A Social Capital Index

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Abstract. We define an index of social capital using game-theoretical concepts. We assume that interests of individuals are presented by means of a cooperative game which take into account possible different players abilities whereas the network of relations is modeled by a graph. The social capital of each actor is then measured as the difference between his Myerson value and his Shapley value.

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INTRODUCTION

During last two decades the concept of social capital has been actively popularized by social scientists: economists, sociologists, political scientists. This notion initially appeared to highlight the importance of social networks relations as a valuable resource for social and economic affairs. However, the current meaning of social capital is wider and nowadays it is usually assumed that social capital describes circumstances in which individuals can use membership in groups and networks to secure benefits. This formulation treats social capital as an attribute of an individual that cannot be evaluated without knowledge of the society in which the individual operates. Our main concern in this paper is to introduce a numerical measure of individual social capital for a society composed by a finite set of individuals. We assume that interests of individuals (players) are presented by means of a cooperative game with transferable utility and players' personal bilateral relations are given by a communication graph.

Given a cooperative game and a communication graph we define an *index of a player's individual social capital* as an excess of the Myerson value over the Shapley value of the player which in turn equals to the Shapley value of the player in the game being the difference between Myerson restricted game of the given game and the given game itself. The consideration of the difference between the Myerson and Shapley values provides a tool for revealing the level of a player's social importance due to his social network relations represented by means of a communication graph. Remark that the such defined social capital index is ideologically close to the centrality measure introduced in Gomez et al. [1]. But while in the above mentioned paper the authors define the centrality measure only for evaluation of a player's positional importance in a graph avoiding a priori differences among players and thus using a symmetric game, we define the social capital index as an index of player's positional importance admitting that players possibly have different cooperative abilities. The use of the difference between Myerson and Shapley value has already appeared (outside the symmetric case) in a quite different context in Moretti et al.[3].

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PRELIMINARIES

A *cooperative game with transferable utility* (*TU game*) is a pair $\langle N, v\rangle$, where $N = \{1,\dots,n\}$ is a finite set of $n \geq 2$ players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ (or $S \in 2^N$) of s players is called a *coalition*, and the associated real number $v(S)$ presents the *worth* of S. We denote by $\mathscr{G}_N$ the set of all games with fixed N. The *Shapley value* [5] of a game $v \in \mathscr{G}_N$ can be given by

$$Sh_i(v) = \sum_{T\subseteq N, T\ni i} \frac{\lambda_T^v}{t}, \qquad \text{for all } i \in N,$$

where λ_T^v are the Harsanyi [2] dividends.

In this paper we study cooperative games with limited cooperation possibilities, represented by an undirected communication graph as introduced by Myerson [4]. An *undirected graph* is a collection of unordered pairs of nodes/players $\gamma \subseteq \gamma_N^c = \{\{i,j\} \mid i,j \in N,\ i \neq j\}$, where γ_N^c is the complete undirected graph without loops on N and an unordered pair $\{i,j\}$ presents a *link* between $i,j \in N$. A pair $\langle v,\gamma\rangle$ of a game $v \in \mathscr{G}_N$ and a communication graph γ on N composes a *game with cooperation (graph) structure*, or simply a Γ-*game*. The set of all Γ-games with fixed N we denote $\mathscr{G}_N^\Gamma$. Given a graph γ, a coalition $S \subseteq N$ is said to be *connected*, if the subgraph $\gamma|_S$ is connected. For a given graph γ and a coalition $S \subseteq N$, denote by $C^\gamma(S)$ the set of all connected subcoalitions of S. Any coalition $S \subseteq N$ splits any graph γ into maximally connected subcoalitions called *components*. By S/γ we denote the set of these components. Following Myerson [4], we assume that for a given $\langle v,\gamma\rangle \in \mathscr{G}_N^\Gamma$, cooperation is possible only among connected players and consider a *restricted game* $v^\gamma \in \mathscr{G}_N$ defined as

$$v^\gamma(S) = \sum_{C\in S/\gamma} v(C), \qquad \text{for all } S \subseteq N.$$

Given $\langle v,\gamma\rangle \in \mathscr{G}_N^\Gamma$ the Myerson value, $\mu(v,\gamma)$, of this game with cooperation structure is given by $Sh(v^\gamma)$.

