \qquad (3) \\ \Big[\big[x(C) = v(C), \forall C \in N_k/\Gamma_k, C \neq N_k \big] \ \& \ \big[x(S) \geq v(S), \forall S \in C^{\Gamma_k}(N_k) \big], \forall k \in M \colon n_k > 1 \Big] \Big\}.$$

Remark 1. Notice that in the above definition the condition of component efficiency on components equal to the entire a priori unions at the level within a priori unions is excluded. The reason is the following. By definition of a quotient game, for any $k \in M$, $v_{\mathcal{P}\Gamma}(\{k\}) = v_k^{\Gamma_k}(N_k)$. If $N_k \in N_k/\Gamma_k$, i.e., if Γ_k is connected, $v_k^{\Gamma_k}(N_k) = v(N_k)$, and therefore, $v_{\mathcal{P}\Gamma}(\{k\}) = v(N_k)$. Besides by definition, $x^{\mathcal{P}}(\{k\}) = x_k^{\mathcal{P}} = x(N_k)$, for all $k \in M$. Furthermore, singleton coalitions are always connected, i.e., $\{k\} \in C^{\Gamma_M}(M)$, for all $k \in M$. Thus, in case when $N_k \in N_k/\Gamma_k$, the presence of a stronger condition $x(N_k) = v(N_k)$ at the level within a priori unions may conflict with a weaker condition $x^{\mathcal{P}}(\{k\}) \geq v_{\mathcal{P}\Gamma}(\{k\})$, which in this case is the same as $x(N_k) \geq v(N_k)$, at the level of a priori unions, that as a result can lead to the emptiness of the core.

The next statement easily follows from the latter definition.

Proposition 1. *For any* $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$ *and* $x \in \mathbb{R}^N$,

$$x \in C(v, \mathcal{P}, \Gamma_{\mathcal{P}}) \iff \big[x^{\mathcal{P}} \in C(v_{\mathcal{P}\Gamma}, \Gamma_M) \big] \ \& \ \big[x_{N_k} \in C(v_k^{x^{\mathcal{P}}}, \Gamma_k), \forall k \in M \colon n_k > 1 \big].$$

Remark 2. The claim $x_{N_k} \in C(v_k^{x^{\mathcal{P}}}, \Gamma_k)$, $k \in M$, is vital only if $N_k \in N_k/\Gamma_k$, i.e., if Γ_k is connected; when Γ_k is disconnected, it can be replaced by $x_{N_k} \in C(v_k, \Gamma_k)$, as well.

[1] In general Γ-values can be applied only to Γ-games determined by graphs of certain types; for more detailed discussion see Sect. 3..

3. Uniform Approach to Component Efficient Γ-Values

We show now that a number of known component efficient Γ-values for games with cooperation structures given by undirected and directed graphs of different types can be approached within a unique framework. This unique approach will be employed later in Section 4. for the two-stage construction of $P\Gamma$-values.

A Γ-value ξ is *component efficient* (CE) if, for any $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$, for all $C \in N/\Gamma$,

$$\sum_{i\in C} \xi_i(v,\Gamma) = v(C).$$

3.1. The Myerson Value

The Myerson value μ (Myerson, 1977) is defined for any Γ-game $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$ with arbitrary undirected graph Γ as the Shapley value of the restricted game v^Γ, i.e.,

$$\mu_i(v,\Gamma) = Sh_i(v^\Gamma), \qquad \text{for all } i \in N.$$

The Myerson value is characterized by two axioms of component efficiency and fairness.

A Γ-value ξ is *fair* (F) if, for any $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$, for every link $\{i,j\} \in \Gamma$,

$$\xi_i(v,\Gamma) - \xi_i(v,\Gamma\backslash\{i,j\}) = \xi_j(v,\Gamma) - \xi_j(v,\Gamma\backslash\{i,j\}).$$

3.2. The Position Value

The position value, introduced in Meessen (1988) and developed in Borm et al. (1992), is defined for any $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$ with arbitrary undirected graph Γ. The position value π attributes to each player in a graph game $\langle v,\Gamma\rangle$ the sum of his individual value $v(i)$ and half of the value of each link he is involved in, where the value of a link is defined as the Shapley payoff to this link in the associated link game on links of Γ. More precisely,

$$\pi_i(v,\Gamma) = v(i) + \frac{1}{2}\sum_{l\in\Gamma_i} Sh_l(\Gamma, v_\Gamma^0), \qquad \text{for all } i \in N,$$

where $\Gamma_i = \{l \in \Gamma | l \ni i\}$, v^0 is the zero-normalization of v, i.e., for all $S \subseteq N$, $v^0(S) = v(S) - \sum_{i\in S} v(i)$, and for any zero-normalized game $v \in \mathcal{G}_N$ and a graph Γ, the associated link game $\langle \Gamma, v_\Gamma\rangle$ between links in Γ is defined as

$$v_\Gamma(\Gamma') = v^{\Gamma'}(N), \qquad \text{for all } \Gamma' \in 2^\Gamma.$$

Slikker (2005) characterizes the position value on the class of all graph games via component efficiency and balanced link contributions.

A Γ-value ξ meets *balanced link contributions* (BLC) if, for any $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$ and $i,j \in N$,

$$\sum_{h|\{i,h\}\in\Gamma} \big[\xi_j(v,\Gamma) - \xi_j(v,\Gamma\backslash\{i,h\})\big] = \sum_{h|\{j,h\}\in\Gamma} \big[\xi_i(v,\Gamma) - \xi_i(v,\Gamma\backslash\{j,h\})\big].$$

3.3. The Average Tree Solution

A new algorithmically very attractive[2] solution concept for undirected cycle-free Γ-games, the so called average tree solution (AT-solution), recently introduced in Herings et al. (2008). Recall the definition. Consider a cycle-free graph game $\langle v, \Gamma\rangle$ and let $i \in N$. Then i belongs to the component $(N/\Gamma)_i$ and induces a unique rooted tree $T(i)$ on $(N/\Gamma)_i$ in the following way. For every $j \in (N/\Gamma)_i \backslash \{i\}$, there is a unique path in the subgraph $\langle (N/\Gamma)_i, \Gamma|_{(N/\Gamma)_i}\rangle$ from i to j. That allows to change undirected links on this path to directed so that the first node in any ordered pair is the node coming first on the path from i to j. The payoff $t^i_j(v,\Gamma)$ associated in the tree $T(i)$ to any player $j \in (N/\Gamma)_i$ (obviously, in this case $(N/\Gamma)_j = (N/\Gamma)_i$) is equal to the worth of the coalition composed of player j and all his subordinates in $T(i)$ minus the sum of the worths of all coalitions composed of any successor of player j and all subordinates of this successor in $T(i)$, i.e.,

$$t^i_j(v,\Gamma) = v(\bar{S}_{T(i)}(j)) - \sum_{h \in F_{T(i)}(j)} v(\bar{S}_{T(i)}(h)), \qquad \text{for all } j \in (N/\Gamma)_i,$$

where for any node $j \in (N/\Gamma)_i$, $F_{T(i)}(j) = \{h \in (N/\Gamma)_i \,|\, (j,h) \in T(i)\}$ is the set of all subordinates of j in $T(i)$, $S_{T(i)}(j)$ is the set of all successors of j in $T(i)$, and $\bar{S}_{T(i)}(j) = S_{T(i)}(j) \cup j$. Every component $C \in N/\Gamma$ in the cycle-free graph Γ induces $|C|$ different trees, one tree for each one of different nodes. The *average tree solution* assigns to each cycle-free graph game $\langle v, \Gamma\rangle$ the payoff vector in which player $j \in N$ receives the average over $i \in (N/\Gamma)_j$ of the payoffs $t^i_j(v,\Gamma)$, i.e.,

$$AT_j(v,\Gamma) = \frac{1}{|(N/\Gamma)_j|} \sum_{i \in (N/\Gamma)_j} t^i_j(v,\Gamma), \qquad \text{for all } j \in N.$$

The average tree solution defined on the class of superadditive cycle-free graph games appears to be stable. On the entire class of cycle-free graph games the average tree solution is characterized via two axioms of component efficiency and component fairness.

A Γ-value ξ is *component fair* (CF) if, for any cycle-free $\langle v, \Gamma\rangle \in \mathcal{G}^\Gamma_N$, for every link $\{i,j\} \in \Gamma$,

$$\frac{1}{|(N/\Gamma \backslash \{i,j\})_i|} \sum_{t \in (N/\Gamma \backslash \{i,j\})_i} \Big(\xi_t(v,\Gamma) - \xi_t(v, \Gamma \backslash \{i,j\}\Big) =$$

$$\frac{1}{|(N/\Gamma \backslash \{i,j\})_j|} \sum_{t \in (N/\Gamma \backslash \{i,j\})_j} \Big(\xi_t(v,\Gamma) - \xi_t(v, \Gamma \backslash \{i,j\}\Big).$$

3.4. Values for Line-Graph Games

Three following values for line-graph Γ-games are studied in Brink et al. (2007), namely, the *upper equivalent solution* given by

$$\xi^{UE}_i(v,\Gamma) = m^u_i(v^\Gamma), \qquad \text{for all } i \in N,$$

[2] In comparison with the Myerson value (the Shapley value) with computational complexity of the order $n!$, the AT-solution has the computational complexity of the order n.

the *lower equivalent solution* given by

$$\xi_i^{LE}(v,\Gamma) = m_i^l(v^\Gamma), \qquad \text{for all } \ i \in N,$$

and the *equal loss solution* given by

$$\xi_i^{EL}(v,\Gamma) = \frac{m_i^u(v^\Gamma) + m_i^l(v^\Gamma)}{2}, \qquad \text{for all } \ i \in N.$$

All of these three solutions for superadditive line-graph games turn out to be stable. Moreover, on the entire class of line-graph games each one of them is characterized via component efficiency and one of the three following axioms expressing different fairness properties.

