VALUATION OF CREDIT DEFAULT SWAPS AND SWAPTIONS

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Abstract. This paper presents a conceptual and general framework for valuation of single-name credit derivatives. The general subfiltration approach of [J-R] to modelling default risk, which includes the Cox-process setting of [L], is integrated with a numeraire invariant approach. Several known results are reformulated and extended in this framework. New concepts and results are presented for change of numeraire in presence of default and valuation of credit swaptions. A new formula on fractional recovery of pre-default value is derived, generalizing that of [D-S]. A Black-Scholes formula for credit default swaptions due to [S] is shown to serve as a least-squares approximation to the general case.

Key words: Credit default swap, swaption, swap rate, subfiltration, conditional survival probability, preprice, prenumeraire, recovery, coadapted numeraires.
JEL Classification: G13

1. Introduction

This paper aims to develop a valuation framework for single-name credit derivatives that combines an arbitrary numeraire with a general conditional probability of survival arising from a subfiltration. We endeavor at a conceptual exposition by starting from first financial principles and proceeding in a self-contained manner within a general probabilistic framework. Along the way, we introduce new concepts and terminology to facilitate the discussion, and to especially highlight our views on cashflow modelling and change of numeraire in presence of default. We hope to have achieved a satisfactory generality in regards to formalization of common single-name structures and their valuation.

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Other important issues like hedging and market completeness, or topics like credit migration, are not addressed.

Aside from reformulation of several known results and various observations of our own, we aim at two main new results in the present framework. The first is a pricing formula for an option to swap any two credit claims which have zero-recovery should default occur before swaption expiration (but they can have non-zero recovery should default occur after expiration). The second is an extension of a result of [D-S] on fractional recovery of pre-default value. It is a non-trivial extension, as it replaces the ordinary exponential in the [D-S] formula by a stochastic exponential, and also involves stochastic integrals.

We model default following the "subfiltration approach" developed in [E-J-Y] and [J-R]. The subfiltration is viewed as a large information subset that explains the likelihood of default but can not predict it exactly. The key quantity in this approach is the *conditional probability of survival* (or of default) given knowledge of the subfiltration. In special cases, it is related to the notions of default hazard rate or preintensity in [L] and [D-S], and in fact equals the exponential of the negative of the integral of the latter.

[E-J-Y] and [J-R] derive a pricing formula, in the subfiltration setting, for credit claims with a promised payoff but zero recovery. Actually, it is a relatively simple generalization of a well known formula in [L] and [D-S]. It serves as the starting point of our main results and is basic to the paper. [J-R] also derive an extension of an equally well-known pricing formula in [L] and [D-S] for credit claims with non-zero recovery. The latter generalization is a significant one, as it replaces an ordinary integral by a stochastic integral. This result of [J-R] serves as the starting point of our second main result, though we modify their technical assumptions to suit our purposes.

Our first swaption pricing formula is with respect to an arbitrary numeraire. Our second swaption result applies the first in a special numeraire. It assumes that at least one of the two zero-recovery swap legs is a "prenumeraire", which implies it has a positive price before default. It then constructs an equivalent "prenumeraire measure" and a "par swap rate" process adapted to the subfiltration. This process is a martingale under the equivalent prenumeraire measure, which, before default, equals the ratio of the two asset prices. After default both leg prices are zero. Here, the (par) swap rate resolves in a unique and natural way the indeterminacy of dividing zero by zero. As such, the swap rate can be thought of as some kind of a "generalized ratio", a process that extends the ratio to all paths.

Valuation of credit default swaptions has also been studied in [S]. Aside from the generality of the present framework and the generality of credit claims to which it is applied, our approach is different and arguably more versatile. [S] utilizes the absolutely continuous measure induced by taking one of the two swap legs as numeraire (which is not a numeraire in our sense, as it is not positive on all paths). Since the swap leg has zero recovery prior to swaption expiration, this induced measure assigns zero probability to the event of default before expiration, and hence is not an equivalent measure. Nevertheless, any extension past default of the ratio of the two legs prices is a martingale under this measure. As such, this approach provides a pricing formula for credit swaptions as an expectation under an inequivalent measure. It leads to the Black-Scholes formula if the ratio is assumed to be lognormal under this measure.

