Cooperative Game Theory
Introduction

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GATE Master's course 2011-2012

Research Master/M2 GAEXA "Game Theory, Experiments, Applied Econometrics"

January 6, 17, 2012

Game Theory

Common features of all games:

1. there is a set of at least two players;
2. players follow a number of of rules;
3. interests of distinct players are different.

Game theory (GT) is a theory of ***rational*** behavior of agents (people) with ***nonidentical*** interests.

Game theory can be defined as the theory of mathematical models of conflict and cooperation between intelligent rational decision-makers.

Its area of applications extends considerably beyond games in the usual sense.

Game theory is applicable whenever at least two individuals–people, companies, political parties, or nations–confront situations where the outcome for each depends on the behavior of all.

The models of game theory are highly abstract representations of real-life situations.

By the term ***game*** we mean any such situation, defined by some set of ***rules***.

The term ***play*** refers to a particular occurrence of a game.

Modern game theory may be said to begin with the work of Zermelo (1913), Borel (1921), von Neumann (1928), and the great seminal book ***"Theory of Games and Economic Behavior"*** of von Neumann and Morgenstern (1944 (1st edition), 1947 (2nd edition), 1953 (3rd edition)).

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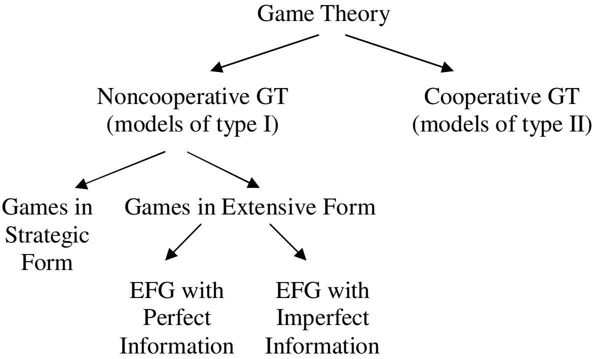
Noncooperative and Cooperative Games

In all GT models the basic entity is a ***player*** (person, firm, agent).

Once we defined the set of players we may distinguish between two types of models:

- primitives are the sets of possible actions of ***individual*** players;
- primitives are the sets of possible ***joint*** actions of ***groups*** of players.


Game Theory

Noncooperative GT (models of type I)

Cooperative GT (models of type II)

Games in Strategic Form

Games in Extensive Form

EFG with Perfect Information

EFG with Imperfect Information

Strategic-Form Games or Games in Normal Form

A *strategic-form* game is $\Gamma = \langle N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, where

$N = \{1, \ldots, n\}$, $n \geq 2$, is a set of *players*,

X_i is a nonempty set of possible *strategies* (or *pure strategies*) of player i. When game Γ is played, each player i must choose $x_i \in X_i$.

Strategy profile $x = (x_1, \ldots, x_n)$ is an *outcome* of the game Γ.

Let $X = \{x = (x_1, \ldots, x_n) \mid x_i \in X_i\}$, the set of all possible outcomes.

$u_i \colon X \to \mathbb{R}$,

The number $u_i(s)$ represents the expected utility *payoff* of player i if the outcome of the game is x.

Equilibrium:

All players in n are happy to find such $x^* \in X$ that

$$u_i(x) \leq u_i(x^*), \qquad \text{for all } \; i \in N, \; x \in X.$$

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Nash Equilibrium

Notation:

Let $x \in X$, $x = (x_1, \dots, x_n)$, $x_i \in X_i$.
$x \| t_i = (x_1, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n)$, i.e. player i replaces his strategy x_i by t_i.

Nash Equilibrium (1950)

An outcome $x^* \in X$ is Nash equilibrium if for all $i \in N$,

$$u_i(x^*) \geq u_i(x^* \| x_i), \qquad \text{for all } x_i \in X_i.$$

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Two-person games

There is a convenient representation of a two-person ($N = \{1, 2\}$) strategic game in which each player has a finite set of strategies.

Let $X_1 = X = \{x_1, \dots, x_n\}$, $X_2 = Y = \{y_1, \dots, y_m\}$,

$$a_{ij} = u_1(x_i, y_j), \; b_{ij} = u_2(x_i, y_j).$$

	y_1	$\dots$	y_m
x_1	(a_{11},b_{11})	$\dots$	(a_{1m},b_{1m})
	$\dots$		$\dots$
x_n	(a_{n1},b_{n1})	$\dots$	(a_{nm},b_{nm})

Battle of the Sexes

This game models a situation in which two players wish to coordinate their behavior but have conflict interests - the wife wants to go to the concert but the husband prefers soccer. But in any case they prefer to spend evening together.

The game has two Nash equilibria: (c,c) and (s,s).

	concert	soccer
concert	2,1	0,0
soccer	0,0	1,2

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Hawk-Dove

Two animals are fighting over some prey. Each can behave like a dove or like a hawk. the best outcome for each animal is that in which it acts like a hawk while the other acts like a dove; the worst outcome is that in which both animals act like hawks. Each animal prefers to be hawkish if its opponent is dovish and dovish if its opponent is hawkish.

The game has two Nash equilibria, (d,h) and (h,d), corresponding to two different conventions about the player who yields.

	dove	hawk
dove	3,3	1,4
hawk	4,1	0,0

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Matching Pennies

Each of two people chooses either Head or Tail. If the choices differ, person 1 pays person 2 one euro; if they are the same, person 2 pays person 1 one euro. Each person cares only about the amount of money that he receives.

The game has **no** Nash equilibria.

	head	tail
head	1,-1	-1,1
tail	-1,1	1,-1

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	head	tail
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The Prisoner's Dilemma

Two suspects in a crime are put into separate cells. If they both confess, each will be sentenced to five years in prison. If only one of them confesses, he will be freed and used as a witness against the other, who will receive a sentence of eight years. If neither confesses, they will both be convicted of a minor offence and spend one year in prison.

The best outcome for the players is that neither confesses, but each player has an incentive to be a "free rider"...

Whatever one player does, the other prefers *confess* to *don't confess*, so the game has unique Nash equilibrium (c,c).

	don't confess	confess
don't confess	-1,-1	-8,0
confess	0,-8	-5,-5

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Cooperative TU Games

$N = \{1, \dots, n\}$ is a finite set of $n \geq 2$ players.

A subset $S \subseteq N$ (or $S \in 2^N$) of s players is a *coalition*.

Given a strategic-form game $\Gamma = \langle N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ with transferable utilities (i.e., utilities which are linear in money) von Neumann and Morgenstern (1953) derive the *worth of coalition* $S \subseteq N$, $S \neq \emptyset$, as its maximum value in the two-person zero-sum game, where S is apposed by its complement $N \backslash S$, and correlated strategies of both S and $N \backslash S$ are used:

$$v(S) = \max_{x_S \in X_S} \min_{x_{N \backslash S} \in X_{N \backslash S}} \sum_{i \in S} u_i(x_1, ..., x_n),$$

when for the grand coalition N,

$$v(N) = \max_{x \in X} \sum_{i \in N} u_i(x_1, ..., x_n).$$

For any $x \in \mathbb{R}^N$ and any $S \subseteq N$, we denote $x(S) = \sum_{i \in S} x_i$.

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A *cooperative game with transferable utilities* (*TU game*) is a pair $\langle N, v\rangle$:

- $N = \{1, \dots, n\}$ is a finite set of $n \geq 2$ players.
- $v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, is the *characteristic function*, $v(S)$ presents the *worth* of the coalition S.

$\mathcal{G}_N$ is the class of TU games with a fixed N.

A TU game may be considered from two different points of view:

- The surplus game interpretation views $v(S)$ as the surplus available to coalition S, namely, the net benefit that agents of S would cash by cooperating.
- In the cost game approach we usually replace notation v for characteristic function by c and consider the worth $c(S)$ of coalition S as a cost of serving all customers in S.