A Γ-value ξ is *upper equivalent* (UE) if, for any line-graph $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$, for any $i = 1,\ldots,n-1$, for all $j = 1,\ldots,i$,

$$\xi_j(v,\Gamma\backslash\{i,i{+}1\}) = \xi_j(v,\Gamma).$$

A Γ-value ξ is *lower equivalent* (LE) if, for any line-graph $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$, for any $i = 1,\ldots,n-1$, for all $j = i+1,\ldots,n$,

$$\xi_j(v,\Gamma\backslash\{i,i{+}1\}) = \xi_j(v,\Gamma).$$

A Γ-value ξ possesses the *equal loss property* (EL) if, for any line-graph $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$, for any $i = 1,\ldots,n-1$,

$$\sum_{j=1}^{i}\Big(\xi_j(v,\Gamma) - \xi_j(v,\Gamma\backslash\{i,i{+}1\})\Big) = \sum_{j=i+1}^{n}\Big(\xi_j(v,\Gamma) - \xi_j(v,\Gamma\backslash\{i,i{+}1\})\Big).$$

3.5. Tree-Type Values for Forest-Graph Games

The tree value

$$t_i(v,\Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j\in T_\Gamma(i)} v(\bar{S}_\Gamma(j)), \qquad \text{for all } \ i \in N$$

and the sink value

$$s_i(v,\Gamma) = v(\bar{P}_\Gamma(i)) - \sum_{j\in O_\Gamma(i)} v(\bar{P}_\Gamma(j)), \qquad \text{for all } \ i \in N$$

respectively for rooted/sink forest Γ-games are studied in Khmelnitskaya (2009). Both these values are stable on the subclass of superadditive games. Moreover, the tree and sink values on the correspondent entire class of rooted/sink forest Γ-games can be characterized via component efficiency and successor equivalence or predecessor equivalence respectively.

A Γ-value ξ is *successor equivalent* (SE) if, for any rooted forest $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$, for every link $\{i,j\} \in \Gamma$, for all k being successors of j, or $k = j$,

$$\xi_k(v,\Gamma\backslash\{i,j\}) = \xi_k(v,\Gamma).$$

A Γ-value ξ is *predecessor equivalent* (PE) if, for any sink forest $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$, for every link $\{i,j\} \in \Gamma$, for all k being predecessors of i, or $k = i$,

$$\xi_k(v,\Gamma\backslash\{i,j\}) = \xi_k(v,\Gamma).$$

3.6. Uniform Framework

Notice that each one of the considered above Γ-values for Γ-games with *suitable* graph structures is characterized by two axioms, CE and one or another *deletion link* (DL) property, reflecting the relevant reaction of a Γ-value on the deletion of a link in the communication graph, i.e.,

$$\begin{aligned}
\text{CE} + \text{F} \quad &\text{for all undirected } \Gamma\text{-games} \Longleftrightarrow \mu(v,\Gamma),\\
\text{CE} + \text{BLC} \quad &\text{for all undirected } \Gamma\text{-games} \Longleftrightarrow \pi(v,\Gamma),\\
\text{CE} + \text{CF} \quad &\text{for undirected cycle-free } \Gamma\text{-games} \Longleftrightarrow AT(v,\Gamma),\\
\text{CE} + \text{UE} \quad &\text{for line-graph } \Gamma\text{-games} \Longleftrightarrow UE(v,\Gamma),\\
\text{CE} + \text{LE} \quad &\text{for line-graph } \Gamma\text{-games} \Longleftrightarrow LE(v,\Gamma),\\
\text{CE} + \text{EL} \quad &\text{for line-graph } \Gamma\text{-games} \Longleftrightarrow EL(v,\Gamma),\\
\text{CE} + \text{SE} \quad &\text{for rooted forest } \Gamma\text{-games} \Longleftrightarrow t(v,\Gamma),\\
\text{CE} + \text{PE} \quad &\text{for sink forest } \Gamma\text{-games} \Longleftrightarrow s(v,\Gamma).
\end{aligned}$$

In the sequel, for the unification of presentation and simplicity of notation, we identify each one of Γ-values with the corresponding DL axiom. For a given DL, let $\mathcal{G}_N^{DL} \subseteq \mathcal{G}_N^{\Gamma}$ be a set of all $\langle v,\Gamma\rangle \in \mathcal{G}_N^{\Gamma}$ with Γ suitable for DL application. To summarize,

$$\text{CE} + \text{DL} \quad \text{on } \mathcal{G}_N^{DL} \iff DL(v,\Gamma),$$

where DL is one of the axioms F, BLC, CF, LE, UE, El, SE, or PE. Whence, $F(v,\Gamma) = \mu(v,\Gamma)$ and $BLC(v,\Gamma) = \pi(v,\Gamma)$ for all undirected Γ-games, $CF(v,\Gamma) = AT(v,\Gamma)$ for all undirected cycle-free Γ-games, $UE(v,\Gamma)$, $LE(v,\Gamma)$, and $EL(v,\Gamma)$ are UE, LE, and EL solutions correspondingly for all line-graph Γ-games, $SE(v,\Gamma) = t(v,\Gamma)$ for all rooted forest Γ-games, and $PE(v,\Gamma) = t(v,\Gamma)$ for all sink forest Γ-games.

4. $P\Gamma$-Values

4.1. Component Efficient $P\Gamma$-Values

We adapt now the notions of component efficiency and discussed above deletion link properties to $P\Gamma$-values and show that similar to component efficient Γ-values, the deletion link properties uniquely define component efficient $P\Gamma$-values on a class of $P\Gamma$-games with suitable graph structure. The involvement of different deletion link properties, depending on the considered graph structure, allows to pick the most favorable among other appropriate combinations of Γ-values applied on both levels between and within a priori unions in the two-stage construction of $P\Gamma$-values discussed below. Moreover, consideration of the only one specific combination of Γ-values restricts the variability of applications, since Γ-values developed for Γ-games defined by undirected graphs are not applicable in Γ-games with, for example, directed rooted forest graph structures, and vice versa.

Introduce first two new axioms of component efficiency with respect to $P\Gamma$-values that inherit the idea of component efficiency for Γ-values and also incorporate the quotient game property[3] of the Owen value in a sense that the vector of total payoffs to a priori unions coincides with the payoff vector in the quotient game.

A $P\Gamma$-value ξ is *component efficient in quotient* (CEQ) if, for any $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$, for each $K \in M/\Gamma_M$,

$$\sum_{k\in K}\sum_{i\in N_k}\xi_i(v,\mathcal{P},\Gamma_{\mathcal{P}}) = v_{\mathcal{P}\Gamma}(K).$$

A $P\Gamma$-value ξ is *component efficient within a priori unions* (CEU) if, for any $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$, for every $k\in M$ and all $C\in N_k/\Gamma_k$, $C\neq N_k$,

$$\sum_{i\in C}\xi_i(v,\mathcal{P},\Gamma_{\mathcal{P}}) = v(C).$$

Reconsider deletion link properties, now with respect to $P\Gamma$-games. Recall that every $P\Gamma$-value is a mapping $\xi\colon \mathcal{G}_N^{\mathcal{P}\Gamma} \to \mathrm{I\!R}^N$. A mapping $\xi = \{\xi_i\}_{i\in N}$ generates on the domain of $P\Gamma$-games a mapping $\xi^{\mathcal{P}}\colon \mathcal{G}_N^{\mathcal{P}\Gamma} \to \mathrm{I\!R}^M$, $\xi^{\mathcal{P}} = \{\xi_k^{\mathcal{P}}\}_{k\in M}$, with $\xi_k^{\mathcal{P}} = \sum_{i\in N_k}\xi_i$, $k\in M$, and m mappings $\xi_{N_k}\colon \mathcal{G}_N^{\mathcal{P}\Gamma} \to \mathrm{I\!R}^{N_k}$, $\xi_{N_k} = \{\xi_i\}_{i\in N_k}$, $k \in M$. Since there are many $P\Gamma$-games $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle$ with the same quotient Γ-game $\langle v_{\mathcal{P}\Gamma}, \Gamma_M\rangle$, there exists a variety of mappings $\psi_{\mathcal{P}}\colon \mathcal{G}_M^{\Gamma} \to \mathcal{G}_N^{\mathcal{P}\Gamma}$ assigning to any Γ-game $\langle u, \Gamma\rangle \in \mathcal{G}_M^{\Gamma}$, some $P\Gamma$-game $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$, such that $v_{\mathcal{P}\Gamma} = u$ and $\Gamma_M = \Gamma$. In general, it is not necessarily that $\psi_{\mathcal{P}}(v_{\mathcal{P}\Gamma}, \Gamma_M) = \langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle$. However, for some fixed $P\Gamma$-game $\langle v^*, \mathcal{P}^*, \Gamma_{\mathcal{P}}^*\rangle$ one can always choose a mapping $\psi_{\mathcal{P}}^*$, such that $\psi_{\mathcal{P}}^*(v_{\mathcal{P}\Gamma}^*, \Gamma_M^*) = \langle v^*, \mathcal{P}^*, \Gamma_{\mathcal{P}}^*\rangle$. Any mapping $\xi^{\mathcal{P}}\circ\psi_{\mathcal{P}}\colon \mathcal{G}_M^{\Gamma} \to \mathrm{I\!R}^M$ by definition represents a Γ-value that, in particular, can be applied to the quotient Γ-game $\langle v_{\mathcal{P}\Gamma}, \Gamma_M\rangle \in \mathcal{G}_M^{\Gamma}$ of some $P\Gamma$-game $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$. Similarly, for a given Γ-value $\phi\colon \mathcal{G}_M^{\Gamma} \to \mathrm{I\!R}^M$, for every $k\in M$, there exists a variety of mappings $\psi_k^{\phi}\colon \mathcal{G}_{N_k}^{\Gamma} \to \mathcal{G}_N^{\mathcal{P}\Gamma}$ assigning to any Γ-game $\langle u, \Gamma\rangle \in \mathcal{G}_{N_k}^{\Gamma}$, some $P\Gamma$-game $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$, such that $v_k^{\phi} = u$ and $\Gamma_k = \Gamma$. For every $k\in M$, a mapping $\xi_{N_k}\circ\psi_k^{\phi}\colon \mathcal{G}_{N_k}^{\Gamma} \to \mathrm{I\!R}^{N_k}$ presents a Γ-value that, in particular, can be applied to Γ-games $\langle v_k^{\phi}, \Gamma_k\rangle \in \mathcal{G}_{N_k}^{\Gamma}$ relevant to some $P\Gamma$-game $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$ together with the given Γ-value ϕ. For a given $(m+1)$-tuple of deletion link axioms $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$ consider a set of $P\Gamma$-games $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}} \subseteq \mathcal{G}_N^{\mathcal{P}\Gamma}$ composed of $P\Gamma$-games $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle$ with graph structures $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$ such that $\langle v_{\mathcal{P}\Gamma}, \Gamma_M\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$, and $\langle v_k^{DL^{\mathcal{P}}}, \Gamma_k\rangle \in \mathcal{G}_{N_k}^{DL^k}$, $k \in M$.

