Two extensions of Young's axiomatization for the Shapley value

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Among single-valued solutions usually called values the most famous and the most appealing is the Shapley value [3]. Different axiomatizations for the Shapley value defined on the entire space of games with fixed set of players are known. Two main of them are the classical one given by Shapley [3] and that of Young [6]. The original Shapley's axiomatization exploits the additivity axiom that being a very beautiful mathematical statement does not express any fairness property. The axiomatization of Young that characterizes the Shapley value by marginality, efficiency, and symmetry appears to be more attractive since all the axioms present different reasonable properties of fair division. The goal of this paper is to present two extensions of the Young's axiomatization for the Shapley value.

Fist, not always we consider the entire space of games. Sometimes due to different reasons we restrict consideration to some subclass of games, e.g. to nonnegative or positive games, to simple games, to convex games, to superadditive games, to constant-sum games, etc. And a reasonable question arises — is Young's axiomatization for the Shapley value on the subclass of games under scrutiny still valid? In [6] Young shows that on the subclass of superadditive games his axiomatization for the Shapley value holds as well. However in general, the answer is negative. For instance, Young's axiomatization for the Shapley value considered on the subclass of all simple games (the Shapley-Shubik power index) is not correct. Indeed, in a simple game with three players the normalized Banzhaf power index presents a counterexample. In a simple game with three players the Shapley-Shubik and normalized Banzhaf indices agree with each other, except in the case where the Shapley-Shubik index assigns to one player the value 2/3 and to each of the other players 1/6; in this case the Banzhaf index assigns to these players the values 3/5 and 1/5 respectively. Yet in the case of simple games with three players the normalized Banzhaf index appears to satisfy the conditions of Young (marginality, efficiency, and symmetry). We prove below that Young's axiomatization is valid for the Shapley value defined on the class of nonnegative constant-sum games with nonzero worth of grand coalition and on the entire class of constant-sum games as well. One might argue that constant-sum games are not that appealing except for their nice mathematical properties. On the other hand, there are indeed relevant classes of nonnegative TU games satisfying the constant sum condition from outset, for example classes of simple majority games. In the literature, other solutions have also been related to the particular class of constant-sum games. For example in [4] it is shown that for this class of games the modified nucleolus coincides with the prenucleolus. The class of nonnegative constant-sum games with nonzero worth of grand coalition and in particular the Shapley value defined on this class appear in the study of semiproportional values in [1].

Second, the Owen value for TU games with coalition structure can be regarded as an expansion of the Shapley value for the situation when a coalition structure is involved. The Owen value was introduced in [2] via a set of axioms it determining. We evolve the Young's approach to the Owen value for games with coalition structure and provide a new axiomatization for the Owen value without additivity axiom that is obtained from the original Owen's one by the replacement of additivity and the null-player property via marginality. We show that the similar axiomatization can be also obtained for the generalization of the Owen value suggested by Winter in [5] for games with level structure.

1 Shapley value for constant-sum games

Recall some definitions and notation. A *cooperative game with transferable utility (TU game)* is a pair $\langle N, v\rangle$, where $N = \{1, \dots, n\}$ is a finite set of $n \geq 2$ players and $v \colon 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ (or $S \in 2^N$) of s players is called a *coalition*, and the associated real number $v(S)$ presents the *worth* of the coalition S. For simplicity of notation and if no ambiguity appears, we write v instead of $\langle N, v\rangle$ when refer to a game, and also omit the braces when writing one-player coalitions such as $\{i\}$. The set of all games with a fixed player set N we denote $\mathcal{G}_N$. For any set of games $\mathcal{G} \subseteq \mathcal{G}_N$, a *value on* $\mathcal{G}$ is a mapping $\psi \colon \mathcal{G} \to \mathbb{R}^n$ that associates with each game $v \in \mathcal{G}$ a vector $\psi(v) \in \mathbb{R}^n$, where the real number $\psi_i(v)$ represents the *payoff* to the player i in the game v.

We say players $i, j \in N$ are *symmetric* with respect to the game $v \in \mathcal{G}$ if they make the same marginal contribution to any coalition, i.e., for any $S \subseteq N \backslash \{i, j\}$, $v(S \cup i) = v(S \cup j)$. A player i is a *null-player* in the game $v \in \mathcal{G}$ if he adds nothing to any coalition non-containing him, i.e., $v(S \cup i) = v(S)$, for every $S \subseteq N \backslash i$.