To a cost game $\langle N, c\rangle$ the associated (surplus) game $\langle N, v\rangle$ is

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S), \qquad \text{for all } \ S \subseteq N.$$

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A glove game

Let N be divided into two disjoint subgroups L and R: $N = L \cup R$, $L \cap R = \emptyset$.

Members of L each have one left hand glove, members of R one right hand glove. The worth of a single glove is nothing while the worth of a (left-right) pair is \$1.

This situation can be described by a TU game

$$v(S) = \min\{|L \cup S|, |R \cup S|\}, \qquad \text{for all } \; S \subseteq N,$$

in particular,

$$v(N) = \min\{l, r\}.$$

A land corn production economy

Consider a corn production economy with $n+1$ agents: one landowner, agent 0, and several landless identical peasants, agents $1, 2, \ldots, n$, i.e., $N = \{0, 1, \ldots, n\}$.

The landowner cannot produce anything by himself and need at least one peasant to cultivate the land.

The monetary value of the crop of the land depends on the number of hired peasants.

Let nondecreasing function $f\colon \{0, 1, \ldots, n\} \to \mathbb{R}$ with $f(0) = 0$ denote the total revenue function, i.e., $f(s)$ represents the monetary value of the production level achieved by hiring s peasants.

The corresponding TU game is given by

$$v(S) = \begin{cases} 0, & S \not\ni 0, \\ f(s-1), & S \ni 0, \end{cases} \qquad \text{for all } \ S \subseteq N,$$

in particular,

$$v(N) = f(n) \quad \text{and} \quad v(\{i\}) = 0 \quad \text{for all } \ i \in N.$$

An airport game

The n airlines share the cost of a runaway, N is the set of airlines.

To serve the planes of company i the length of the runaway (roughly proportional to its cost) must be c_i.

Without loss of generality, we assume $c_n \leq c_{n-1} \leq \ldots \leq c_2 \leq c_1$.

This yields the cost sharing game:

$$c(S) = \max_{i \in S} c_i, \qquad \text{for all } \ S \subseteq N.$$

The corresponding cost savings game is

$$v(S) = \sum_{i \in S} c_i - \max_{i \in S} c_i, \qquad \text{for all } \ S \subseteq N.$$

A bankruptcy game

A *bankruptcy problem* $(E; d)$ is defined by a set of claimants N, an estate $E \in \mathbb{R}_+$ and a vector of claims $d = (d_1, ..., d_n \in \mathbb{R}^N_+$ assuming that the total sum of claims of the creditors exceeds the estate,

$$d(N) = \sum_{i \in N} d_i > E.$$

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One Mishnah in the Babylonian Talmud discusses three bankruptcy problems of the division of the estate E of the died person, $E = 100,\ 200,$ and 300 respectively, among his three widows that according to his testament should get $d_1 = 100$, $d_2 = 200$, and $d_3 = 300$ correspondingly. The Mishnah prescribes the following division

		Estate		
		100	200	300
Claim	d_1=100	33.33	50	50
	d_2=200	33.33	75	100
	d_3=300	33.33	75	150

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	d_3=300	33.33	75	150

$$x_1 \leq x_2 \leq \ldots \leq x_n$$

$$(d_1 - x_1) \leq (d_2 - x_2) \leq \ldots \leq (d_n - x_n)$$

A bankruptcy game

The *bankruptcy game* $v_{E;d} \in G_N$ corresponding to bankruptcy problem $(E; d)$ is defined by Aumann and Mashler (1985) as

$$v_{E;d}(S) = \begin{cases} \max\{0,\ E - d(N \backslash S)\}, & S \subseteq N, S \neq \emptyset, \\ 0, & S = \emptyset. \end{cases}$$

		Estate		
		100	200	300
S	1	0	0	0
	2	0	0	0
	3	0	0	0
	12	0	0	0
	13	0	0	100
	23	0	100	200
	123	100	200	300

A simple majority game

Consider a parliament of m seats composed by n parties having correspondingly $m_1, \dots, m_n$ seats.

The threshold on the number of seats to pass a bill is $\left[\frac{\sum\limits_{i \in N} m_i}{2}\right] + 1.$

This voting situation can be described by the game:

N is the set of parties;

$$v(S) = \begin{cases} 1, & \sum\limits_{i \in S} m_i \geq \left[\frac{\sum\limits_{i \in N} m_i}{2}\right] + 1, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for all } S \subseteq N.$$

In this case the game v does not reflect monetary gains but voting power instead.

A coalition is assigned a value of 1 if and only if this coalition has a majority in parliament and therefore is able to pass a bill.

If $v(S) = 1$ then coalition S is a *winning*, otherwise $v(S) = 0$ and S is a *losing*.

Notice that if S is winning then $N \backslash S$ is losing.

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This voting situation can be described by the game:

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$$v(S) = \begin{cases} 1, & \sum_{i\in S} m_i \geq \left[\frac{\sum_{i\in N} m_i}{2}\right] + 1, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{for all } S \subseteq N.$$

In this case the game v does not reflect monetary gains but voting power instead.

A coalition is assigned a value of 1 if and only if this coalition has a majority in parliament and therefore is able to pass a bill.

If $v(S) = 1$ then coalition S is a *winning*, otherwise $v(S) = 0$ and S is a *losing*.

Notice that if S is winning then $N\backslash S$ is losing.

TU Games: properties

Most savings TU games derived from practical situation satisfy *superadditivity*.

A game v is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ such that $S \cap T = \emptyset$.

For cost games the subadditivity replaces the superadditivity.

A cost game c is *subadditive* if $c(S \cup T) \leq c(S) + c(T)$ for all $S, T \subseteq N$ such that $S \cap T = \emptyset$.

All just discussed savings games are superadditive.

The airport cost sharing game is subadditive.

A game v is *additive* if $v(S) + v(T) = v(S \cup T)$ for all $S, T \subseteq N$ such that $S \cap T = \emptyset$.

A game v is *inessential* if $\sum\limits_{i \in N} v(\{i\}) = v(N)$, otherwise v is *essential*.

Theorem

- *Every inessential superadditive game is additive.*
- *For every essential superadditive game v it holds that*

$$v(N) > \sum_{i \in N} v(\{i\}).$$

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TU Games: properties

Proof (1).

a) It is obvious that every additive game is inessential.

b) Let v is inessential and consider $S, T \subseteq N$, $S \cap T = \emptyset$,
From superadditivity of v it follows that

$$v(S) + v(T) \leq v(S \cup T),$$

$$\sum_{i \in S} v(\{i\}) \leq v(S),$$

$$\sum_{i \in T} v(\{i\}) \leq v(T),$$

$$\sum_{i \in N \setminus (S \cup T)} v(\{i\}) \leq v(N \setminus (S \cup T)),$$

$$v(S \cup T) + v(N \setminus (S \cup T)) \leq v(N),$$

and since v is inessential, $v(N) = \sum_{i \in N} v(\{i\})$.

$\Longrightarrow \quad v(S) + v(T) = v(S \cup T).$ ∎

TU Games: properties

A game v is *nonnegative* if $v(S) \geq 0$ for all $S \subseteq N$.

A game v is *monotonic* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$.

Every monotonic game is nonnegative.

Every nonnegative superadditive game is monotonic.

A monotonic game v is said to be *simple* if it takes only the values 0 and 1.

In particular, the *zero* game, i.e., the game v such that $v(S) = 0$ for all $S \subseteq N$, is a simple game.

A simple majority game is simple.

A game v is *symmetric*, if the worth of each coalition depends only on its cardinality.

A land corn production economy game is not symmetric.