A $P\Gamma$-value ξ defined on $\mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ *satisfies* $(m+1)$*-tuple of deletion link axioms* $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$, if Γ-value $\xi^{\mathcal{P}}\circ\psi_{\mathcal{P}}$ meets $\mathrm{DL}^{\mathcal{P}}$ and every Γ-value $\xi_{N_k}\circ\psi_k^{DL^{\mathcal{P}}}$, $k\in M$, meets the corresponding DL^k.

We focus on $P\Gamma$-values that reflect the two-stage distribution procedure when at first the quotient Γ-game $\langle v_{\mathcal{P}\Gamma}, \Gamma_M\rangle$ is played between a priori unions, and then the total payoffs y_k, $k \in M$, obtained by each N_k are distributed among their members by playing Γ-games $\langle v_k^y, \Gamma_k\rangle$. To ensure that benefits of cooperation between a priori

[3] A P-value ξ satisfies *the quotient game property*, if for any $\langle v, P\rangle \in \mathcal{G}_N^{\mathcal{P}}$, for all $k \in M$,

$$\xi_k(v_{\mathcal{P}}, \{M\}) = \xi_k(v_{\mathcal{P}}, \langle M\rangle) = \sum_{i\in N_k}\xi_i(v,\mathcal{P}).$$

unions can be fully distributed among single players, we assume that solutions in all Γ-games $\langle v_k^y, \Gamma_k\rangle$, $k \in M$, are efficient. Since we concentrate on component efficient solutions, it is important to ensure that the requirement of efficiency does not conflict with component efficiency, which is equivalent to the claim that for every $k \in M$,

$$\sum_{C \in N_k/\Gamma_k} v_k^y(C) = y_k.$$

If Γ_k is connected, i.e. if N_k is the only element of N_k/Γ_k, then the last equality holds automatically since by definition $v_k^y(N_k) = y_k$. Otherwise, for every $k \in M$, for which Γ_k is disconnected, it is necessary to require that

$$\sum_{C \in N_k/\Gamma_k} v(C) = y_k. \tag{4}$$

We say that in $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$ the graph structure $\{\Gamma_k\}_{k\in M}$ is *compatible* with a payoff $y \in \mathbb{R}^M$ in $\langle v_{\mathcal{P}\Gamma}, \Gamma_M\rangle$, if for every $k \in M$, either Γ_k is connected, or (4) holds.

For applications involving disconnected graphs Γ_k, the requirement of compatibility (4) appears to be too demanding. But it is worth to emphasize the following.

Remark 3. If all Γ_k, $k \in M$, are connected, then $\{\Gamma_k\}_{k\in M}$ is always compatible with any payoff $y \in \mathbb{R}^M$ in $\langle v_{\mathcal{P}\Gamma}, \Gamma_M\rangle$, and efficiency follows from component efficiency automatically.

Denote by $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ the set of all $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ with graph structures $\{\Gamma_k\}_{k\in M}$ compatible with $DL^{\mathcal{P}}(v_{\mathcal{P}\Gamma}, \Gamma_M)$.

Theorem 1. *There is a unique $P\Gamma$-value defined on $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$, that meets CEQ, CEU, and $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$, and for any $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ it is given by*

$$\xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = \begin{cases} DL^{\mathcal{P}}_{k(i)}(v_{\mathcal{P}\Gamma}, \Gamma_M), & N_{k(i)} = \{i\}, \\ DL_i^{k(i)}(v_{k(i)}^{DL^{\mathcal{P}}}, \Gamma_{k(i)}), & n_{k(i)} > 1, \end{cases} \quad \textit{for all } i \in N. \tag{5}$$

From now on we refer to the $P\Gamma$-value ξ as to the $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$*-value.*

Proof. I. First prove that the $P\Gamma$-value given by (5) is the unique one on $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ that satisfies CEQ, CEU, and $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$. Take a $P\Gamma$-value ξ on $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ meeting CEQ, CEU, and $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$. Let $\langle v^*, \mathcal{P}^*, \Gamma^*_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ with $\Gamma^*_{\mathcal{P}} = \langle \Gamma^*_M, \{\Gamma^*_k\}_{k\in M}\rangle$, and let $v^*_{\mathcal{P}\Gamma}$ denote its quotient game. Notice that by choice of $\langle v^*, \mathcal{P}^*, \Gamma^*_{\mathcal{P}}\rangle$, it holds that $\langle v^*_{\mathcal{P}\Gamma}, \Gamma^*_M\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$ and $\langle (v^*)_k^{DL^{\mathcal{P}}}, \Gamma^*_k\rangle \in \mathcal{G}_{N_k}^{DL^k}$, for all $k \in M$.

Step 1. Level of a priori unions.

Consider the mapping $\psi^*_{\mathcal{P}}\colon \mathcal{G}_M^{DL^{\mathcal{P}}} \to \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ that assigns to any Γ-game $\langle u, \Gamma\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$, the $P\Gamma$-game $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$, such that $v_{\mathcal{P}\Gamma} = u$ and

$\Gamma_M = \Gamma$, and satisfies the condition $\psi^*_{\mathcal{P}}(v^*_{\mathcal{P}\Gamma}, \Gamma^*_M) = \langle v^*, \mathcal{P}^*, \Gamma^*_{\mathcal{P}}\rangle$. By definition of $\xi^{\mathcal{P}}$, for any $\langle u, \Gamma\rangle \in \mathcal{G}^{DL^{\mathcal{P}}}_M$ and $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle = \psi^*_{\mathcal{P}}(u, \Gamma)$, it holds that

$$(\xi^{\mathcal{P}} \circ \psi^*_{\mathcal{P}})_k(u, \Gamma) = \sum_{i\in N_k} \xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}), \qquad \text{for all } k \in M. \tag{6}$$

Since ξ meets CEQ, for any $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}_N$, for all $K \in M/\Gamma_M$,

$$\sum_{k\in K} \sum_{i\in N_k} \xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = v_{\mathcal{P}\Gamma}(K).$$

Combining the last two equalities and taking into account that by definition of $\psi^*_{\mathcal{P}}$, $v_{\mathcal{P}\Gamma} = u$ and $\Gamma_M = \Gamma$, we get that for any $\langle u, \Gamma\rangle \in \mathcal{G}^{DL^{\mathcal{P}}}_M$, for every $K \in M/\Gamma$,

$$\sum_{k\in K} (\xi^{\mathcal{P}} \circ \psi^*_{\mathcal{P}})_k(u, \Gamma) = u(K),$$

i.e., the Γ-value $\xi^{\mathcal{P}} \circ \psi^*_{\mathcal{P}}$ on $\mathcal{G}^{DL^{\mathcal{P}}}_M$ satisfies CE. From the characterization results for Γ-values, discussed above in Sect. 3., it follows that CE and $\text{DL}^{\mathcal{P}}$ together guarantee that for any $\langle u, \Gamma\rangle \in \mathcal{G}^{DL^{\mathcal{P}}}_M$,

$$(\xi^{\mathcal{P}} \circ \psi^*_{\mathcal{P}})_k(u, \Gamma) = DL^{\mathcal{P}}_k(u, \Gamma), \qquad \text{for all } k \in M.$$

In particular, the last equality is valid for $\langle u, \Gamma\rangle = \langle v^*_{\mathcal{P}\Gamma}, \Gamma^*_M\rangle \in \mathcal{G}^{DL^{\mathcal{P}}}_M$, i.e.,

$$(\xi^{\mathcal{P}} \circ \psi^*_{\mathcal{P}})_k(v^*_{\mathcal{P}\Gamma}, \Gamma^*_M) = DL^{\mathcal{P}}_k(v^*_{\mathcal{P}\Gamma}, \Gamma^*_M), \qquad \text{for all } k \in M.$$

Wherefrom, because of (6) and by choice of $\psi^*_{\mathcal{P}}$,

$$\sum_{i\in N_k} \xi_i(v^*, \mathcal{P}^*, \Gamma^*_{\mathcal{P}}) = DL^{\mathcal{P}}_k(v^*_{\mathcal{P}\Gamma}, \Gamma^*_M), \qquad \text{for all } k \in M.$$

Hence, due to arbitrary choice of the $P\Gamma$-game $\langle v^*, \mathcal{P}^*, \Gamma^*_{\mathcal{P}}\rangle$, it follows that for any $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}_N$,

$$\sum_{i\in N_k} \xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = DL^{\mathcal{P}}_k(v_{\mathcal{P}\Gamma}, \Gamma_M), \qquad \text{for all } k \in M. \tag{7}$$

Notice that for $k \in M$ such that $N_k = \{i\}$, equality (7) reduces to

$$\xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = DL^{\mathcal{P}}_{k(i)}(v_{\mathcal{P}\Gamma}, \Gamma_M), \qquad \text{for all } i \in N \text{ s.t. } N_{k(i)} = \{i\}. \tag{8}$$

Step 2. Level of single players within a priori unions.