INDEX OF SOCIAL CAPITAL

For every Γ-game $\langle v,\gamma\rangle \in \mathscr{G}_N^\Gamma$ we define a *social capital index* of a player $i \in N$ as

$$SC_i(v,\gamma) = \mu_i(v,\gamma) - Sh_i(v) = Sh_i(v^\gamma - v). \tag{1}$$

Due to its definition via the Shapley and Myerson values, the social capital index satisfies

(i) *Linearity*: for any two Γ-games $\langle v,\gamma\rangle, \langle v',\gamma\rangle \in \mathscr{G}_N^\Gamma$ with the same graph γ and all $\alpha,\beta \in \mathbb{R}$ it holds that for every $i \in N$

$$SC_i(\alpha v + \beta v', \gamma) = \alpha\, SC_i(v,\gamma) + \beta\, SC_i(v',\gamma);$$

(ii) *Fairness*: for any $\langle v,\gamma\rangle \in \mathscr{G}_N^\Gamma$, for any link $\{i,j\} \in \gamma$, $i,j \in N$, it holds that

$$SC_i(v,\gamma) - SC_i(v,\gamma\backslash\{i,j\}) = SC_j(v,\gamma) - SC_j(v,\gamma\backslash\{i,j\}).$$

Proposition 1 *For any Γ-game $\langle v,\gamma\rangle \in \mathscr{G}_N^\Gamma$ the SC-index meets the following properties:*

(i) *Shift Independence:* $SC(v,\gamma) = SC(v_0,\gamma)$, *$v_0$ being the zero normalization of v.*

(ii) *Non-Positivity in Total:* $\sum_{i\in N} SC_i(v,\gamma) \leq 0$;

(iii) *Neutrality for connected graphs: if graph γ is connected, then*

$$\sum_{i\in N} SC_i(v,\gamma) = 0;$$

(iv) *Link Monotonicity for superadditive games: if v is superadditive, then for any link $\{i,j\} \in \gamma$, $i,j \in N$, it holds*

$$SC_i(v,\gamma) \geq SC_i(v,\gamma\backslash\{i,j\}).$$

The following theorem tells us that in the star, the maximum of social capital (with a monotonic game) is obtained at the hub.

Theorem 1 *Given a monotonic Γ-game $\langle v,\sigma^i\rangle \in \mathscr{G}_N^\Gamma$ with σ^i being the star with the hub at $i \in N$, it holds that*

$$SC_i(v,\sigma^i) \geq SC_j(v,\sigma^i), \text{ for all } j \in N \setminus \{i\}.$$

Theorem 2 *For any two superadditive Γ-games $\langle v,\gamma\rangle, \langle v,\sigma^i\rangle \in \mathscr{G}_N^\Gamma$ with the same game v and graph σ^i being a star with the hub at $i \in N$, it holds that*

$$SC_i(v,\sigma^i) \geq SC_i(v,\gamma).$$

The meaning of Theorem 2 is that playing a superadditive game the best option for any player is, if possible, to take the central position among the others.

Theorem 3 *For any two superadditive Γ-games $\langle v,\gamma\rangle, \langle v,\gamma^i\rangle \in \mathscr{G}_N^\Gamma$ with the same game v and graph γ^i on N in which $i \in N$ is an isolated node, it holds that*

$$SC_i(v,\gamma^i) \leq SC_i(v,\gamma).$$

The meaning of Theorem 3 is that playing a superadditive game for any player the worst option is to be isolated.

The next corollary provides upper and lower bounds of the *SC*-index for superadditive Γ-games.

Corollary 1 For any superadditive Γ-game $\langle v,\gamma\rangle \in \mathscr{G}_N^\Gamma$ it holds that

$$-\max_{j\in N} Sh_j(v) \leq SC_i(v,\gamma) \leq \max_{j\in N} \sum_{S \not\ni j} p_s\, v(S), \qquad \text{for all } i \in N,$$

p_s being the coefficient of the marginal contribution of a player to coalition S in the Shapley formula.

Theorem 4 *For any two superadditive Γ-games $\langle v,\gamma\rangle, \langle v,\sigma^i\rangle \in \mathscr{G}_N^\Gamma$ with the same game v and graph σ^i being a star with the hub at $i \in N$, if for every $j \in N$ and all $S \subseteq N\backslash\{i,j\}$ it holds that $v(S\cup\{i\}) \leq v(S\cup\{j\})$, then*

$$SC_i(v,\sigma^i) \geq SC_j(v,\gamma), \qquad \text{for all } j \in N.$$

$$SC_i(v,\gamma^i) \leq SC_j(v,\gamma), \qquad \text{for all } j \in N.$$

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