A game v is a *constant-sum* game if for all $S \subseteq N$ it holds

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Strategic equivalence of TU games

Two TU games $\langle N, v\rangle$ and $\langle N, v'\rangle$ are called *strategic equivalent* or *S-equivalent* if there exists a positive number $k > 0$ and a vector $a \in \mathbb{R}^N$ such that

$$v'(S) = kv(S) + \sum_{i\in S} a_i, \qquad \text{for all } \ S \subseteq N.$$

If v is S-equivalent to v' we write $v \sim v'$.

S-equivalence is the equivalence relation, i.e., it is reflexive ($v \sim v$), symmetric ($v \sim v' \ \longrightarrow \ v' \sim v$) and transitive ($v \sim v', v' \sim v" \ \longrightarrow \ v \sim v"$).

A game v is $(0, 1)$*-normalized* if

- $v(\{i\}) = 0$ for all $i \in N$;
- $v(N) = 1$.

Theorem

Every essential superadditive game v is S-equivalent to exactly one $(0, 1)$-normalized game.

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Proof.

Assume that $(0,1)$-normalized game v' is S-equivalent to v. Then we have $n+1$ equations

$$v'(\{i\}) = kv(\{i\}) + a_i = 0, \qquad \text{for all } \ i \in N,$$
$$v'(N) = kv(N) + a(N) = 1.$$

$$\Longrightarrow \qquad k = \frac{1}{v(N) - \sum\limits_{i\in N} v(\{i\})} > 0,$$

$$a_i = -\frac{v(\{i\})}{v(N) - \sum\limits_{i\in N} v(\{i\})} \qquad \text{for all } \ i \in N.$$

∎

Payoff vector, imputation set

Every $x = (x_1, ..., x_n) \in \mathbb{R}^N$ can be considered as a *payoff vector* to players in N.

$x \in \mathbb{R}^N$ is *efficient* in the game v if $x(N) = v(N)$.

$x \in \mathbb{R}^N$ is *individually rational* in the game v if $x_i \geq v(\{i\})$ for all $i \in N$.

The *imputation set* of a game $v \in \mathcal{G}_N$ is
$I(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x_i \geq v(i),\ i \in N\}$.

Proposition

In every superadditive game v the imputation set $I(v)$ is nonempty.

Proof. Consider $x \in \mathbb{R}^N$:

$$x_i = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{n}, \qquad \text{for all } i \in N.$$

$x(N) = v(N)$, and from the superadditivity of v it follows that for all $i \in N$, $x_i \geq v(\{i\})$. ∎

In every (0,1)-normalized game v the imputation set $I(v)$ coincides with the unit simplex in $\mathbb{R}^N$:

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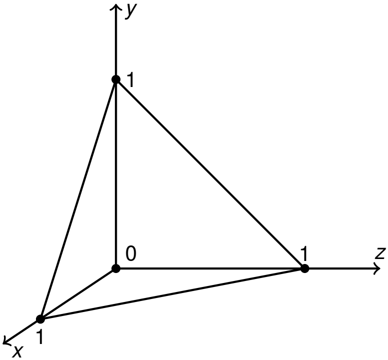
Imputation set for a 3-person (0, 1)-normalized game



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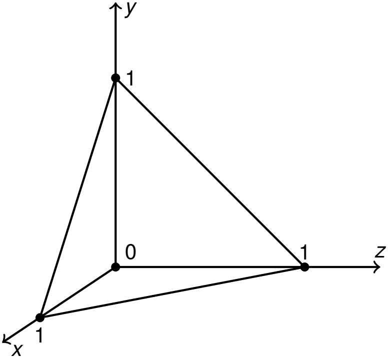

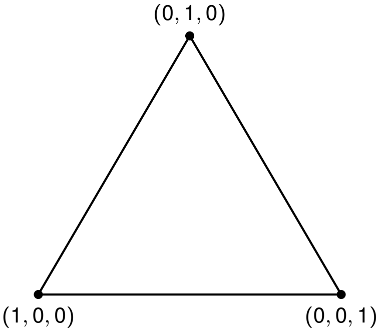

Core

The *core* (Gillies, 1959) of a game $v \in \mathcal{G}_N$ is

$$C(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(S) \geq v(S), \text{ for all } S \subseteq N,\ S \neq \emptyset\}.$$

Example. In a three-person game v with $v(\{i\}) = 0$ for all $i = 1, 2, 3$ and $v(S) = 1$ otherwise, the core $C(v)$ is empty.

Indeed, let $C(v) \neq \emptyset$ and let $x \in C(v)$.

$\Longrightarrow$ $x_1 + x_2 + x_3 = 1$, $x_i \geq 0$, and $x_i + x_j \geq 1$.

$\Longrightarrow$ $x_i = 0$ for all $i = 1, 2, 3$.

$\Longrightarrow$ $x_1 + x_2 + x_3 = 0$ which contradicts the first equality.

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Core

Proposition

In every essential constant-sum game the core is empty.

Proof.

If $x \in C(v)$ then

- $x(N) = v(N)$,
- $x_i \geq v(\{i\})$ and $x(N \backslash \{i\}) \geq v(N \backslash \{i\})$ for all $i \in N$.

By hypothesis

- $\sum_{i \in N} v(\{i\}) \neq v(N)$, since v is essential,
- $v(\{i\}) + v(N \backslash \{i\}) = v(N)$, $i \in N$, since v is constant-sum.

If $\sum_{i \in N} v(\{i\}) > v(N)$ then $C(v) = \emptyset$.

Consider the case when $\sum_{i \in N} v(\{i\}) < v(N)$ and let $x \in C(v)$.

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Core

Proposition

The core of a symmetric game is nonempty if and only if for all $S \subseteq N$ it holds that

$$\frac{v(N)}{n} \geq \frac{v(S)}{s}. \tag{1}$$

Proof.

1. Let (1) holds true and consider $x = \left(\frac{v(N)}{n}, ..., \frac{v(N)}{n}\right) \in \mathbb{R}^N$.

$\Longrightarrow \quad x(N) = v(N)$, and $x(S) = s\frac{v(N)}{n} \geq v(S)$.

$\Longrightarrow \quad x \in C(v)$.

2. Let $C(v) \neq \emptyset$ and $x \in C(v)$.

$\Longrightarrow \quad x(S) \geq v(S)$, $S \subseteq N$, and for any permutation $\pi\colon N \to N$, $x(\pi S) \geq v(S)$, since v is symmetric.

$\Longrightarrow \quad \hat{x} = \dfrac{\sum_{\pi} x(\pi S)}{n!} \in C(v)$, since $C(v)$ is a convex set.

Obviously, $\hat{x} = x = \left(\frac{v(N)}{n}, ..., \frac{v(N)}{n}\right)$ and $\hat{x}(S) = s\frac{v(N)}{n} \geq v(S)$.

Core

Proposition

The core of a symmetric game is nonempty if and only if for all $S \subseteq N$ it holds that

$$\frac{v(N)}{n} \geq \frac{v(S)}{s}. \tag{1}$$

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Core

A necessary (but not sufficient) condition for the core $C(v)$ of game v to be nonempty.

Proposition

If $C(v) \neq \emptyset$, then for any partition of N given by coalitions $S_1, \ldots, S_m \subseteq N$, i.e., $S_i \cap S_j = \emptyset$, $i \neq j$, and $\bigcup_{i=1}^{m} S_i = N$, it holds that

$$\sum_{i=1}^{m} v(S_i) \leq v(N).$$

Core

For a game v we consider a *marginal worth vector* $m^v \in \mathbb{R}^N$ equal to the vector of marginal contributions to the grand coalition,

$$m_i^v = v(N) - v(N \backslash \{i\}), \qquad \text{for all } \ i \in N,$$

and the *gap* vector $g^v \in \mathbb{R}^{2^N}$,

$$g^v(S) = \begin{cases} \sum_{i \in S} m_i^v - v(S), & S \subseteq N, S \neq \emptyset, \\ 0, & S = \emptyset, \end{cases}$$

that for every coalition $S \subseteq N$ measures the total coalitional surplus of marginal contributions to the grand coalition over its worth.