Consider $k' \in M$ for which $n_{k'} > 1$. Let the mapping $\psi^*_{k'} : \mathcal{G}^{DL^{k'}}_{N_{k'}} \to \bar{\mathcal{G}}^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}_N$ assign to $\langle u, \Gamma\rangle \in \mathcal{G}^{DL^{k'}}_{N_{k'}}$, the $P\Gamma$-game $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}_N$, such that $v^{DL^{\mathcal{P}}}_{k'} = u$ and $\Gamma_{k'} = \Gamma$, and let $\psi^*_{k'}$ meet the condition $\psi^*_{k'}((v^*)^{DL^{\mathcal{P}}}_{k'}, \Gamma^*_{k'}) = \langle v^*, \mathcal{P}^*, \Gamma^*_{\mathcal{P}}\rangle$. By definition of $\xi_{N_{k'}}$, for any $\langle u, \Gamma\rangle \in \mathcal{G}^{DL^{k'}}_{N_{k'}}$ and $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle = \psi^*_{k'}(u, \Gamma)$, it holds that

$$(\xi_{N_{k'}} \circ \psi^*_{k'})_i(u, \Gamma) = \xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}), \qquad \text{for all } i \in N_{k'}. \tag{9}$$

Since ξ meets CEU, for any $\langle v,\mathcal{P},\Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$, for all $C\in N_{k'}/\Gamma_{k'}$, $C\neq N_{k'}$,

$$\sum_{i\in C}\xi_i(v,\mathcal{P},\Gamma_{\mathcal{P}})=v(C).$$

From (7) it follows, in particular, that for any $\langle v,\mathcal{P},\Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$, such that $N_{k'}\in N_{k'}/\Gamma_{k'}$,

$$\sum_{i\in N_{k'}}\xi_i(v,\mathcal{P},\Gamma_{\mathcal{P}})=DL^{\mathcal{P}}_{k'}(v_{\mathcal{P}\Gamma},\Gamma_M).$$

Combining the last two equalities with (9) and recalling that by choice of $\psi^*_{k'}$, $v^{DL^{\mathcal{P}}}_{k'}=u$ and $\Gamma_{k'}=\Gamma$, and therefore for any $C\in N_{k'}/\Gamma$, $C\neq N_{k'}$, $v(C)=v|_{N_{K'}}(C)=v^{DL^{\mathcal{P}}}_{k'}(C)=u(C)$, we obtain that for any $\langle u,\Gamma\rangle\in\mathcal{G}^{DL^{k'}}_{N_{k'}}$, for every $C\in N_{k'}/\Gamma$,

$$\sum_{i\in C}(\xi_{N_{k'}}\circ\psi^*_{k'})_i(u,\Gamma)=\begin{cases} DL^{\mathcal{P}}_{k'}(v_{\mathcal{P}\Gamma},\Gamma_M), & C=N_{k'},\\ u(C), & C\neq N_{k'},\end{cases}$$

with $\langle v_{\mathcal{P}\Gamma},\Gamma_M\rangle$ being the quotient Γ-game for $\langle v,\mathcal{P},\Gamma_{\mathcal{P}}\rangle=\psi^*_{k'}(u,\Gamma)$. Whence, on a set of Γ-games $\mathcal{G}^{DL^{k'}}_{N_{k'}}(DL^{\mathcal{P}}_{k'})$ defined as

$$\mathcal{G}^{DL^{k'}}_{N_{k'}}(DL^{\mathcal{P}}_{k'})=\big\{\langle u,\Gamma\rangle\in\mathcal{G}^{DL^{k'}}_{N_{k'}}\mid u(N_{k'})=DL^{\mathcal{P}}_{k'}(v_{\mathcal{P}\Gamma},\Gamma_M)\text{ for }\langle v,\mathcal{P},\Gamma_{\mathcal{P}}\rangle=\psi^*_{k'}(u,\Gamma)\big\},$$

the Γ-value $\xi_{N_{k'}}\circ\psi^*_{k'}$ meets CE. CE together with $\mathrm{DL}^{k'}$ guarantee that for any $\langle u,\Gamma\rangle\in\mathcal{G}^{DL^{k'}}_{N_{k'}}(DL^{\mathcal{P}}_{k'})$,

$$(\xi_{N_{k'}}\circ\psi^*_{k'})_i(u,\Gamma)=DL^{k'}_i(u,\Gamma),\qquad \text{for all }\ i\in N_{k'}.$$

Observe that by choice of $\psi^*_{k'}$, $\langle (v^*)^{DL^{\mathcal{P}}}_{k'},\Gamma^*_{k'}\rangle\in\mathcal{G}^{DL^{k'}}_{N_{k'}}(DL^{\mathcal{P}}_{k'})$. Hence, in particular, the last equality holds on the Γ-game $\langle (v^*)^{DL^{\mathcal{P}}}_{k'},\Gamma^*_{k'}\rangle$, i.e.,

$$(\xi_{N_{k'}}\circ\psi^*_{k'})_i((v^*)^{DL^{\mathcal{P}}}_{k'},\Gamma^*_{k'})=DL^{k'}_i((v^*)^{DL^{\mathcal{P}}}_{k'},\Gamma^*_{k'}),\qquad \text{for all }\ i\in N_{k'}.$$

Wherefrom, since (9) and by choice of $\psi^*_{k'}$, we obtain that

$$\xi_i(v^*,\mathcal{P}^*,\Gamma^*_{\mathcal{P}})=DL^{k'}_i((v^*)^{DL^{\mathcal{P}}}_{k'},\Gamma^*_{k'}),\qquad \text{for all }\ i\in N_{k'}.$$

Due to the arbitrary choice of both, $\langle v^*,\mathcal{P}^*,\Gamma^*_{\mathcal{P}}\rangle$ and $k'\in M$ for which $n_{k'}>1$, it holds that for any $\langle v,\mathcal{P},\Gamma_{\mathcal{P}}\rangle\in\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$,

$$\xi_i(v,\mathcal{P},\Gamma_{\mathcal{P}})=DL^{k(i)}_i(v^{DL^{\mathcal{P}}}_{k(i)},\Gamma_{k(i)}),\qquad \text{for all }\ i\in N\ \text{ s.t. }\ n_{k(i)}>1. \tag{10}$$

Observe that the proof of equality (10) is based on equality (7) only when $N_k\in N_k/\Gamma_k$, but (7) holds for all N_k, $k\in M$. To exclude any conflict, we show now that on $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$, (10) agrees with (7), when $N_k\notin N_k/\Gamma_k$, as well. Let $\langle v,\mathcal{P},\Gamma_{\mathcal{P}}\rangle\in\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ be such that for some $k''\in M$, $n_{k''}>1$ and $N_{k''}\notin N_{k''}/\Gamma_{k''}$. Then,

$$\sum_{i\in N_{k''}}\xi_i(v,\mathcal{P},\Gamma_{\mathcal{P}})=\sum_{C\in N_{k''}/\Gamma_{k''}}\sum_{i\in C}\xi_i(v,\mathcal{P},\Gamma_{\mathcal{P}})\overset{(10)}{=}\sum_{C\in N_{k''}/\Gamma_{k''}}\sum_{i\in C}DL^{k''}_i(v^{DL^{\mathcal{P}}}_{k''},\Gamma_{k''}).$$

Whence, due to component efficiency of $DL^{k''}$-value and since, for every $C \in N_{k''}/\Gamma_{k''}$, $C \subsetneq N_{k''}$, it holds that $v_{k''}^{DL^{\mathcal{P}}}(C) = v_{k''}(C) = v|_{N_{k''}}(C) = v(C)$, we obtain

$$\sum_{i \in N_{k''}} \xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = \sum_{C \in N_{k''}/\Gamma_{k''}} v(C).$$

By definition of $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}}, \{DL^k\}_{k \in M}}$, the graph structure within a priori unions $\{\Gamma_k\}_{k \in M}$ in $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}} \rangle$ is compatible with $DL^{\mathcal{P}}(v_{\mathcal{P}\Gamma}, \Gamma_M)$, which means that

$$\sum_{C \in N_k/\Gamma_k} v(C) = DL_k^{\mathcal{P}}(v_{\mathcal{P}\Gamma}, \Gamma_M), \qquad \text{for all } k \in M \colon N_k \notin N_k/\Gamma_k. \tag{11}$$

Combining the last two equalities we obtain that (7) holds for k'' as well.

Notice now that (8) and (10) together produce formula (5).

II. To complete the proof we verify that the $P\Gamma$-value ξ on $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}}, \{DL^k\}_{k \in M}}$ given by (5) meets all axioms CEQ, CEU, and $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k \in M} \rangle$. Consider arbitrary $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}} \rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}}, \{DL^k\}_{k \in M}}$. To simplify discussion and w.l.o.g. we assume that for all $k \in M$, $n_k > 1$. Consider some $k \in M$ and let $C \in N_k/\Gamma_k$. Because of component efficiency of DL^k-value, from (5) it follows that

$$\sum_{i \in C} \xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = v_k^{DL^{\mathcal{P}}}(C). \tag{12}$$

If $C \neq N_k$, then $v_k^{DL^{\mathcal{P}}}(C) = v_k(C) = v|_{N_k}(C) = v(C)$. Hence, due to arbitrary choice of k, ξ satisfies CEU. Moreover, from (12) and by definition of $DL_k^{\mathcal{P}}$-game $v_k^{DL^{\mathcal{P}}}$, it also follows that

$$\sum_{i \in N_k} \xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = DL_k^{\mathcal{P}}(v_{\mathcal{P}\Gamma}, \Gamma_M), \qquad \text{for all } k \in M \colon N_k \in N_k/\Gamma_k.$$

Observe that on $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}}, \{DL^k\}_{k \in M}}$, due to validity of equality (11), just proved CEU provides that for all $k \in M$, for which $N_k \notin N_k/\Gamma_k$, the last equality holds as well:

$$\sum_{i \in N_k} \xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = \sum_{C \in N_k/\Gamma_k} \sum_{i \in C} \xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}) \overset{CEU}{=} \sum_{C \in N_k/\Gamma_k} v(C) \overset{(11)}{=} DL_k^{\mathcal{P}}(v_{\mathcal{P}\Gamma}, \Gamma_M).$$

Hence,

$$\sum_{i \in N_k} \xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = DL_k^{\mathcal{P}}(v_{\mathcal{P}\Gamma}, \Gamma_M), \qquad \text{for all } k \in M. \tag{13}$$

Consider $K \in M/\Gamma_M$.

$$\sum_{k \in K} \sum_{i \in N_k} \xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}) \overset{(13)}{=} \sum_{k \in K} DL_k^{\mathcal{P}}(v_{\mathcal{P}\Gamma}, \Gamma_M).$$