A value ξ is *marginalist* if, for all $v \in \mathcal{G}$, for every $i \in N$, $\xi_i(v)$ depends only upon the ith marginal utility vector $\{v(S \cup i) - v(S)\}_{S \subseteq N \backslash i}$, i.e.,

$$\xi_i(v) = \phi_i(\{v(S \cup i) - v(S)\}_{S \subseteq N \backslash i}),$$

where $\phi_i \colon \mathbb{R}^{2^{n-1}} \to \mathbb{R}^1$.

A value ξ is *efficient* if, for all $v \in \mathcal{G}$,

$$\sum_{i \in N} \xi_i(v) = v(N).$$

A value ξ is *symmetric* if, for all $v \in \mathcal{G}$, for any permutation $\pi\colon N \to N$, and for all $i \in N$,

$$\xi_{\pi(i)}(v^\pi) = \xi_i(v),$$

where $v^\pi(S) = v(\pi(S))$ for all $S \subseteq N$, $S \neq \emptyset$.

We prove that the Shapley value defined on the class of nonnegative constant-sum games with nonzero worth of grand coalition

$$\mathcal{G}_N^{+c} = \{v \in \mathcal{G}_N \mid v(N) \neq 0,\ v(S) \geq 0,\ v(S) + v(N \backslash S) = v(N), \text{ for all } S \subseteq N\},$$

can be characterized by three Young's axioms of marginality, efficiency, and symmetry.

Theorem 1.1. *The only efficient, symmetric, and marginalist value defined on $\mathcal{G}_N^{+c}$ is the Shapley value.*

It is reasonable to note that the Young's axiomatization is valid as well for the Shapley value defined on the entire class of constant-sum games

$$\mathcal{G}_N^c = \{v \in \mathcal{G}_N \mid v(S) + v(N \backslash S) = v(N), \text{ for all } S \subseteq N\}.$$

2 Owen value for games with coalition structure

Now we restrict our consideration to games with coalition structure. A coalition structure $\mathcal{B} = \{B_1, ..., B_m\}$ on a player set N is a partition of the player set N, i.e., $B_1 \cup ... \cup B_m = N$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. Denote by $\mathfrak{B}_N$ a set of all coalition structures on a fixed player set N. In this context a value is an operator that assigns a vector of payoffs to any pair $(v, \mathcal{B})$ of a game and a coalitional structure on N. More precisely, for any set of games $\mathcal{G} \subseteq \mathcal{G}_N$ and any set of coalition structures $\mathfrak{B} \subseteq \mathfrak{B}_N$, a *coalitional value on $\mathcal{G}$ with a coalition structure from $\mathfrak{B}$* is a mapping $\xi\colon \mathcal{G} \times \mathfrak{B} \to \mathbb{R}^n$ that associates with each pair $\langle v, \mathcal{B}\rangle$ of a game $v \in \mathcal{G}$ and a coalition structure $\mathcal{B} \in \mathfrak{B}$ a vector $\xi(v, \mathcal{B}) \in \mathbb{R}^n$, where the real number $\xi_i(v, \mathcal{B})$ represents the *payoff* to the player i in the game v with the coalition structure $\mathcal{B}$.

A coalitional value ξ is *efficient* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$,

$$\sum_{i \in N} \xi_i(v, \mathcal{B}) = v(N).$$

A coalitional value ξ is *marginalist* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$, for every $i \in N$, $\xi_i(v, \mathcal{B})$ depends only upon the ith marginal utility vector $\{v(S \cup i) - v(S)\}_{S \subseteq N \backslash i}$, i.e.,

$$\xi_i(v, \mathcal{B}) = \phi_i(\{v(S \cup i) - v(S)\}_{S \subseteq N \backslash i}),$$

where $\phi_i\colon \mathbb{R}^{2^{n-1}} \to \mathbb{R}^1$.

A coalitional value ξ possesses the *null-player property* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$, every null-player i in game v gets nothing, i.e., $\xi_i(v, \mathcal{B}) = 0$.

A coalitional value ξ is *additive* if, for any two $v, w \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$, for every $i \in N$,

$$\xi_i(v + w, \mathcal{B}) = \xi_i(v, \mathcal{B}) + \xi_i(w, \mathcal{B}),$$

where $(v + w)(S) = v(S) + w(S)$, for all $S \subseteq N$.