Proposition

In any game v, the vector m^v provides upper bounds of the core: for any $x \in C(v)$,

$$x_i \leq m_i^v, \qquad \text{for all } \ i \in N.$$

In particular, for an arbitrary game v, the condition

$$v(N) \leq \sum_{i \in N} m_i^v$$

is a necessary condition for the core $C(V)$ of game v to be nonempty,

i.e., a strictly negative gap of the grand coalition $g^v(N) < 0$ implies $C(v) = \emptyset$.

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Necessary and sufficient conditions for the nonemptyness of the core

$C(v) \neq \emptyset$ means that there exists $x \in \mathbb{R}^N$: $x(N) = v(N)$ and $x(S) \geq v(S)$, $S \subsetneq N$.

Consider a linear program (LP): *Minimize* $x(N)$ (2)

subject to $x(S) \geq v(S)$ for all $S \subsetneq N$. (3)

LP (2)-(3) is feasible $(x \in \mathbb{R}^N\colon x_i = \max\{0, \max_{S \subsetneq N} v(S)\}$ meets (3))

and bounded from below $(x(N) \geq \sum_{i \in N} v(\{i\}))$

$\Longrightarrow$ LP (2)-(3) always has a solution, let its value be z^*.

Then

- $z^* = v(N) \Longrightarrow C(v)$ coincides with the solution set of LP (2)-(3).
- $z^* < v(N) \Longrightarrow$ any $y \in \mathbb{R}^N$: $y(N) = v(N)$ and $y_i \geq x_i$ with x being a solution of LP (2)-(3) (i.e., $x(N) = z^*$ and x meets (3)), belongs to $C(v)$.
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Necessary and sufficient conditions for the nonemptyness of the core

Consider the dual LP: *Maximize*

$$\sum_{S\subsetneq N} \lambda_S\, v(S) \tag{4}$$

subject to

$$\sum_{\substack{S\subsetneq N\\ S\ni i}} \lambda_S = 1 \quad \text{for all} \quad i \in N. \tag{5}$$

$$\lambda_S \geq 0 \qquad \text{for all} \quad S \subsetneq N. \tag{6}$$

Both LPs (2)-(3) and (4)-(6) are solvable and by the duality theorem they have the same value.

The challenge of the dual LP (4)-(6) is that it contains v only in the objective function and not in the constraints.

Moreover, for all games $v \in \mathcal{G}_N$ the set of feasible vectors $\lambda \in \mathbb{R}^{2^N \setminus \{N\}}$ defined by (5)-(6) is the same.

Proposition

$C(v) \neq \emptyset$ if and only if for all $\lambda \in \mathbb{R}^{2^N \setminus \{N\}}$ that meet constraints (5)-(6) it holds

$$\sum_{S\subsetneq N} \lambda_S\, v(S) \leq v(N). \tag{7}$$

Notice that it is enough to consider the constraints (5) only for $\lambda_S > 0$.

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Necessary and sufficient conditions for the nonemptyness of the core

A collection of coalitions $\mathcal{B} \subset 2^N \backslash \{N\}$ is called *balanced*, if positive numbers $\lambda_S > 0$, $S \in \mathcal{B}$, called *balancing weights*, exist such that

$$\sum_{S \in \mathcal{B}:\ S \ni i} \lambda_S = 1, \qquad \text{for all } \ i \in N.$$

Given a balanced collection of coalitions the system of balancing weights in general might be not unique.

Theorem (Bondareva, 1963; Shapley, 1967, weak form)

A necessary and sufficient condition that the core of a game $\langle N, v \rangle$ is nonempty is that for each balanced collection $\mathcal{B}$ and each system of balancing weights $(\lambda_S)_{S \in \mathcal{B}}$

$$\sum_{S \subsetneq \mathcal{B}} \lambda_S \, v(S) \leq v(N).$$

Necessary and sufficient conditions for the nonemptyness of the core

For fixed N the set of balancing weights forms a convex, compact polyhedron in $\mathbb{R}^{2^N \setminus \{N\}}$.

$\Longrightarrow$ it is enough to check inequality (7) only at its extreme points that correspond minimal balanced collections.

A balanced collection is called *minimal balanced* if it does not contain a proper balanced subcollection.

Every minimal balanced collection has a unique system of balancing weights.

Theorem (Bondareva, 1963; Shapley, 1967, sharp form)

A necessary and sufficient condition that the core of a game $\langle N, v \rangle$ is nonempty is that for each minimal balanced collection $\mathcal{B}$

$$\sum_{S \subsetneq \mathcal{B}} \lambda_S \, v(S) \leq v(N),$$

where $(\lambda_S)_{S \in \mathcal{B}}$ is the system of balancing weights for $\mathcal{B}$.

Necessary and sufficient conditions for the nonemptyness of the core

For $N = \{1,2,3\}$ there are 5 minimal balanced collections:

$\{\{1\},\{2\},\{3\}\}$, $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$, $\{\{3\},\{2,1\}\}$, $\{\{1,2\},\{2,3\},\{3,1\}\}$.

The corresponding systems of balancing weights are:

1. $\lambda_{\{i\}} = 1$, $i = 1,2,3$, $\lambda_S = 1$ otherwise.

2.-4. $i = 1,2,3$: $\lambda_{\{i\}} = 1$, $\lambda_{\{j,k\}} = 1$, $j,k \neq i$, $j \neq k$.

5. $\lambda_{\{i,j\}} = \frac{1}{2}$, $i,j = 1,2,3$, $i \neq j$, $\lambda_S = 1$ otherwise.

Proposition

If $N = \{1,2,3\}$ then $C(v) \neq \emptyset$ if and only if

$$v(N) \geq \max\{v(\{1\}) + v(\{2\}) + v(\{3\}),\ v(\{1\}) + v(\{2,3\}),\ v(\{2\}) + v(\{1,3\}),$$

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For a superadditive game v it is enough to check only the last inequality.

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Interpretation of balancing weights

Balancing weight λ_S can be interpreted as the intensity of coalition (firm) S in its participation in total production.

In case of 3-person games, the first four minimal balanced collections are composed by firms working the full day.

The fifth minimal balanced collection is composed by firms working half a day.

A player i is active in every firm $S \ni i$.

The total intensity of each player is equal to 1:

$$\sum_{\substack{S \subsetneq N \\ S \ni i}} \lambda_S = 1, \quad \text{for all } \ i \in N.$$

The value z^* of both LPs presents the maximal total benefit of N under the full total intensity of all players while the intensity of firms might be not full.

Core of a 3-person game

Let v be a 3-person $(0,1)$-normalized superadditive game.

$$\Longrightarrow \quad v(\{1,2,3\}) = 1, \;\; v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \;\; 0 \leq v(\{i,j\}) \leq 1.$$

Denote $v(\{i,j\}) = c_k$, $i \neq j \neq k$.

Proposition

$C(v) \neq \emptyset$ if and only if $c_1 + c_2 + c_3 \leq 2$.

Proof.

1. Let $x \in C(v)$.

$$\Longrightarrow \quad x_1 + x_2 + x_3 = 1, \;\; x_i \geq 0, \;\; x_1 + x_2 \geq c_3, \;\; x_2 + x_3 \geq c_1, \;\; x_3 + x_1 \geq c_2.$$

$$\Longrightarrow \quad x_1 \leq 1 - c_1, \;\; x_2 \leq 1 - c_2, \;\; x_3 \leq 1 - c_3.$$

Summing up the last three inequalities we obtain $x_1 + x_2 + x_3 \leq 3 - (c_1 + c_2 + c_3)$.

$$\Longrightarrow \quad c_1 + c_2 + c_3 \leq 2.$$

2. Let $c_1 + c_2 + c_3 \leq 2$.

Consider $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \geq 0$: $\displaystyle\sum_{i=1}^{3}(c_i + \epsilon_i) = 2$ and $1 - c_i - \epsilon_i \geq 0$, $i = 1,2,3$.