Whence and due to component efficiency of $DL^{\mathcal{P}}$-value, we obtain that ξ meets CEQ. Next, let a mapping $\psi_{\mathcal{P}} \colon \mathcal{G}_M^{DL^{\mathcal{P}}} \to \bar{\mathcal{G}}_N^{DL^{\mathcal{P}}, \{DL^k\}_{k \in M}}$ assign to any $\langle u, \Gamma \rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$, the $P\Gamma$-game $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}} \rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}}, \{DL^k\}_{k \in M}}$, such that $v_{\mathcal{P}\Gamma} = u$ and $\Gamma_M = \Gamma$. Then, for any $\langle u, \Gamma\rangle \in \mathcal{G}_M^{DL^{\mathcal{P}}}$ and $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle = \psi^*_{\mathcal{P}}(u, \Gamma)$, by definition of $\xi^{\mathcal{P}}$ and due to (13), it holds

$$(\xi^{\mathcal{P}}\circ\psi_{\mathcal{P}})_k(u,\Gamma)=\xi^{\mathcal{P}}_k(v,\mathcal{P},\Gamma_{\mathcal{P}})=\sum_{i\in N_k}\xi_i(v,\mathcal{P},\Gamma_{\mathcal{P}})\overset{(13)}{=}DL^{\mathcal{P}}_k(v_{\mathcal{P}\Gamma},\Gamma_M),\ \text{for all}\ k\in M.$$

Hence, $(\xi^{\mathcal{P}}\circ\psi_{\mathcal{P}})(u,\Gamma) = DL^{\mathcal{P}}(u,\Gamma)$, i.e., Γ-value $\xi^{\mathcal{P}}\circ\psi_{\mathcal{P}}$ meets DL$^{\mathcal{P}}$. Similarly we can show that for every $k \in M$, Γ-value $\xi_{N_k}\circ\psi_k^{DL^{\mathcal{P}}}$ satisfies DL^k. □

A simple algorithm for computing the $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-value of a $P\Gamma$-game $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ follows from Theorem 1:

- compute the $DL^{\mathcal{P}}$-value of $\langle v_{\mathcal{P}\Gamma}, \Gamma_M\rangle$;
- distribute the rewards $DL^{\mathcal{P}}_k(v_{\mathcal{P}\Gamma},\Gamma_M)$, $k\in M$, obtained by a priori unions among single players applying the DL^k-values to Γ-games $\langle v_k^{DL^{\mathcal{P}}}, \Gamma_k\rangle$ within a priori unions.

Example 1. Consider a numerical example for the $\langle LE, \underbrace{CF,\dots,CF}_{m}\rangle$-value ξ of a $P\Gamma$-game $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle$ with cooperation structure $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$ given by line-graph Γ_M and undirected trees Γ_k, $k \in M$. As we will see below in Sect. 6., the $\langle LE, \underbrace{CF,\dots,CF}_{m}\rangle$-value provides a reasonable solution for the river game with multiple users.

Assume that N contains 6 players, the game v is defined as follows:

$v(\{i\}) = 0$, for all $i \in N$;
$v(\{2,3\}) = 1$, $v(\{4,5\}) = v(\{4,6\}) = 2.8$, $v(\{5,6\}) = 2.9$,
otherwise $v(\{i,j\}) = 0$, for all $i,j\in N$;
$v(\{1,2,3\}) = 2$, $v(\{1,2,3,i\}) = 3$, for $i=4,5,6$; otherwise $v(S) = |S|$, if $|S| \geq 3$;

and the coalition and cooperation structures, respectively, are given by Fig. 2.

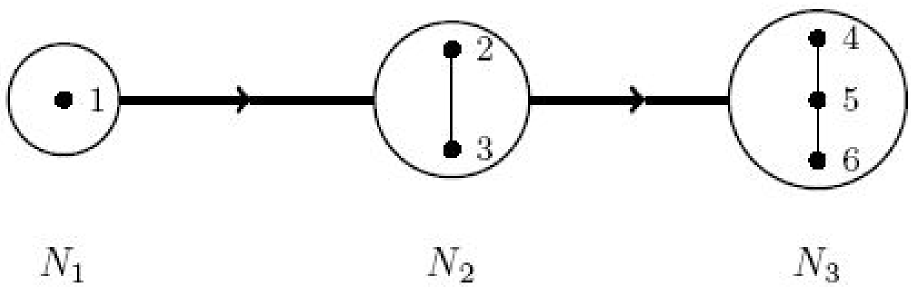

Figure2.

In this case $N = N_1 \cup N_2 \cup N_3$;

$N_1=\{1\}$, $N_2=\{2,3\}$, $N_3=\{4,5,6\}$; $\Gamma_1=\emptyset$, $\Gamma_2=\{\{2,3\}\}$, $\Gamma_3=\{\{4,5\},\{5,6\}\}$;
$M = \{1,2,3\}$; $\Gamma_M = \{(1,2),(2,3)\}$;

the quotient game $v_{\mathcal{P}\Gamma}$ is given by

$$v_{\mathcal{P}\Gamma}(\{1\}) = 0,\ v_{\mathcal{P}\Gamma}(\{2\}) = 1,\ v_{\mathcal{P}\Gamma}(\{3\}) = 3,$$
$$v_{\mathcal{P}\Gamma}(\{1,2\}) = 2,\ v_{\mathcal{P}\Gamma}(\{2,3\}) = 5,\ v_{\mathcal{P}\Gamma}(\{1,3\}) = 4,\ v_{\mathcal{P}\Gamma}(\{1,2,3\}) = 6;$$

the restricted quotient game $v_{\mathcal{P}\Gamma}^{\Gamma_M}$ is

$$v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{1\}) = 0,\ v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{2\}) = 1,\ v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{3\}) = 3,$$
$$v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{1,2\}) = 2,\ v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{2,3\}) = 5,\ v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{1,3\}) = v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{1\}) + v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{3\}) = 3,$$
$$v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{1,2,3\}) = 6;$$

the games v_k, $k = 1, 2, 3$, within a priori unions N_k are given respectively by

$$v_1(\{1\}) = 0;$$
$$v_2(\{2\}) = v_2(\{3\}) = 0,\ v_2(\{2,3\}) = 1;$$
$$v_3(\{4\}) = v_3(\{5\}) = v_3(\{6\}) = 0,\ v_3(\{4,5\}) = v_3(\{4,6\}) = 2.8,\ v_3(\{5,6\}) = 2.9,$$
$$v_3(\{4,5,6\}) = 3;$$

and the restricted games $v_k^{\Gamma_k}$, $k = 1, 2, 3$, within a priori unions N_k are

$$v_1^{\Gamma_1}(\{1\}) = 0;$$
$$v_2^{\Gamma_2}(\{2\}) = v_2^{\Gamma_2}(\{3\}) = 0,\ v_2^{\Gamma_2}(\{2,3\}) = 1;$$
$$v_3^{\Gamma_3}(\{4\}) = v_3^{\Gamma_3}(\{5\}) = v_3^{\Gamma_3}(\{6\}) = 0,\ v_3^{\Gamma_3}(\{4,5\}) = 2.8,\ v_3^{\Gamma_3}(\{4,6\}) = 0,$$
$$v_3^{\Gamma_3}(\{5,6\}) = 2.9,\ v_3^{\Gamma_3}(\{4,5,6\}) = 3.$$

Due to the above algorithm, the PG-value ξ can be obtained by finding of the lower equivalent solution in the line-graph quotient game $\langle v_{\mathcal{P}\Gamma}, \Gamma_M\rangle$ and thereafter the total payoffs to the a priori unions $LE_k(v_{\mathcal{P}\Gamma}, \Gamma_M)$, $k \in M$, should be distributed according to the average-tree solution applied to cycle-free graph LE-games within a priori unions, i.e., for all $i \in N$, $\xi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = AT_i(v_{k(i)}^{LE}, \Gamma_{k(i)})$. Simple computations show that

$$LE_1(v_{\mathcal{P}\Gamma}, \Gamma_M) = v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{1,2,3\}) - v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{2,3\}) = 1,$$
$$LE_2(v_{\mathcal{P}\Gamma}, \Gamma_M) = v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{2,3\}) - v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{3\}) = 2,$$
$$LE_3(v_{\mathcal{P}\Gamma}, \Gamma_M) = v_{\mathcal{P}\Gamma}^{\Gamma_M}(\{3\}) = 3;$$

$$AT_1(v_1^{LE}, \Gamma_1) = LE_1 = 1,$$
$$AT_2(v_2^{LE}, \Gamma_2) = [[LE_2 - v_2(\{3\})] + v_2(\{2\})]/2 = (2+0)/2 = 1,$$
$$AT_3(v_2^{LE}, \Gamma_2) = [v_2(\{3\}) + [LE_2 - v_2(\{2\})]]/2 = (0+2)/2 = 1,$$

$$AT_4(v_3^{LE}, \Gamma_3) = [[LE_3 - v_3(\{5,6\})] + v_3(\{4\}) + v_3(\{4\})]/3 = \\ = [(3-2.9)+0+0]/3 = \frac{1}{30},$$
$$AT_5(v_3^{LE}, \Gamma_3) = [[v_3(\{5,6\}) - v_3(\{6\})] + [LE_3 - v_3(\{4\}) - v_3(\{6\})] + \\ + [v_3(\{4,5\}) - v_3(\{4\})]]/3 = (2.9+3+2.8)/3 = 2\frac{27}{30},$$
$$AT_6(v_3^{LE}, \Gamma_3) = [v_3(\{6\}) + v_3(\{6\}) + [LE_3 - v_3(\{4,5\})]]/3 = \\ = [0+0+(3-2.8)]/3 = \frac{2}{30}.$$

Thus, $\xi(v,\mathcal{P},\Gamma_{\mathcal{P}}) = (1,\,1,\,1,\,\frac{1}{30},\,2\frac{27}{30},\,\frac{2}{30})$.