The approach of [S] bypasses certain theoretically significant constructs. It is theoretically desirable to establish that there exists an extension of the ratio that is a martingale not just in the inequivalent measure associated with the swap leg, but also in an *equivalent* measure naturally constructed from the swap leg. This has practical significance too, as one can now incorporate into computation established dynamics for interest rates and other relevant market variables, which in practice are invariably modelled under equivalent measures. To summarize our view, our joint construction of a generalized ratio and an equivalent measure penetrates a bit deeper than the comparatively plain construction of the absolutely continuous measure in [S].

A numeraire-invariant approach is properly built on the notion of state price density or deflator, a financially intuitive concept that embodies both interest rate and risk premium information. In this context, the price of a contingent claim is considered to be a process whose *deflated* value (that is, its value times the deflator) is a martingale in the actual probability measure. We define a numeraire to be a claim with positive price. Any numeraire induces a probability measure, the so-called numeraire measure, under which the ratio of other claim prices over the numeraire price is a martingale. This reduces pricing to (conditional) expectation valuation. In practice, one can employ a numeraire whose price is deemed observable.

Valuation theory under arbitrary numeraire is conducive to the freedom of choosing appropriate numeraires for different categories of claims. Many such applications are known, a chief instance being an option to swap two assets. Nevertheless, with almost full unanimity, existing approaches to credit risk modelling employ only one numeraire, namely, the continuous money market numeraire, defined as the exponential of the integral of the instantaneous interest rate. Even [S], who changes numeraire, begins the initial formulation in terms of the continuous money market numeraire. ([E-J-Y] assume for simplicity that interest rates are zero throughout, and [J-R] do the same except in few cases.) We find it important to treat general numeraires, and we do not impose such restrictions. At the least, a general numeraire approach offers a more compact and convenient notation over the relatively long expression that the money market numeraire entails.

The Cox process approach to credit risk modelling, developed in [L], is an important special case of the general subfiltration framework. The subfiltration in this setting turns out to be a conditionally independent subfiltration: every process adapted to the subfiltration which is a martingale under the subfiltration is also a martingale under the original filtration. [B-S-W] develop a default framework which is more general than the Cox-process setting, but also assumes that the subfiltration is conditionally independent. Their framework encompasses structural models too. Ours excludes structural models, but does not require the subfiltration to be conditionally independent. [B-S-W] develop their framework in mostly different directions than we develop ours.

The conditional survival probability in the Cox-process setting turns out to be absolutely continuous and decreasing. We do not require that the conditional survival probability be either continuous or of finite variation, and allow it to be a general (suitably integrable) semimartingale in all except a single one of our results. We make note in passing how some of the general results simplify if those assumptions are made.

Esthetics aside, the generality of the subfiltration approach makes it useful for some interesting cases which violate the assumptions of the Cox-process setting. Some examples are given in [J-R] and [E-J-Y] (and the references there). Abundant examples of conditional survival probabilities that are neither continuous nor decreasing emerge from multi-name correlated default models. And, the recent economically feasible model of [G-G] furnishes an example in which continuity holds, but absolute continuity fails. Theirs is a hybrid, incomplete information, structural model, in which default occurs when firm value drops below a non-observable random threshold.

A byproduct of the theory is several new credit risk concepts that encapsulate main ideas of the paper and orient them for application. Chief among them are the notions of *preprice*, *prenumeraire*, *prenumeraire measure*, *swap rate* , and *coadapted numeraires*. The preprice of a claim can be interpreted as the price calculated by an agent who can only observe the incomplete subfiltration information. A main task is to express prices in terms of preprices. A prenumeraire is a claim with a positive preprice. To any prenumeraire we associate in a natural way an equivalent measure called prenumeraire measure. We will show that a survival swaption price equals the prenumeraire times the truncated first moment of the swap rate in the prenumeraire measure. Coadapted numeraires form equivalent classes under each of which key quantities remain invariant.