Let $x_i = 1 - c_i - \epsilon_i \geq 0$, $i = 1,2,3 \;\; \Longrightarrow \;\; x = (x_1, x_2, x_3) \in C(v)$. ∎

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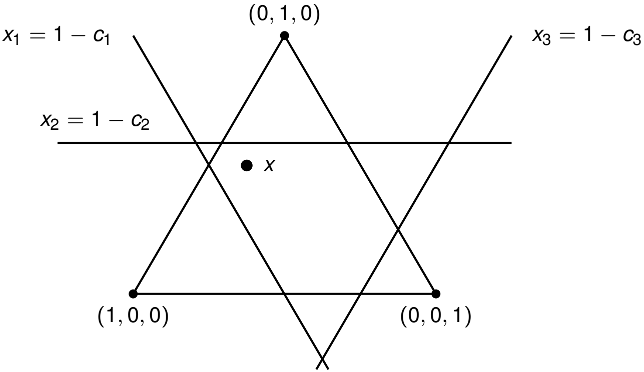
Let $x \in C(v)$, $\Longrightarrow \quad x_i \leq 1 - c_i, \;\; i = 1,2,3.$



Shapley value

For any $\mathcal{G} \subseteq \mathcal{G}_N$, a *value* on $\mathcal{G}$ is a mapping $\xi \colon \mathcal{G} \to \mathbb{R}^N$

The most reasonable approach to the choice of a solution concept is the *axiomatic approach* that allows choosing a solution satisfying a number of a priori chosen properties stated as axioms reflecting reasonable under the circumstances criteria, such as social efficiency, fairness, marginality, simplification of computational aspects etc,.

A value ξ is *efficient* if, for all $v \in \mathcal{G}$, $\sum_{i \in N} \xi_i(v) = v(N)$.

A value ξ possesses the *null-player property* if, for all $v \in \mathcal{G}$, for every null-player i in game v, $\xi_i(v) = 0$.
A player i is a *null-player* in the game $v \in \mathcal{G}$ if
for every $S \subseteq N \backslash i$, $v(S \cup i) = v(S)$.

A value ξ is *symmetric* if, for all $v \in \mathcal{G}$, for any $\pi \colon N \to N$, and for all $i \in N$,

$$\xi_{\pi(i)}(v^{\pi}) = \xi_i(v),$$

where $v^{\pi}(S) = v(\pi(S))$ for all $S \subseteq N$, $S \neq \emptyset$.

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Shapley value

For any $\mathcal{G} \subseteq \mathcal{G}_N$, a *value* on $\mathcal{G}$ is a mapping $\xi\colon \mathcal{G} \to \mathbb{R}^N$

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Shapley value

Theorem (Shapley, 1953)

There is a unique value defined on the class $\mathcal{G}_N$ that satisfies efficiency, symmetry, null-player property, and additivity, and for all $v \in \mathcal{G}_N$, for every $i \in N$, it is given by

$$Sh_i(v) = \sum_{s=0}^{n-1} \frac{s!(n-s-1)!}{n!} \sum_{\substack{S \subseteq N \setminus \{i\} \\ |S|=s}} \big(v(S \cup \{i\}) - v(S)\big). \tag{8}$$

Shapley value

Let Π be a set of all $n!$ permutations $\pi\colon\ N \to N$ of N.
Denote by $\pi^i = \{j \in N \mid \pi(j) \leq \pi(i)\}$ the set of players with rank number not greater than the rank number of i, including i itself.

The *marginal contribution vector* $m^\pi(v) \in \mathbb{R}^N$ of a game v and a permutation π is given by

$$m_i^\pi(v) = v(\pi^i) - v(\pi^i \backslash \{i\}), \qquad \text{for all } \ i \in N.$$

Consider the following *probabilistic distribution model*:

Players enter the room in an arbitrary order, let $\pi = (i_1, ..., i_n)$.
There is $n!$ equally possible orderings.

The first player i_1 gets $v(\{i_1\}) = m_{i_1}^\pi(v)$,

The second player i_2 gets $v(\{i_1, i_2\}) - v(\{i_1\}) = m_{i_2}^\pi(v)$, etc.

The last player i_n gets $v(N) - v(N\backslash\{i_n\}) = m_{i_n}^\pi(v)$.

$$Sh_i(v) = \psi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi} m_i^\pi(v). \tag{9}$$

Players $i, j \in N$ are symmetric with respect to $v \in \mathcal{G}$ if for any $S \subseteq N\backslash\{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$.

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Shapley value

Proof.

1. We show that (9) satisfies efficiency, symmetry, null-player property, and additivity.

Efficiency: $$\sum_{i=1}^{n}\psi_i(v) = \frac{1}{n!}\sum_{\pi\in\Pi}\sum_{i=1}^{n} m_i^{\pi}(v) = \frac{1}{n!}\, n!\, v(N) = v(N).$$

Null-player property: follows straightforwardly.

Additivity: is obvious since everything is linear.

Symmetry:

First show that (9) can be transformed into the form (8).

Fix a coalition S and check how many times $S = \pi^i$.

$$S \qquad i \qquad N\backslash(S\cup\{i\})$$

$$\Longrightarrow \quad \psi_i(v) = \sum_{S\subseteq N\backslash\{i\}} \frac{s!(n-s-1)!}{n!}\big(v(S\cup\{i\})-v(S)\big) =$$

$$= \sum_{s=0}^{n-1}\frac{s!(n-s-1)!}{n!}\sum_{\substack{S\subseteq N\backslash\{i\}\\ |S|=s}}\big(v(S\cup\{i\})-v(S)\big).$$

Let $\pi\in\Pi$,

$$\psi_{\pi(i)}(v^{\pi}) = \sum_{\pi(S)\subseteq N\backslash\{\pi(i)\}}\frac{|\pi(S)|!(n-|\pi(S)|-1)!}{n!}\big(v(\pi(S)\cup\{\pi(i)\})-v(\pi(S))\big) = \psi_i(v).$$

Shapley value

2. Assume that a value ψ on $\mathcal{G}_N$ meets all four axioms.

For every $T \subseteq N$, $T \neq \emptyset$, define the *unanimity* game as

$$u_T(S) = \begin{cases} 1, & T \subseteq S, \\ 0, & T \not\subseteq S, \end{cases} \qquad \text{for all } \ S \subseteq N.$$

Let Δ be arbitrary real number. First we find $\psi(\Delta\, u_T)$.

Every $i \notin T$ is a null-player and therefore gets nothing.

Any $i, j \in T$ are symmetric and by symmetry they get equal payoffs.

$$\Longrightarrow \qquad \psi_i(\Delta\, u_T) = \begin{cases} \dfrac{\Delta}{t}, & i \in T, \\ 0, & i \notin T. \end{cases}$$

It turns out that unanimity games $\{u_T\}_{\substack{T \subseteq N \\ T \neq \emptyset}}$ create a basis in $\mathcal{G}_N$.

To prove that it is enough to show that all u_T, $T \subseteq N$, $T \neq \emptyset$, are linear independent.

Let $\sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \alpha_T\, u_T = 0$, and let not all $\alpha_T = 0$.

$\Longrightarrow$ there is a coalition T_0 such that $\alpha_{T_0} \neq 0$ and for all $T \subset T_0$, $\alpha_T = 0$.

$$\Longrightarrow \quad u_{T_0} = \sum_{T \not\subseteq T_0} -\frac{\alpha_T}{\alpha_{T_0}} u_T \quad \Longrightarrow \quad u_{T_0}(T_0) = \sum_{T \not\subseteq T_0} -\frac{\alpha_T}{\alpha_{T_0}} u_T(T_0).$$

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Shapley value

Corollary.