It was already mentioned before that the $P\Gamma$-games $\langle v,\langle N\rangle,\Gamma_{\langle N\rangle}\rangle$ and $\langle v,\{N\},\Gamma_{\{N\}}\rangle$ reduce to the Γ-game $\langle v,\Gamma_N\rangle$. Whence, any $\langle F,\{DL^k\}_{k\in N}\rangle$-value of $\langle v,\langle N\rangle,\Gamma_{\langle N\rangle}\rangle$ and any $\langle DL,F\rangle$-value of $\langle v,\{N\},\Gamma_{\{N\}}\rangle$ coincide with the Myerson value of $\langle v,\Gamma_N\rangle$; moreover, if the graph Γ_N is complete, they coincide also with the Shapley value and the Owen value. Thereafter note that in a $P\Gamma$-game $\langle v,\mathcal{P},\Gamma_{\mathcal{P}}\rangle$ with any coalition structure $\mathcal{P}$, empty graph Γ_M, and complete graphs Γ_k, $k\in M$, any $\langle DL^{\mathcal{P}},\underbrace{F,\ldots,F}_{m}\rangle$-value coincides with the Aumann-Drèze value of the P-game $\langle v,\mathcal{P}\rangle$. However, the $\langle DL^{\mathcal{P}},\{DL^k\}_{k\in M}\rangle$-value of a $P\Gamma$-game $\langle v,\mathcal{P},\Gamma_{\mathcal{P}}\rangle$ with non-trivial coalition structure $\mathcal{P}$ never coincides with the Owen value (and therefore with the value of Vázquez-Brage et al. (1996), as well) because in our model no cooperation is allowed between a proper subcoalition of any a priori union with members of other a priori unions. On the contrary, the Owen model assumes that every subcoalition of any chosen a priori union may represent this union in the negotiation procedure with other entire a priori unions.

4.2. Stability

Theorem 2. *If the set of DL axioms is restricted to CF, LE, UE, EL, SE, and PE, then the $\langle DL^{\mathcal{P}},\{DL^k\}_{k\in M}\rangle$-value of any superadditive $\langle v,\mathcal{P},\Gamma_{\mathcal{P}}\rangle\in\bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$ belongs to the core $C(v,\mathcal{P},\Gamma_{\mathcal{P}})$.*

Remark 4. Under the hypothesis of Theorem 2, all $\langle DL^{\mathcal{P}},\{DL^k\}_{k\in M}\rangle$-values are combinations of the AT solution for undirected cycle-free Γ-games, the UE, LE, and EL solutions for line-graph Γ-games, and the tree/sink value for rooted/sink forest Γ-games, that are stable on the class of superadditive Γ-games (cf. Herings et al. (2008), Brink et al. (2007), Demange (2004), Khmelnitskaya (2009)).

Proof. For any superadditive $P\Gamma$-game $\langle v,\mathcal{P},\Gamma_{\mathcal{P}}\rangle$, the quotient game $v_{\mathcal{P}\Gamma}$ and games v_k, $k\in M$, within a priori unions are superadditive as well. Due to Remark 4, $DL(v,\Gamma)\in C(v,\Gamma)$, for every superadditive $\langle v,\Gamma\rangle)\in\mathcal{G}_N^{DL}$. Whence,

$$DL^{\mathcal{P}}(v_{\mathcal{P}\Gamma},\Gamma_M)\in C(v_{\mathcal{P}\Gamma},\Gamma_M), \tag{14}$$

$$DL^k(v_k,\Gamma_k)\in C(v_k,\Gamma_k), \qquad \text{for all } k\in M\colon n_k>1. \tag{15}$$

From (14) and because every singleton coalition is connected it follows that

$$DL_k^{\mathcal{P}}(v_{\mathcal{P}\Gamma},\Gamma_M)\geq v_{\mathcal{P}\Gamma}(\{k\})\overset{(2)}{=}v_k^{\Gamma_k}(N_k), \qquad \text{for all } k\in M\colon n_k>1.$$

Observe that, if $N_k\in N_k/\Gamma_k$, the games $v_k^{\Gamma_k}$ and v_k coincide, and therefore, because of the last inequality, the $DL_k^{\mathcal{P}}$-game $v_k^{DL^{\mathcal{P}}}$ is superadditive as well. Thus,

$$DL^k(v_k^{DL^{\mathcal{P}}},\Gamma_k)\in C(v_k^{DL^{\mathcal{P}}},\Gamma_k), \qquad \text{for all } k\in M\colon n_k>1 \ \&\ N_k\in N_k/\Gamma_k. \tag{16}$$

If $N_k\notin N_k/\Gamma_k$, then by definition $C(v_k^{DL^{\mathcal{P}}},\Gamma_k)\overset{(1)}{=}C(v_k,\Gamma_k)$. Besides, by definition any of the following Γ-values: the AT solution for undirected cycle-free Γ-games, the UE, LE, and EL solutions for line-graph Γ-games, and the tree/sink values for rooted/sink forest Γ-games, is defined via the correspondent restricted game. Hence, if $N_k \notin N_k/\Gamma_k$, then $DL^k(v_k^{DL^{\mathcal{P}}}, \Gamma_k) = DL^k(v_k, \Gamma_k)$. Wherefrom, together with the previous equality and because of (16) and (15), we arrive at

$$DL^k(v_k^{DL^{\mathcal{P}}}, \Gamma_k) \in C(v_k^{DL^{\mathcal{P}}}, \Gamma_k), \qquad \text{for all } k \in M\colon n_k > 1. \tag{17}$$

As it is shown in part II of the proof of Theorem 1 (equality (13)), the vector

$$\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle^{\mathcal{P}}(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = \Big\{\sum_{i\in N_k} \langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle_i(v, \mathcal{P}, \Gamma_{\mathcal{P}})\Big\}_{k\in M}$$

is the $DL^{\mathcal{P}}$-value for the quotient Γ-game $\langle v_{\mathcal{P}\Gamma}, \Gamma_M\rangle$. Therefore, from (14),

$$\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle^{\mathcal{P}}(v, \mathcal{P}, \Gamma_{\mathcal{P}}) \in C(v_{\mathcal{P}\Gamma}, \Gamma_M). \tag{18}$$

Further,

$$\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle|_{N_k}(v, \mathcal{P}, \Gamma_{\mathcal{P}}) \overset{(5)}{=} DL^k(v_k^{DL^{\mathcal{P}}}, \Gamma_k), \quad \text{for all } k \in M\colon n_k > 1.$$

Whence together with (17), it follows that

$$\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle|_{N_k}(v, \mathcal{P}, \Gamma_{\mathcal{P}}) \in C(v_k^{DL^{\mathcal{P}}}, \Gamma_k), \qquad \text{for all } k \in M\colon n_k > 1. \tag{19}$$

Due to Proposition 1, (18) and (19) ensure that

$$\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle(v, \mathcal{P}, \Gamma_{\mathcal{P}}) \in C(v, \mathcal{P}, \Gamma_{\mathcal{P}}). \qquad \square$$

Return back to Example 1 and notice that it illustrates Theorem 2 as well. Observe, that v is superadditive, and $\xi(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = \langle LE, CF, CF, CF\rangle(v, \mathcal{P}, \Gamma_{\mathcal{P}}) \in C(v, \mathcal{P}, \Gamma_{\mathcal{P}})$. But $\phi(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = \langle F, F, F, F\rangle(v, \mathcal{P}, \Gamma_{\mathcal{P}})$ being the combination of the Myerson values, i.e., $\phi_i(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = \mu_i(v^{\mu}_{k(i)}, \Gamma_{k(i)})$, $i \in N$, does not belong to $C(v, \mathcal{P}, \Gamma_{\mathcal{P}})$. Indeed, $\phi(v, \mathcal{P}, \Gamma_{\mathcal{P}}) = (0.5, 1, 1, \frac{2}{3}, 2\frac{7}{60}, \frac{43}{60})$. However, since $\phi_4 + \phi_5 = 2\frac{47}{60} < v_3^{\Gamma_3}(\{4,5\}) = 2.8 = 2\frac{48}{60}$, $\phi_{N_3} \notin C(v_3^{\mu}, \Gamma_3)$. Whence, due to Proposition 1, $\phi(v, \mathcal{P}, \Gamma_{\mathcal{P}}) \notin C(v, \mathcal{P}, \Gamma_{\mathcal{P}})$.

Due to Proposition 1, every core selecting $\mathcal{P}\Gamma$-value meets the weaker properties of CEQ and CEU together. Whence and from Theorem 2 the next theorem follows.

Theorem 3. *If the set of DL axioms is restricted to CF, UE, LE, EL, SE, and PE, then the* $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$*-value of a superadditive* $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}}, \{DL^k\}_{k\in M}}$ *is the unique core selector that satisfies* $(m+1)$*-tuple of axioms* $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$.

Now let $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle$ be a superadditive $\mathcal{P}\Gamma$-game in which all graphs in $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$ are either undirected cycle-free, or directed line-graphs or rooted/sink forests, and besides all Γ_k, $k \in M$, are connected. Then there exists a $(m+1)$-tuple of $\langle \mathrm{DL}^{\mathcal{P}}, \{\mathrm{DL}^k\}_{k\in M}\rangle$ axioms of types CF, UE, LE, EL, SE, or PE, for which the cooperation structure $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$ is suitable. Due to Remark 3, $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}}, \{DL^k\}_{k\in M}}$. Whence applying Theorem 2, we obtain that Theorem 4 below holds true. It is worth to note that it is impossible to guarantee that $\{\Gamma_k\}_{k\in M}$, is compatible with $DL^{\mathcal{P}}(v_{\mathcal{P}\Gamma}, \Gamma_M)$, when among Γ_k, $k \in M$, some graphs are disconnected.

Theorem 4. *For every superadditive* $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}},\{DL^k\}_{k\in M}}$*, for which all graphs in* $\Gamma_{\mathcal{P}} = \langle \Gamma_M, \{\Gamma_k\}_{k\in M}\rangle$ *are either undirected cycle-free, or directed line-graphs or rooted/sink forests, and all graphs* Γ_k*,* $k \in M$*, are connected,* $C(v, \mathcal{P}, \Gamma_{\mathcal{P}}) \neq \emptyset$*.*

4.3. Harsanyi Dividends

Consider now $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-values with respect to the distribution of Harsanyi dividends. Since for every $v \in \mathcal{G}_N$ and $S \subseteq N$, it holds that $v(S) = \sum_{T\subseteq N, T\neq\emptyset} \lambda_T^v\, u_T(S)$, where λ_T^v is the dividend of T in v, the Harsanyi dividend of a coalition has a natural interpretation as the extra revenue from cooperation among its players that they could not realize staying in proper subcoalitions. How the value under scrutiny distributes the dividend of a coalition among the players provides the important information concerning the interest of different players to create the coalition. This information is especially important in games with limited cooperation when it might happen that one player (or some group of players) is responsible for the creation of a coalition. In this case, if such a player obtains no quota from the dividend of the coalition, she may simply block at all the coalition creation. This happens, for example, with some values for line-graph games (see discussion in Brink et al. (2007)).