The introductory subsections that follow discuss informally the subfiltration approach to modelling default risk, and compare it to the earlier Cox-process approach of [L] and preintensity approach of [D-S]. For a more complete overview of different approaches, we refer the reader to the relatively recent survey of [J-R (b)].

Formal modelling begins in section 2, which defines the framework, beginning with claims and numeraires. Theorem 2.1 sets an agenda for the paper. The subfiltration approach is introduced next to carry out this agenda. The concept of conditional probability of survival and related notions are defined and remarked on. The last subsection collects the two aforementioned results of [J-R] that we need, one for zero-recovery claims, the other for claims with recovery. The rest of the paper develops this framework in three directions, and is new to our best knowledge.

Section 3 develops a credit swaption pricing theory, beginning with notion of preprices and their transformation under change of numeraire. First, a valuation result is established for an option to swap two survival claims in an arbitrary numeraire. Second, when one of the survival claims is a prenumeraire, the said result is applied to the prenumeraire measure to get a pricing formula in terms of the swap rate. Next, a Black-Scholes approximation, useful for practical applications, is defined by orthogonally projecting the stochastic logarithm of the par swap rate on the space of square-integrable deterministic volatility (Gaussian) martingales. Finally, examples are presented, including definition and pricing formulae for the two legs of a credit default swap.

Section 4 formalizes a notion of the recovery process of a claim, and constructs claims with prescribed recovery process. The [J-R] formula for claims with recovery is then reformulated and proved under suitable technical assumptions on the conditional survival probability as a semimartingale integrator, to ensure stochastic integrals of suitable integrands are integrable in the numeraire measure. An "abstract version" of pricing claims with recovery is also presented. The pricing formula for claims with recovery is next used in our extension of the [D-S] result on fractional recovery of predefault value. Essentially, it constructs a claim with a given predefault recovery ratio and promised payoff, and simultaneously provides an alternative pricing formula. It is noted that when the conditional survival probability is absolutely continuous and of finite variation, then the stochastic exponential and integral in our formula reduce to their ordinary counterparts, specializing it to the [D-S] formula. A related but separate result on the predefault recovery ratio is also presented.

Section 5 introduces the notion of coadaptability, and points out its elementary properties and connection to conditional independence. The conditional survival probability is invariant under coadapted change of numeraire, as are preprices of all claims.

1.1. **Subfiltration approach to modelling probability of default.** Default time τ is customarily modelled as a $\mathcal{F}_t$-stopping time, where filtration $\mathcal{F}_t$ represents the evolution of all information. The behavior of default time τ boils down in some sense to how the filtration $\mathcal{F}_t^\tau$ generated by τ sits within filtration $\mathcal{F}_t$. Different modelling paradigms can be distinguished and compared by their differing assumptions on the properties of this embedding.

[E-J-Y] and [J-R] propose $\mathcal{F}_t = \mathcal{F}_t^\tau \vee \mathcal{H}_t$ for some subfiltration $\mathcal{H}_t$. They further assume that the *conditional survival probability* $\alpha_t := E[1_{\tau>t} \,|\, \mathcal{H}_t]$ is (strictly) positive.

Positivity of α_t essentially implies that default time τ is unpredictable, and thus excludes standard structural models. But, it not imposed because of any prejudice against the structural approach, but rather to make the Bayes rule work. Namely, with $\alpha_t > 0$, the Bayes rule applies to yield $P[\tau > T \,|\, \mathcal{F}_t] = 1_{\tau>t} P[\tau > T \,|\, \mathcal{H}_t]/\alpha_t$ for $t \leq T$. (Moreover, $P[\tau > T \,|\, \mathcal{H}_t] = E[\alpha_T \,|\, \mathcal{H}_t]$.) A slightly more general version of this formula constitutes the aforementioned generalization in [E-J-Y] and [J-R] of the earlier pricing formulae for zero-recovery claims.