Any game $v \in \mathcal{G}_N$ can be presented via unanimity basis $\{u_T\}_{\substack{T \subseteq N \\ T \neq \emptyset}}$,

$$v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_T^v \, u_T, \tag{10}$$

Δ_T^v is called the *dividend* of the coalition T in the game v.

$$v(S) = \sum_{\substack{T \subseteq S \\ T \neq \emptyset}} \Delta_T^v, \qquad \text{for all } \; S \subseteq N, \; S \neq \emptyset.$$

Moreover, we obtain another formula representation for the Shapley value:

$$Sh_i(v) = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \frac{\Delta_T^v}{t} \qquad \text{for all } \; i \in N.$$

Shapley value

A value ξ is *marginalist* if, for all $v \in \mathcal{G}$, for every $i \in N$,

$$\xi_i(v) = \phi_i(\{v(S \cup i) - v(S)\}_{S \subseteq N \setminus i}),$$

where $\phi_i \colon \mathbb{R}^{2^{N-1}} \to \mathbb{R}^1$.

Theorem (Young, 1985)

The only efficient, symmetric, and marginalist value defined on the class $\mathcal{G}_N$ is the Shapley value.

Proof. Let index I of a game $v \in \mathcal{G}_N$ be the number of nonzero terms under the summation in (10), i.e.,

$$v = \sum_{r=1}^{I} \Delta_{T_r} u_{T_r},$$

where all $\Delta_{T_r} \neq 0$.

The rest of the proof is done by induction on I.

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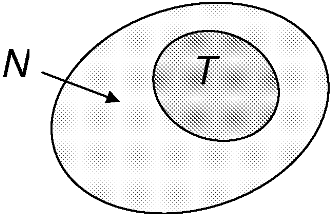
Shapley value

Let ξ be an efficient, symmetric, and marginalist value on $\mathcal{G}_N$.

If $I = 0$, then $v \equiv 0$, and for symmetric 0-game due to efficiency and symmetry of both values, $\xi_i(v) = Sh_i(v) = 0$ for all $i \in N$.

Assume that $\xi(v) = Sh(v)$ if index of v is at most I, and consider v with the index $I + 1$.

Consider $T = \bigcap_{r=1}^{I+1} T_r$.



Shapley value

For any $i \notin T$ consider a game $v^{(i)} = \sum_{r:\ T_r \ni i} \Delta_{T_r} u_{T_r}$.

Index of $v^{(i)}$ is at most I $\xRightarrow{\text{induction hypothesis}}$ $\xi(v^{(i)}) = Sh(v^{(i)})$.

ith marginal utility vectors in both v and $v^{(i)}$ coincide

$$\xRightarrow{\text{marginality}} \quad \xi_i(v) = \xi_i(v^{(i)}), \ \ Sh_i(v) = Sh_i(v^{(i)}).$$

Therefore, $\xi_i(v) = Sh_i(v)$, for all $i \notin T$.

If $T \neq \emptyset$ then all $i, j \in T$ are symmetric and due to symmetry of both values,

$$\xi_i(v) = \xi_j(v), \ \ Sh_i(v) = Sh_j(v), \qquad \text{for all } \ i, j \in N.$$

Thus, because of efficiency,

$$\xi_i(v) = Sh_i(v), \qquad \text{for all } \ i \in N.$$

Shapley value

For any $i \notin T$ consider a game $v^{(i)} = \sum_{r:\ T_r \ni i} \Delta_{T_r} u_{T_r}$.

Index of $v^{(i)}$ is at most I $\overset{\textit{induction hypothesis}}{\Longrightarrow}$ $\xi(v^{(i)}) = Sh(v^{(i)})$.

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$$\overset{\textit{marginality}}{\Longrightarrow} \quad \xi_i(v) = \xi_i(v^{(i)}), \quad Sh_i(v) = Sh_i(v^{(i)}).$$

Therefore, $\xi_i(v) = Sh_i(v)$, for all $i \notin T$.

If $T \neq \emptyset$ then all $i, j \in T$ are symmetric and due to symmetry of both values,

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Shapley value, potential

Let $\mathcal{G}$ denote the set of all TU games.

Given a function $P\colon \mathcal{G} \to \mathbb{R}$, the *marginal contribution* of a player i in a game $\langle N, v\rangle$ is

$$D^i P(N, v) = P(N, v) - P(N\backslash\{i\}, v),$$

where game $\langle N\backslash\{i\}, v\rangle$ is the restriction of $\langle N, v\rangle$ to $N\backslash\{i\}$.

A function $P\colon \mathcal{G} \to \mathbb{R}$ with $P(\emptyset, v) = 0$ is called a *potential function* if for all games $\langle N, v\rangle \in \mathcal{G}$ it satisfies the condition

$$\sum_{i\in N} D^i P(N, v) = v(N). \tag{11}$$

Marginals of a potential function are always efficient.

Theorem (Hart and Mas-Colell, 1988)

There exists a unique potential function P. For every game $\langle N, v\rangle \in \mathcal{G}$

$$P(N, v) = \frac{1}{n}\left[v(N) + \sum_{i\in N} P(N\backslash\{i\}, v)\right],$$

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Convex games

A game v is *convex* if for all $i \in N$ and all $S \subseteq T \subseteq N \backslash \{i\}$,

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T), \tag{12}$$

or equivalently, if for all $S, T \subseteq N$,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T). \tag{13}$$

Proof of the equivalence (12) to (13).

1. (12) is a particular case of (13) for coalitions $S \cup \{i\}$ and T.

2. Assume (12) and rewrite (13) in the equivalent form

$$v(S) - v(S \cap T) \leq v(S \cup T) - v(T). \tag{14}$$

Let $R = S \backslash T$, then $S = (S \cap T) \cup R$ and $S \cup T = T \cup R$.
Assume that $R = \{i_1, ..., i_r\}$.
Applying (12) successively, we get

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Lemma (Shapley, 1971)

Let $v \in \mathcal{G}_N$ be convex. Then every marginal contribution vector $m^\pi(v)$, $\pi \in \Pi$, belongs to the core $C(v)$.

Proof. For simplicity of notation and without loss of generality consider $\pi(N) = \{1, ..., n\}$.

Pick an arbitrary $S \subseteq N$ and let $S = \{i_1, ..., i_s\}$.

For every $1 \leq k \leq s$, apply (12) to $S' = \{i_1, ..., i_{k-1}\}$, $T' = \{1, ..., i_{k-1}\}$, and $i = i_k$:

$$v(i_1, ..., i_k) - v(i_1, ..., i_{k-1}) \leq v(1, ..., i_k) - v(1, ..., i_k - 1) = m^\pi_{i_k}(v).$$

Summing up these inequalities from $k = 1$ to $k = s - 1$, we get

$$v(i_1, ..., i_s) = v(S) \leq \sum_{k=1}^{s} m^\pi_{i_k}(v) = \sum_{i \in S} m^\pi_i(v).$$

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Corollary If $v \in \mathcal{G}_N$ is convex, then $Sh(v) \in C(v)$.

In fact Shapley (1971) proved that in a convex game $C(v) = co\{m^\pi_i(v)\}_{\pi \in \Pi}$.

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Theorem (Ichiishi, 1981)

If for $v \in \mathcal{G}_N$ all marginal contribution vectors $m^\pi(v)$, $\pi \in \Pi$, belong to the core $C(v)$, then v is a convex game.

Proof. Consider arbitrary coalitions $S, T \subseteq N$ and fix any permutation $\pi \in \Pi$ of N that provides a successive ordering of coalitions $S \cap T$, $T \backslash S$, $S \backslash T$, $N \backslash (S \cup T)$.