Because of Theorem 1, every $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$-value is a combination of the $DL^{\mathcal{P}}$-value in the quotient Γ-game and DL^k-values, $k \in M$, in the corresponding Γ-games within a priori unions. Whence and by definition of a $P\Gamma$-game we obtain

Proposition 2. *In any* $\langle v, \mathcal{P}, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{\mathcal{P}\Gamma}$ *the only feasible coalitions are either* $S = \bigcup_{k\in Q} N_k$*,* $Q \subseteq M$*, or* $S \subset N_k$*,* $k \in M$*. Every* $\langle DL^{\mathcal{P}}, \{DL^k\}_{k\in M}\rangle$*-value distributes* λ_S^v *of* $S = \bigcup_{k\in Q} N_k$ *according to the* $DL^{\mathcal{P}}$*-value and of* $S \subset N_k$ *according to the* DL^k*-value.*

5. Generalization on Games with Level Structures

Games with (multi)level (coalition) structures were first considered in Winter (1989). A *level structure* on N is a finite sequence of partitions $\mathcal{L} = (\mathcal{P}_1, ..., \mathcal{P}_q)$ such that every $\mathcal{P}_r$, is a refinement of $\mathcal{P}_{r+1}$, that is, if $P \in \mathcal{P}_r$, then $P \subset Q$ for some $Q \in \mathcal{P}_{r+1}$. Similarly as for games with coalition structures, for games with level structures it is assumed that cooperation possible only either between single players within a priori unions $N_k^1 \in \mathcal{P}_1$, $k \in M_1$, at the first level, or at each level $r = 1, \ldots, q-1$ among entire a priori unions $N_k^r, N_l^r \in \mathcal{P}_r$, $k, l \in M_r$, that simultaneously belong to the same element of $\mathcal{P}_{r+1}$, or among entire a priori unions $N_k^q \in \mathcal{P}_q$, $k \in M_q$, at the upper level q, and besides no cooperation is allowed between elements from different levels. It is worth to stress that when we consider cooperation among a priori unions we bear in mind a priori unions as entire units and not as collections of single players or smaller subunions belonging to coalition structures at the lower levels. A *multilevel graph (cooperation) structure* on N is specified by a tuple of graphs $\Gamma_{\mathcal{L}} = \langle \Gamma_{M_q}, \{\{\Gamma_k^r\}_{k\in M_r}\}_{r=1}^q\rangle$, where Γ_{M_q} defines links between a priori unions $N_k^q \in \mathcal{P}_q$, $k \in M_q$ at the upper level q; any Γ_k^r, $k \in M_r$, $r = 2, \ldots, q$, presents links between a priori unions $N_k^{r-1} \in \mathcal{P}_{r-1}$ at the level $r-1$ that belong to the same a priori union $N_k^r \in \mathcal{P}_r$ at the level r; and graphs Γ_k^1, $k \in M_1$, connect single players within a priori unions $N_k^1 \in \mathcal{P}_1$, $k \in M_1$, at the first level. Fig. 3 provides a possible

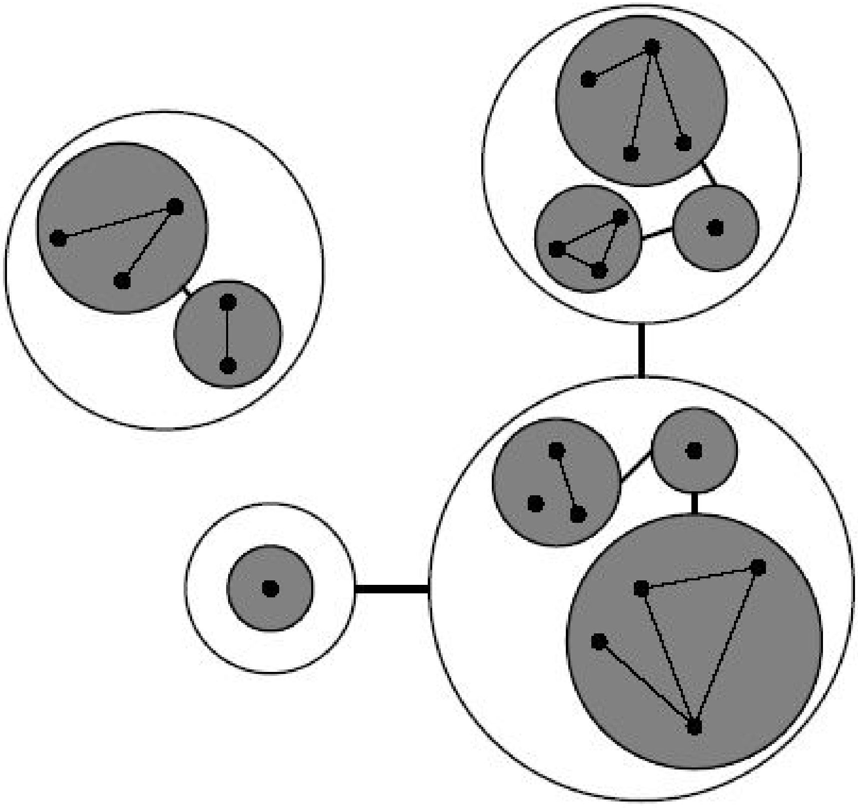

Figure3.

example of the (two-)level (coalition) structure endowed with the three-level graph structure.

A triple $\langle v, \mathcal{L}, \Gamma_{\mathcal{L}}\rangle$ presenting a combination of a TU game $v \in \mathcal{G}_N$ with level structure $\mathcal{L}$ and with limited cooperation possibilities presented via multilevel graph structure $\Gamma_{\mathcal{L}}$ constitutes a *graph game with level structure* or simply *$L\Gamma$-game*. The set of all $L\Gamma$-games with a fixed player set N we denote $\mathcal{G}_N^{\mathcal{L}\Gamma}$. A *$L\Gamma$-value* is defined as a mapping $\xi\colon \mathcal{G}_N^{\mathcal{L}\Gamma} \to \mathbb{R}^N$ that associates with every $\langle v, \mathcal{L}, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{\mathcal{L}\Gamma}$ a vector $\xi(v, \mathcal{L}, \Gamma_{\mathcal{L}}) \in \mathbb{R}^N$.

We extend now the approach suggested to $P\Gamma$-values on $L\Gamma$-values. First adapt the notion of component efficiency. Introduce some extra notation. Let $k_r(i)$ is such that $i \in N^r_{k_r(i)} \in \mathcal{P}_r$, for all $r = 1, \ldots, q$. For every $r = 2, ..., q-1$ and $k_r \in M_r$, let $\mathcal{P}^{k_r}_{r-1} = \{N_k \in \mathcal{P}_{r-1} \mid N_k \subseteq N_{k_r} \in \mathcal{P}_r\}$, $M^{k_r}_{r-1} = \{k \in M_{r-1} \mid N_k \subseteq N_{k_r} \in \mathcal{P}_r\}$, and define a game $v^{k_r}_{r-1}$, $k_r \in M_r$, on $M^{k_r}_{r-1}$ as follows:

$$v^{k_r}_{r-1}(Q) = \begin{cases} v^{r-2,k_{r-1}}_{\mathcal{P}\Gamma}(N_{k_r(k_{r-1})}), & Q = \{k_r(k_{r-1})\}, \\ v(\bigcup\limits_{k\in Q} N_k), & |Q| > 1, \end{cases} \qquad \text{for all } Q \subseteq M^{k_r}_{r-1},$$

where $v^{r-1,k_r}_{\mathcal{P}\Gamma}$ is the quotient restricted game in $P\Gamma$-game $\langle v^{k_r}_{r-1}, \mathcal{P}^{k_r}_{r-1}, \langle \Gamma_{k_r}, \{\Gamma_k\}_{k\in M^{k_r}_{r-1}}\rangle\rangle$. Define a game v_q on M_q as

$$v_q(Q) = \begin{cases} v^{q-2,k_{q-1}}_{\mathcal{P}\Gamma}(N_{k_q(k_{q-1})}), & Q = \{k_q(k_{q-1})\}, \\ v(\bigcup\limits_{k\in Q} N_k), & |Q| > 1, \end{cases} \qquad \text{for all } Q \subseteq M_q,$$

where $v^q_{\mathcal{P}\Gamma}$ is the quotient restricted game in $P\Gamma$-game $\langle v_q, \mathcal{P}_q, \langle \Gamma_{M_q}, \{\Gamma_{k_q}\}_{k_q\in M_q}\rangle\rangle$.

A $L\Gamma$-value ξ is *component efficient in levels* (CEL) if, for any $\langle v, \mathcal{L}, \Gamma_{\mathcal{P}}\rangle \in \mathcal{G}_N^{\mathcal{L}\Gamma}$,

(i) for all $k_1 \in M_1$, for any $C \in N_{k_1}/\Gamma_{k_1}$, $C \neq N_{k_1}$,

$$\sum_{i\in C} \xi_i(v, \mathcal{L}, \Gamma_{\mathcal{L}}) = v(C),$$

(ii) for every level $r = 2, \ldots, q-1$, for all $k_r \in M_r$, for any $C \in N_{k_r}/\Gamma_{k_r}$, $C \neq N_{k_r}$,

$$\sum_{k_r\in C} \sum_{i\in N_{k_r}} \xi_i(v, \mathcal{L}, \Gamma_{\mathcal{L}}) = v^{r-1,k_r}_{\mathcal{P}\Gamma}(C),$$

(iii) for any component $C \in M_q/\Gamma_{M_q}$,

$$\sum_{k_q\in C} \sum_{i\in N_{k_q}} \xi_i(v, \mathcal{L}, \Gamma_{\mathcal{L}}) = v^q_{\mathcal{P}\Gamma}(C).$$

Notice that for $L\Gamma$-games with at least two levels there are three conditions of component efficiency instead of two given by CEU and CEQ for $P\Gamma$-games. This happens because the graph structures within a priori unions at quotient levels $r = 2, \ldots, q-1$ possess peculiarities of both graph structures, within a priori unions containing single players at the first level and among a priori unions at the upper level.