A deeper formula in [J-R] generalizes an earlier pricing formula for a claim with $\mathcal{H}_t$-adapted recovery process R_t. It is: $E[1_{t<\tau\leq T} R_\tau \,|\, \mathcal{F}_t] = 1_{\tau>t} E[-\int_t^T R_s d\alpha_s \,|\, \mathcal{H}_t]/\alpha_t$. Here, α_t is a semimartingale, and as opposed to the earlier formula, the integral is a stochastic integral.

The subfiltration $\mathcal{H}_t$ may be viewed as partial information that contains everything except the fact of default itself. It encompasses information on interest rates, economic and financial fundamentals, and other factors influencing the health of companies in general, as well as any market data relevant to the trade at hand, such as may be, equity indices, currency exchange rates, inflation swap rates, etc. Name-specific data, like firm equity price or earnings, should be included in $\mathcal{H}_t$ only with care, for this data eventually cease to arrive, perhaps abruptly, after the firm defaults, or at least at liquidation or restructuring, whereas information in $\mathcal{H}_t$ continues to evolve indefinitely. (This is comparable to the puzzle in the Cox-process setting that the hazard rate is defined even after default. But, at a more basic level, the unconditional survival probability too is defined at all times, including after default.)

The conditional survival probability α_t may then be interpreted as an agent's best probabilistic estimate as to whether the firm has survived up to time t or not, which agent observes everything in $\mathcal{H}_t$, but is somehow denied the exact information in $\mathcal{F}_t^\tau$ as to whether or not default has actually occurred by time t.

As shown in [E-J-Y] and [J-R], when α_t is continuous and decreasing (as in the Cox process setting), the compensator Λ_t of the default process $1_{\tau\leq t}$ (i.e., the unique predictable increasing process such that $1_{\tau\leq t} - \Lambda_t$ is a martingale) satisfies $\Lambda_t = -\log(\alpha_{t\wedge\tau})$. The compensator

Λ_t always has the property that Λ_τ is unit exponentially distributed. So, when α_t is continuous and decreasing, it follows that α_τ is (0,1)-uniformly distributed.

Another interesting construct is $\overline{\alpha}_t := E[1_{\tau>t} \,|\, \mathcal{H}]$, where $\mathcal{H} = \bigvee_{t\in(0,\infty)} \mathcal{H}_t$. As noted in section 2.4, $\overline{\alpha}_\tau$ is always (0,1)- uniformly distributed and independent of subfiltration $\mathcal{H}_t$. For conditionally independent subfiltrations, $\alpha_t = \overline{\alpha}_t$, as shown in [J-R].

1.2. **Cox process approach to modelling default hazard rate.** In the Cox-process approach developed by [L], default time τ is not a given primitive. Rather, he constructs it from other primitives, namely, a given filtration $\mathcal{H}_t$, an independent and unit exponentially distributed random variable E, and a positive $\mathcal{H}_t$-adapted process h_t that is viewed as the hazard rate. Then, default time τ is defined as the first time that accumulated hazard rate $\int_0^t h_s ds$ reaches a draw from E. In other words, τ is defined as the solution to the equation $\int_0^\tau h_s ds = E$.

As shown by [L], default time τ thus constructed satisfies $\alpha_t = \exp(-\int_0^t h_s)ds$. It follows that $\alpha_\tau = \exp(-E)$ is $(0,1)$-uniformly distributed and independent of $\mathcal{H}_t$. As such, the hazard rate setting of [L] can be considered as the special case of the subfiltration approach in which conditional survival probability α_t is absolutely continuous and decreasing and α_τ is (0,1)-uniformly distributed and independent of $\mathcal{H}_t$.