Let $S \cap T = \{i_1, ..., i_r\}$, $T \backslash S = \{i_{r+1}, ..., i_t\}$, $S \backslash T = \{i_{t+1}, ..., i_q\}$, and $N \backslash (S \cup T) = \{i_{q+1}, ..., i_n\}$, where $r = |S \cap T|$ and $q = |S \cup T|$.

Since $m^\pi(v) \in C(v)$,

$$\sum_{i \in S} m^\pi_i(v) \geq v(S).$$

But

$$\sum_{i \in S} m^\pi_i(v) = \sum_{i \in S} (v(\pi^i) - v(\pi^i \backslash \{i\})) =$$

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Proposition

In any convex game $v \in \mathcal{G}_N$,

$$g^v(N) \geq 0,$$

$$g^v(N) \geq g^v(S), \qquad \textit{for all } S \subseteq N.$$

Proof. The inequality $g^v(N) \geq 0$ follows from the nonemptiness of the core of any convex game.

Next notice that for any $S \subseteq N$,

$$g^v(N) - g^v(S) = \sum_{i \in N \setminus S} \left[v(N) - v(N \setminus \{i\})\right] - \left[v(N) - v(S)\right].$$

Let $N \setminus S = \{i_1, i_2, \ldots i_{n-s}\}$. Then,

$$v(N) - v(S) =$$

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Applying successively $n - s$ times the inequality (12), we obtain that for all $S \subseteq N$, $g^v(N) - g^v(S) \geq 0$. ▮

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A glove game

Let N be divided into two disjoint subgroups L and R: $N = L \cup R$, $L \cap R = \emptyset$.

Members of L each have one left hand glove, members of R one right hand glove.
The worth of a single glove is nothing while the worth of a (left-right) pair is \$1.

This situation can be described by a TU game

$$v(S) = \min\{|L \cup S|, |R \cup S|\}, \qquad \text{for all } \ S \subseteq N,$$

in particular,

$$v(N) = \min\{l, r\}.$$

Superadditivity

Let $S, S' \subseteq N$, $S \cap S' = \emptyset$.
Denote $l_S = |L \cap S|$, $l_{S'} = |L \cap S'|$, $r_S = |R \cap S|$, $r_{S'} = |R \cap S'|$.

a) $l_S \leq r_S$ and $l_{S'} \leq r_{S'} \implies v(S) + v(S') = l_S + l_{S'} = |L \cap (S \cup S')| = v(S \cup S')$.

Similarly, if $l_S \geq r_S$ and $l_{S'} \geq r_{S'}$, $v(S) + v(S') = r_S + r_{S'} = v(S \cup S')$.

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Let N be divided into two disjoint subgroups L and R: $N = L \cup R$, $L \cap R = \emptyset$.

Members of L each have one left hand glove, members of R one right hand glove.
The worth of a single glove is nothing while the worth of a (left-right) pair is \$1.

This situation can be described by a TU game

$$v(S) = \min\{|L \cup S|, |R \cup S|\}, \qquad \text{for all } \; S \subseteq N,$$

in particular,

$$v(N) = \min\{l, r\}.$$

Superadditivity

Let $S, S' \subseteq N$, $S \cap S' = \emptyset$.
Denote $l_S = |L \cap S|$, $l_{S'} = |L \cap S'|$, $r_S = |R \cap S|$, $r_{S'} = |R \cap S'|$.

a) $l_S \leq r_S$ and $l_{S'} \leq r_{S'} \implies v(S) + v(S') = l_S + l_{S'} = |L \cap (S \cup S')| = v(S \cup S')$.

Similarly, if $l_S \geq r_S$ and $l_{S'} \geq r_{S'}$, $v(S) + v(S') = r_S + r_{S'} = v(S \cup S')$.

b) $l_S \leq r_S$, $r_{S'} \leq l_{S'}$ and $l_S + l_{S'} \leq r_S + r_{S'} \implies$
$v(S) + v(S') = l_S + r_{S'} \leq l_S + l_{S'} = v(S \cup S')$.

Similarly, if $l_S \leq r_S$, $r_{S'} \leq l_{S'}$ and $l_S + l_{S'} \geq r_S + r_{S'}$,
$v(S) + v(S') = l_S + r_{S'} \leq r_S + r_{S'} = v(S \cup S')$.

A glove game

Core

If $l \neq r$ then $|C(v)| = 1$.

If $l < r \implies C(v) = \{x^* \in \mathbb{R}^N \mid x_i^* = 1,\ i \in L,\ \text{and}\ x_i^* = 0,\ i \in R\}$.

(i) Show that $x^* \in C(v)$

Let $S \subseteq N$, $l_S = |L \cap S|$, $r_S = |R \cap S| \implies v(S) = \min\{l_S, r_S\}$ and $x^*(S) = l_S$.

a) $l_S \leq r_S \implies v(S) = l_S = x^*(S)$.

b) $l_S > r_S \implies v(S) = r_S < l_S = x^*(S)$.

(ii) Assume that $x \in C(v) \implies x_i \geq 0,\ i \in N$.

$$m^v_{i_r} = v(N) - v(N \backslash \{i_r\}) = l - l = 0 \implies x_{i_r} \leq 0 \implies x_{i_r} = 0.$$

$$m^v_{i_l} = v(N) - v(N \backslash \{i_l\}) = l - (l-1) = 1 \implies 0 \leq x_{i_l} \leq 1 \implies \sum_{i \in L} x_i \leq l.$$

But $\sum\limits_{i \in L} x_i = \sum\limits_{i \in N} x_i = v(N) = l \implies x_{i_l} = 1$.

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A land corn production economy

Consider a corn production economy with $n+1$ agents: one landowner, agent 0, and several landless identical peasants, agents $1, 2, ..., n$, i.e., $N = \{0, 1, ..., n\}$.

The landowner cannot produce anything by himself.

The monetary value of the crop of the land depends on the number of hired peasants.

Let nondecreasing function $f\colon \{0, 1, ..., n\} \to \mathbb{R}$ with $f(0) = 0$ denote the total revenue function, i.e., $f(s)$ represents the monetary value of the production level achieved by hiring s peasants.

The corresponding TU game is given by

$$v(S) = \begin{cases} 0, & S \not\ni 0, \\ f(s-1), & S \ni 0, \end{cases} \qquad \text{for all } \ S \subseteq N,$$

in particular,

$$v(N) = f(n) \quad \text{and} \quad v(\{i\}) = 0 \quad \text{for all } \ i \in N.$$

Superadditivity

Let $S, T \subseteq N$, $S \cap T = \emptyset$.

a) If $S \cup T \not\ni 0 \implies S \not\ni 0$ and $T \not\ni 0 \implies v(S) + v(T) = 0 + 0 = v(S \cup T)$.

b) If $S \cup T \ni 0$ then it is enough to consider only the case $s, t \geq 2$

$$\implies v(S) + v(T) = \begin{cases} f(s-1), & S \ni 0,\ T \not\ni 0, \\ f(t-1), & S \not\ni 0,\ T \ni 0, \end{cases} \leq f(s+t-1) = v(S \cup T).$$

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A land corn production economy

Shapley value

All orderings of players $0, 1, ..., n$ are uniformly distributed.

$\Longrightarrow$ in any random ordering of players the landowner 0 is at kth position, $k = 1, ..., n+1$, with probability $\frac{1}{n+1}$ and his marginal contribution is equal to $f(k-1)$ because $(k-1)$ peasants are before him.

$$\Longrightarrow \quad Sh_0(v) = \frac{1}{n+1}\sum_{k=1}^{n+1} f(k-1) = \frac{1}{n+1}\sum_{k=1}^{n} f(k).$$

All peasants $i = 1, ..., n$ are identical

$$\Longrightarrow \quad Sh_i(v) = \frac{1}{n}\Big(f(n) - \frac{1}{n+1}\sum_{k=1}^{n} f(k)\Big), \qquad \text{for all } i = 1, ..., n.$$

Core

$C(v) \neq \emptyset$: $C(v) \ni x = (x_0, x_1, ..., x_n), x_0 = f(n),\ x_i = 0,\ i = 1, ..., n.$

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An airport game

N is the set of n airlines.

c_i is the cost to serve the company i, $i \in N$; w.l.g. $\quad c_n \leq c_{n-1} \leq \ldots \leq c_2 \leq c_1$.