In case of $L\Gamma$-games a few issues, such as the consideration of a tuple of DL axioms $\langle \mathrm{DL}^{\mathcal{P}_q}, \{\{\mathrm{DL}^k\}_{k\in M_r}\}_{r=1}^{q-1}\rangle$ with respect to a $L\Gamma$-value, the compatibility of cooperation structures $\{\Gamma_k\}_{k\in M_r}$, $r=1,\ldots,q-1$, with the payoffs in quotient Γ-games at all upper levels, and the definition of a set of $L\Gamma$-games $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}_q},\{\{DL^k\}_{k\in M_r}\}_{r=1}^{q-1}}$, are much similar to their analogs for $P\Gamma$-games. So, without any loss we skip the detailed discussion over these matters.

Theorem 5. *There is a unique $L\Gamma$-value defined on $\bar{\mathcal{G}}_N^{DL^{\mathcal{P}_q},\{\{DL^k\}_{k\in M_r}\}_{r=1}^{q-1}}$ that meets CEL, and $\langle DL^{\mathcal{P}_q}, \{\{DL^k\}_{k\in M_r}\}_{r=1}^{q-1}\rangle$, and for any $\langle v,\mathcal{L},\Gamma_{\mathcal{L}}\rangle \in \bar{\mathcal{G}}_N^{DL^{\mathcal{P}_q},\{\{DL^k\}_{k\in M_r}\}_{r=1}^{q-1}}$ it is given by*

$$\xi_i(v,\mathcal{L},\Gamma_{\mathcal{L}}) = DL_i^{k_1(i)}(v_{k_1(i)}^{DL^{k_2(i)}}, \Gamma^1_{k_1(i)}), \qquad \textit{for all } i \in N,$$

where for all $r=2,\ldots,q-1$, for any $S \subseteq N_{k_{r-1}(i)}$,

$$v_{k_{r-1}(i)}^{DL^{k_r(i)}}(S) = \begin{cases} DL_{k_{r-1}(i)}^{k_r(i)}(v_{k_r(i)}^{DL^{k_{r+1}(i)}}, \Gamma^r_{k_r(i)}), & S = N_{k_{r-1}(i)}, \\ v_{\mathcal{P}_{r-1}\Gamma}(S), & S \subsetneqq N_{k_{r-1}(i)}, \end{cases}$$

and

$$v_{k_{q-1}(i)}^{DL^{k_q(i)}}(S) = \begin{cases} DL_{k_q(i)}^{\mathcal{P}_q}(v_{\mathcal{P}_q\Gamma}, \Gamma_{M_q}), & S = N_{k_{q-1}(i)}, \\ v_{\mathcal{P}_{q-1}\Gamma}(S), & S \subsetneqq N_{k_{q-1}(i)}, \end{cases} \quad \textit{for all } S \subseteq N_{k_{q-1}(i)}.$$

The proof of Theorem 5 is a straightforward generalization of the proof of Theorem 1 and we leave it to the careful reader.

Theorems 2-4 for $P\Gamma$-values also admit natural extensions on $L\Gamma$-values.

6. Sharing a River with Multiple Users

Ambec and Sprumont (2002) approach the problem of optimal water allocation for a given river with certain capacity over the agents (countries) located along the river from the game theoretic point of view. Their model assumes that between each pair of neighboring agents there is an additional inflow of water. Each agent, in principal, can use all the inflow between itself and its upstream neighbor, however, this allocation in general is not optimal in respect to total welfare. To obtain more profitable allocation it is allowed to allocate more water to downstream agents which in turn can compensate the extra water obtained by side-payments to upstream ones. The problem of optimal water allocation is approached as the problem of optimal welfare distribution. Brink et al. (2007) show that the Ambec-Sprumont river game model can be naturally embedded into the framework of a line-graph Γ-game. In Khmelnitskaya (2009) the line-graph river model is extended to the rooted-tree and sink-tree digraph model of a river with a delta or with multiple sources respectively. All these models consider each agent as a single unit. We extend the model to multiple agents assuming that each agent represents a community of users. However, in our model no cooperation between single users or proper subgroups of users belonging to different agents is allowed, i.e., the presence of international firms having branches at different levels along the river is excluded.

Let $N = \bigcup_{k \in M} N_k$ be a set players (users of water) composed of the communities of users N_k, $k \in M$, located along the river and numbered successively from upstream to downstream. Let $e_{lk} \geq 0$, $k \in M$, l is a predecessor of k, be the inflow of water in front of the most upstream community(ies) (in this case $l = 0$) or the inflow of water entering the river between neighboring communities in front of N_k. Moreover, we assume that each N_k is equipped by a connected pipe system binding all its members. Without loss of generality we may assume that all graphs Γ_k, $k \in M$, presenting pipe systems within communities N_k are cycle free; otherwise it is always possible to close some pipes responsible for cycles. Indeed, for a graph with cycles there is a final set of cycle-free subgraphs with the same set of nodes as in the original graph. It is not a problem to choose an optimal subgraph from this set with respect to minimizing technological costs of water transportation within the community. Fig. 4–6 illustrate the model.

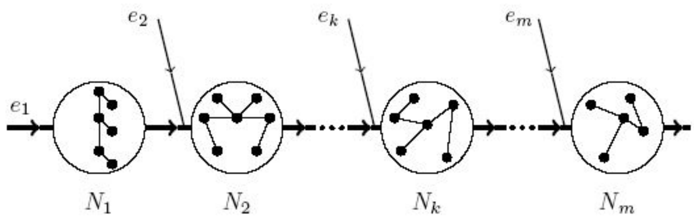

Figure4. Line-graph river

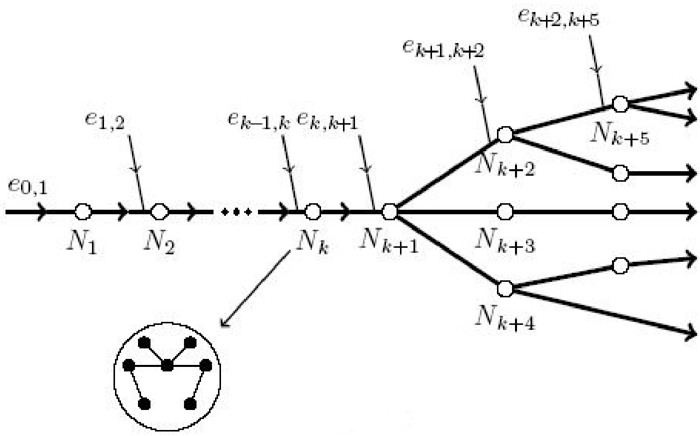

Figure5. River with delta

Following Ambec and Sprumont (2002) it is assumed that for each N_k there is a quasi-linear utility function representing the utility of N_k as a single unit and which is given by $u^k(x_k, t_k) = b^k(x_k) + t_k$ where x_k is the amount of water allocated to N_k, $b^k \colon \mathbb{R}_+ \to \mathbb{R}$ is a continuous nondecreasing function providing benefit $b^k(x_k)$ to N_k through the consumption of x_k of water, and t_k is a monetary compensation to N_k. Moreover, in case of a river with a delta it is also assumed that, if splitting

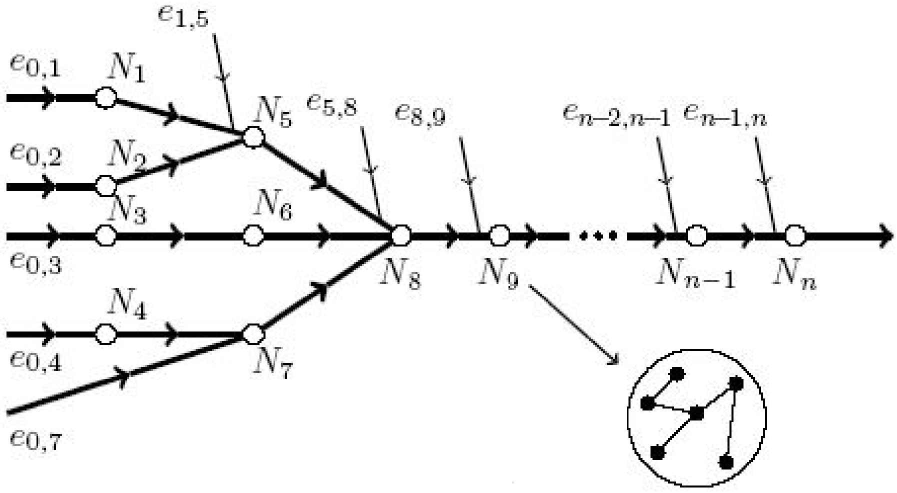

Figure6. River with multiple sources

of the river into branches happens to occur after a certain N_k, then this comunity takes, besides his own quota, also the responsibility to split the rest of the water flow to the branches such to guarantee the realization of the water distribution plan to his successors. Further, we assume that, if the total shares of water for all N_k, $k \in M$, are fixed, then for each N_k there is a mechanism presented in terms of a TU game v_k that allocates the obtained share of water optimally to the players in N_k. We do not discuss how the games v_k, $k \in M$, are constructed and leave it open outside the scope of the paper.

In the model under scrutiny, no cooperation is allowed among single users from different levels along the course of the river. Thus, the problem of optimal water allocation fits the framework of the introduced above $P\Gamma$-game which solution is given by a $P\Gamma$-value that in turn is a combination of solutions for a line-graph, rooted-tree, or sink-tree Γ-game among N_k, $k \in M$, and cycle-free graph games within each N_k. In accordance with the results obtained in Ambec and Sprumont (2002), Brink et al. (2007), Khmelnitskaya (2009), the optimal water distribution among N_k, $k \in M$, can be modeled as a line-graph, rooted-tree, or sink-tree superadditive river game. If all games v_k, $k \in M$, determining water distribution within communities are superadditive as well, then all discussed in the paper $P\Gamma$-values for such type of $P\Gamma$-games are selectors of the core of the river game with multiple users.

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