Subfiltration $\mathcal{H}_t$ in the Cox-process setting turns out to be a conditionally independent subfiltration, meaning that every $\mathcal{H}_t$-martingale is also a $\mathcal{F}_t$-martingale. Ironically, this somewhat degrades the role played by subfiltration $\mathcal{H}_t$, for all $\mathcal{H}_t$-conditional expectations of $\mathcal{H}$-measurable random variables become replaceable with corresponding $\mathcal{F}_t$-conditional expectations (e.g., $E[\alpha_T \,|\, \mathcal{H}_t] = E[\alpha_T \,|\, \mathcal{F}_t]$).

1.3. **Preintensity approach to modelling compensator of default process.** The preintensity approach developed in [D-S] does not incorporate any subfiltration. Instead it focuses attention on the compensator Λ_t of the default process $1_{\tau\le t}$. It assumes that compensator Λ_t is absolutely continuous, and takes as a given primitive a positive process $h_t > 0$, referred to as preintensity, such that $1_{\tau>t} h_t = \lambda_t$, where $\lambda_t := \Lambda_t'$ is the default intensity. This implies $\Lambda_t = \int_0^t 1_{\tau>s} h_s ds = \int_0^{t\wedge\tau} h_s ds$, a situation reminiscent of the Cox-process setting.

At any time and state that default has not occurred, the preintensity equals the intensity by definition. But at times that default has previously occurred, the intensity itself is zero, i.e., $1_{\tau\le t}\lambda_t = 0$. So the preintensity is by definition a positive extension of the default intensity to paths and times where default has previously occurred.

At times that default has not occurred, the preintensity enjoys the same interpretation as does the hazard rate of the Cox-process setting, namely it measures the instantaneous probability of default "right after" time t given firm survival up to time t. For this reason, both approaches are often collectively referred to as the "intensity approach", and together they embrace a large majority of reduced-form models of default risk.

[D-S] briefly mention that the Cox-process construction furnishes an example consistent with their preintensity approach. However, they do not elaborate on how one might best go about choosing an appropriate extension of intensity λ_t in general. Presumably, the choice of extension is immaterial. At any rate, this choice must satisfy certain constraints if it is to specialize to the Cox-process setting. One requirement is that the chosen preintensity h_t must satisfy $E[\exp(-\int_0^t h_s ds)] = P[\tau > t]$. Another constraint is that h_t be independent of Λ_τ. Indeed, as previously mentioned, both these conditions hold in the Cox setting.

The preintensity approach of [D-S] does not take any subfiltration as a given primitive. As such, it is not a special case of the subfiltration approach. But, the two approaches do intersect in the Cox-process setting, for in this case, as already mentioned, compensator Λ_t equals $-\log(\alpha_{t\wedge\tau})$, and therefore the process $-\alpha'_t/\alpha_t$ serves as a preintensity.

As a whole, the subfiltration approach seems more convenient to us, at least for valuation purposes, for the pricing formulae can be written directly and succinctly in terms of the conditional survival probability α_t alone, without any need to resort to compensator Λ_t.

2. Transversal Default valuation models

This section introduces notation and terminology and formalizes the framework. The two results that we need from [J-R] are collected in the last subsection. For the purposes of this paper, we introduce definitions throughout in italic $Words$ with first letters capitalized.

2.1. **State price density and claims.**

A *(Contingent-Claim) Valuation Model* $\mathcal{N}$ is a 5-tuple $\mathcal{N} = (\overline{T}, \Omega, \mathcal{F}_t, P, \xi_t)$, where

$0 < \overline{T} < \infty$ is the terminal date,

$\{\mathcal{F}_t\}_{0\le t\le \overline{T}}$ is a right-continuous filtration on state space Ω.

P is a complete probability measure on $\mathcal{F}_{\overline{T}}$, and $\mathcal{F}_0$ equals subalgebra of events of probability 0 or 1,

$\xi_t > 0$ is an a.s. positive, integrable semimartingale with $\xi_0 = 1$ a.s.

For the rest of the paper we fix a given valuation model $\mathcal{N}$.

Throughout, we set $\mathcal{F} := \mathcal{F}_{\overline{T}}$ and $\xi := \xi_{\overline{T}}$.