The cost sharing game: $\quad c(S) = \max_{i \in S} c_i, \quad$ for all $S \subseteq N$.

The corresponding cost savings game: $\quad v(S) = \sum_{i \in S} c_i - \max_{i \in S} c_i, \quad$ for all $S \subseteq N$.

Superadditivity

Let $S, T \subseteq N$, $S \cap T = \emptyset$.

$$v(S) + v(T) = \sum_{i \in S} c_i - \max_{i \in S} c_i + \sum_{i \in T} c_i - \max_{i \in T} c_i = \sum_{i \in S \cup T} c_i - \max_{i \in S} c_i - \max_{i \in T} c_i$$

$$\max_{i \in S \cup T} c_i = \max\{\max_{i \in S} c_i, \max_{i \in T} c_i\} \quad \Longrightarrow \quad v(S) + v(T) \leq v(S \cup T).$$

Convexity

Let $S, T \subseteq N$.

$$v(S) + v(T) = \sum_{i \in S} c_i - \max_{i \in S} c_i + \sum_{i \in T} c_i - \max_{i \in T} c_i = \sum_{i \in S} c_i + \sum_{i \in T \setminus S} c_i + \sum_{i \in S \cap T} c_i - \max_{i \in S} c_i - \max_{i \in T} c_i.$$

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An airport game

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An airport game

Shapley value

Let $c_{n+1} = 0 \implies c_{n+1} = 0 \leq c_n \leq c_{n-1} \leq \ldots \leq c_2 \leq c_1$.

For every $1 \leq k \leq n$ consider the game c^k:

$$c^k(S) = \begin{cases} c_k - c_{k+1}, & S \cap \{1, ..., k\} \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{for all } S \subseteq N,$$

Show that $c = \sum_{k=1}^{n} c^k$. Let $S \subseteq N$ and let $\max_{i \in S} c_i = c_j$, $j \in S$.

$$\implies \sum_{k=1}^{n} c^k(S) = \sum_{k=j}^{n} c^k(S) = (c_j - c_{j+1}) + ... + (c_n - c_{n+1}) = c_j = c(S).$$

Every $i \in \{k+1, ..., n\}$ is a 0-player in game $c^k \implies Sh_i(c^k) = 0$.

Any $i, j \in \{1, ..., k\}$ are symmetric players in game $c^k \implies Sh_i(c^k) = Sh_j(c^k)$.

$c^k(N) = c_k - c_{k=1}$ for all $1 \leq k \leq n$.

$$\implies \qquad Sh_i(c^k) = \begin{cases} \dfrac{c_k - c_{k+1}}{k}, & i \in \{1, ..., k\}, \\ 0, & i \in \{k+1, ..., n\}, \end{cases}$$

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A bankruptcy game

The *bankruptcy game* $v_{E;d} \in G_N$ corresponding to bankruptcy problem $(E; d)$, $E < d(N)$, is defined as

$$v_{E;d}(S) = \begin{cases} \max\{0,\ E - d(N \backslash S)\}, & S \subseteq N, S \neq \emptyset, \\ 0, & S = \emptyset. \end{cases}$$

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$\Longrightarrow \quad v(S) + v(T) = \max\{0, d(S) - \alpha\} + \max\{0, d(T) - \alpha\}$.

a) $d(S) - \alpha < 0$ and $d(T) - \alpha < 0$

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since it is possible that $v(S \cap T) - \alpha < 0$.

A bankruptcy game

Convexity

Let $S, T \subseteq N$.

$$v(S) + v(T) = \max\{0, d(S) - \alpha\} + \max\{0, d(T) - \alpha\}.$$

a) $d(S) - \alpha < 0$ and $d(T) - \alpha < 0$

$\Longrightarrow \quad v(S) + v(T) = 0 + 0 \leq v(S \cup T) + v(S \cap T)$.

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Core

$$C(v) = \{x \in \mathbb{R}^N \mid x(N) = E,\ 0 \le x_i \le d_i,\ i \in N\}$$

Proof.

(*i*) Let $x \in C(v) \implies x(N) = v(N) = \max\{0, E - d(N \setminus N)\} = E$,

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Nucleolus

For a game v, the *excess* of a coalition $S \subseteq N$ with respect to a payoff vector $x \in \mathbb{R}^N$ is

$$e^v(S,x) = v(S) - x(S).$$

$x \in C(v) \Leftrightarrow e^v(S,x) \leq 0$ for all $S \subseteq N$.

For $x, y \in \mathbb{R}^m$, we say that x is *lexicographically smaller* than y (notation $x \prec_{lex} y$) if $x = y$ or if there exists $j \in \{1, ..., m\}$: $x_i = y_i$ for all $i < j$ and $x_j < y_j$.

For example, for $m = 3$ we have $(0, 100, 100) \prec_{lex} (1, -10, -10)$ and $(10, 4, 100) \prec_{lex} (10, 5, 6)$.

The *nucleolus* of a game v (Schmeidler, 1969) is a minimizer of the lexicographic ordering of components of the excess vector of a given game v arranged in decreasing order of their magnitude over the imputation set $I(v)$:

$$\nu(v) = x \in I(v) : \ \theta(x) \preceq_{lex} \theta(y), \ \forall y \in I(v),$$

where $\theta(x) = (e(S_1, x), e(S_2, x), ..., e(S_{2^n-1}, x))$,
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Nucleolus

Proposition

If $C(v) \neq \emptyset$ *then* $\nu(v) \in C(v)$.

Proof.

$x \in C(v) \Leftrightarrow e^v(S,x) \leq 0$ for all $S \subseteq N \Leftrightarrow \theta(x) \leq 0$.

Take $x \in C(v) \implies$ by definition of the nucleolus $\theta(\nu(v)) \preceq_{lex} \theta(x)$,

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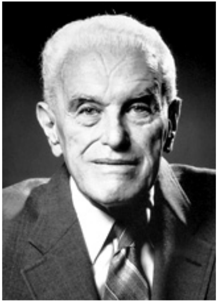
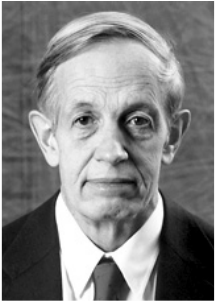
The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1994

"for their pioneering analysis of equilibria in the theory of non-cooperative games"



John C. Harsanyi
(1920-2000)

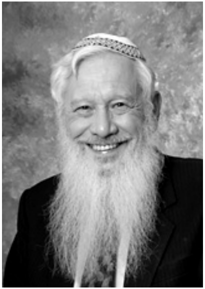


John F. Nash Jr.
b. 1928

Reinhard Selten
b. 1930

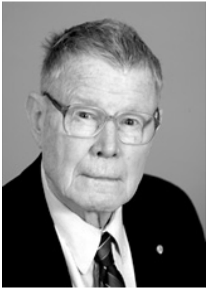
The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2005

"for having enhanced our understanding of conflict and cooperation through game-theory analysis"



Robert J. Aumann

b. 1930

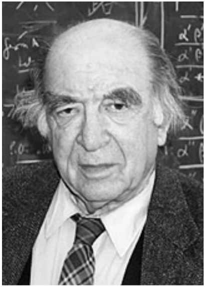


Thomas C. Schelling

b. 1921

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2007

"for having laid the foundations of mechanism design theory"



Leonid Hurwicz
(1917-2008)

Eric S. Maskin
b. 1950

Roger B. Myerson
b. 1951

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