We consider two symmetry axioms. First note that for a given game $v \in \mathcal{G}$ and coalition structure $\mathcal{B} = \{B_1, ..., B_m\} \in \mathfrak{B}$, we can define a game between coalitions or in other terms a *quotient game* $\langle M, v^{\mathcal{B}} \rangle$ with $M = \{1, \ldots, m\}$ in which each coalition B_k acts as a player. We define the quotient game $v^{\mathcal{B}}$ as:

$$v^{\mathcal{B}}(Q) = v(\bigcup_{k \in Q} B_k), \qquad \text{for all } Q \subseteq M.$$

A coalitional value ξ is *symmetric across coalitions* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$, for any two symmetric in $v^{\mathcal{B}}$ players $k, l \in M$, the total payoffs for coalitions B_k, B_l are equal, i.e.,

$$\sum_{i \in B_k} \xi_i(v, \mathcal{B}) = \sum_{i \in B_l} \xi_i(v, \mathcal{B}).$$

A coalitional value ξ is *symmetric within coalitions* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$, any two players who are symmetric in v and belong to the same coalition in B get the same payoffs, i.e., for any $i, j \in B_k \in \mathcal{B}$ that are symmetric in v,

$$\xi_i(v, \mathcal{B}) = \xi_j(v, \mathcal{B}).$$

The Owen value was introduced in Owen [2] as the unique efficient, additive, symmetric across coalitions, and symmetric within coalitions coalitional value that possesses the null-player property.[1]

We prove that the Owen value defined on entire set of games $\mathcal{G}_N$ with any possible coalition structure from $\mathfrak{B}_N$ can be characterized by four axioms of efficiency, marginality, symmetry across coalitions, and symmetry within coalitions.

Theorem 2.1. *The only efficient, marginalist, symmetric across coalitions, and symmetric within coalitions coalitional value defined on $\mathcal{G}_N \times \mathfrak{B}_N$ is the Owen value.*

It is reasonable to note that for some subclasses of games, for example for the subclass of superadditive games or for the subclasses of constant-sum games $\mathcal{G}_N^c$ or $\mathcal{G}_N^{+c}$, if it is desired to stay entirely within one of these subclasses and not in the entire set of games $\mathcal{G}_N$, the same axiomatization for the Owen value via efficiency, marginality, symmetry across coalitions, and symmetry within coalitions is still valid.

Winter [5] introduced a generalization of the Owen value for games with level structure. A *level structure* is a finite sequence of partitions $\mathcal{L} = (\mathcal{B}_1, ..., \mathcal{B}_p)$ such that every $\mathcal{B}_r$ is a refinement of $\mathcal{B}_{r+1}$. Denote by $\mathfrak{L}_N$ the set of all level structures

[1] We present above the original Owen's axioms in the formulation of Winter [5].

on N. In this context, for any set of games $\mathcal{G} \subseteq \mathcal{G}_N$ and any set of level structures $\mathfrak{L} \subseteq \mathfrak{L}_N$, a *level structure value on* $\mathcal{G}$ *with a level structure from* $\mathfrak{L}$ is an operator defined on $\mathcal{G} \times \mathfrak{L}$ that assigns a vector of payoffs to any pair $(v, \mathcal{L})$ of a game $v \in \mathcal{G}$ and a level structure $\mathcal{L} \in \mathfrak{L}$. It is not difficult to see that the Winter's extension of the Owen value for games with level structure admits the similar axiomatization with the replacement of two above mentioned symmetry axioms by the following two captured from [5].

A level structure value ξ is *coalitionally symmetric* if, for all $v \in \mathcal{G}$ and any level structure $\mathcal{L} = (\mathcal{B}_1, ..., \mathcal{B}_p)$, for each level $1 \leq r \leq p$ for any two symmetric in $v^{\mathcal{B}_r}$ players $k, l \in M_r$ such that $B_k, B_l \in \mathcal{B}_r$ are subsets of the same component in $\mathcal{B}_t$ for all $t > r$, the total payoffs for coalitions B_k, B_l are equal, i.e.,

$$\sum_{i \in B_k} \xi_i(v, \mathcal{L}) = \sum_{i \in B_l} \xi_i(v, \mathcal{L}).$$

A level structure value ξ is *symmetric within coalitions* if, for all $v \in \mathcal{G}$ and any level structure $\mathcal{L} = (\mathcal{B}_1, ..., \mathcal{B}_p)$, any two players i, j who are symmetric in v and for every level $1 \leq r \leq p$ simultaneously belong or not to the same non-singleton coalition in $\mathcal{B}_r$, get the same payoffs, i.e., $\xi_i(v, \mathcal{L}) = \xi_j(v, \mathcal{L})$.

Theorem 2.2. *The only efficient, marginalist, coalitionally symmetric, and symmetric within coalitions level structure value defined on* $\mathcal{G}_N \times \mathfrak{L}_N$ *is the Winter value for games with level structure.*

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