In what follows t and T will denote times in $[0, \overline{T}]$.

We denote $E_t[\cdot\,] := E[\cdot\,|\,\mathcal{F}_t]$.

In this setting, $\mathcal{F}_t$ represents all information up to and including time t. For simplicity, we restrict the domain of t to the compact interval $[0, \overline{T}]$ and assume $\mathcal{F}_0$ is the trivial σ-subalgebra.

P represents the *actual* probability measure, variously also referred to as the subjective, later, objective, historical, empirical, or physical probability measure.

The process ξ_t represents a *state price density* or a *deflator*. The price at time 0 of a claim that pays 1 if an event $E \in \mathcal{F}_t$ occurs is thought to equal $\int_E \xi_t(\omega) P(d\omega)$.

We distinguish between a claim C and its price process C_t.

A *Claim* C is a $\mathcal{F}$-measurable random variable $C \in \mathcal{F}$ such that ξC is integrable.[1] We write $C \in \mathcal{C}$.

The *Price Process* of claim C is the unique process C_t such that $C_{\overline{T}} = C$ and $\xi_t C_t$ is a (right-continuous) P-martingale. In other words, price process C_t is defined by (the unique right continuous version of)

$$C_t =: \mathcal{V}_t(C) := E_t[\xi C]/\xi_t, \qquad C \in \mathcal{C}.$$

2.2. **Numeraire relative valuation under numeraire measure.** A *Numeraire* β is an a.s. positive claim. We write $\beta \in \mathcal{C}^+$.

Note, a.s. $\beta_t > 0$, all t.

The *(β-Associated)Numeraire Measure* P^β is defined by

$$\frac{dP^\beta}{dP} := \frac{\xi\beta}{\beta_0}.$$

Note, P^β is an equivalent *probability* measure, because $E[\xi\beta] = \beta_0$. By the Radon Nikodym theorem, every P-equivalent probability measure arises in this way.

Note, $C \in \mathcal{F}$ is a claim if and only if C/β is P^β-integrable.

Note, if B is any other numeraire , then $dP^B/dP^\beta = (\beta_0/B_0)B/\beta$.

Note, $1/\xi$ is a numeraire, $\mathcal{V}_t(1/\xi) = 1/\xi_t$, and $P^{1/\xi} = P$.

A well known property of change of measure is that for any σ-subalgebra $\mathcal{G}$ of $\mathcal{F}$, and any P^β-integrable $F \in \mathcal{F}$, we have

$$E^\beta[F \mid \mathcal{G}] = \frac{E[F\xi\beta \mid \mathcal{G}]}{E[\xi\beta \mid \mathcal{G}]}.$$

This implies $C \in \mathcal{F}$ is a claim if and only if C_t/β_t is a P^β-martingale:

$$E^\beta[C/\beta \mid \mathcal{F}_t] = E[\xi C \mid \mathcal{F}_t]/E[\xi\beta \mid \mathcal{F}_t] = C_t/\beta_t.$$

For this reason the numeraire measure P^β is also referred to as the *martingale measure* associated to numeraire β. To reemphasize,

$$C_t = \beta_t E_t^\beta[C/\beta]. \qquad (C \in \mathcal{C})$$

[1] We follow the usual convention of denoting a σ-algebra and the set of measurable real-valued functions on it by the same symbol.

2.3. **Default time τ.** Default time is modelled as a $\mathcal{F}_t$-stopping time τ, meaning $\{\tau > t\} \in \mathcal{F}_t$ for $t \leq \overline{T}$.

Note, $\mathcal{V}_t(1_{\tau>t}C) = 1_{\tau>t}C_t$ for any claim C.

The following result is not crucial to the paper, but it has an intuitive financial interpretation, and in a way sets the agenda for introducing additional structure, such as a subfiltration.

Theorem 2.1. *Let C be a claim, $\beta > 0$ be a numeraire, and τ be a $\mathcal{F}_t$-stopping time. Then, for $t \leq T$,*

$$1_{\tau>t}C_t = \mathcal{V}_t(1_{t<\tau\leq T}\beta C_{\tau\wedge T}/\beta_{\tau\wedge T} + 1_{\tau>T}C).$$

Or equivalently,

$$1_{\tau>t}C_t = \beta_t E_t^\beta[1_{t<\tau\leq T}C_{\tau\wedge T}/\beta_{\tau\wedge T} + 1_{\tau>T}C_T/\beta_T].$$

Proof. Set $M_t := C_t/\beta_t$ and $X := 1_{t<\tau\leq T}M_{\tau\wedge T} + 1_{\tau>T}M_T$. We must show $E_t^\beta[X] = 1_{\tau>t}M_t$. Clearly, $X = M_{\tau\wedge T} - 1_{\tau\leq t}M_{\tau\wedge t}$. Hence, $E_t^\beta[X] = E_t^\beta[M_{\tau\wedge T}] - 1_{\tau\leq t}M_{\tau\wedge t}$, because $1_{\tau\leq t}M_{\tau\wedge t} \in \mathcal{F}_t$. Since M_t is a P^β martingale, by (a simple version of) the optional stopping theorem, $M_{\tau\wedge t} = E_t^\beta[M_{\tau\wedge T}]$. Hence, $E_t^\beta[X] = M_{\tau\wedge t} - 1_{\tau\leq t}M_{\tau\wedge t} = 1_{\tau>t}M_t$. □

We can interpret this result as follows. Consider the claim $C^{\beta,t}$ obtained by investing the "recovery" C_τ of C in the numeraire β if default occurs between t and T, and investing payoff C_T in the numeraire β if default does not occur by time T. Then, the theorem says that $C_t(\omega) = C_t^{\beta,t}(\omega)$ on paths ω that have not defaulted prior to time t.

The result is a tautological representation of a claim's price in terms of its promised payoff in case of survival and its recovery in case of default. In credit risk modelling one introduces enough auxiliary structure as to give substance to this representation in terms of quantities derived from that structure.

2.4. **Subfiltration $\mathcal{H}_t$ and conditional survival probability α_t^β.** A *Default Valuation Model* is a 3-tuple $\mathcal{M} = (\mathcal{N}, \tau, \mathcal{H}_t)$, where $\mathcal{N}$ is a valuation model, τ is a $\mathcal{F}_t$ stopping time, and $\mathcal{H}_t$ is a subfiltration of $\mathcal{F}_t$ satisfying the usual hypothesis and $\mathcal{H}_0 = \mathcal{F}_0$.

Set $\mathcal{H} := \mathcal{H}_{\overline{T}}$ throughout the paper.

Given a numeraire $\beta > 0$, $\mathcal{H}_t$ is a *β-Conditionally-Independent Subfiltration of $\mathcal{F}_t$* if every $(\mathcal{H}_t, P^\beta)$ martingale is also a $(\mathcal{F}_t, P^\beta)$ martingale.[2]

[2]This condition is referred to as the martingale invariance property in [J-R]. (They actually state it for square integrable martingales. However, in our setting with $\overline{T} < \infty$, this restriction is not necessary.) As mentioned in [J-R], it is equivalent to the σ-algebras, $\mathcal{F}_t$ and $\mathcal{H}$ be P^β-conditionally independent given $\mathcal{H}_t$, all t, meaning either (hence all) of the following three equivalent conditions is satisfied:

(i) $\forall t$, $\forall$ bounded $F_t \in \mathcal{F}_t$, $H \in \mathcal{H}$, $E^\beta[F_tH \,|\, \mathcal{H}_t] = E^\beta[F_t \,|\, \mathcal{H}_t]E^\beta[H \,|\, \mathcal{H}_t]$.

(ii) $\forall t$, $\forall$ bounded $F_t \in \mathcal{F}_t$, $E^\beta[F_t \,|\, \mathcal{H}] = E^\beta[F_t \,|\, \mathcal{H}_t]$.

(iii) $\forall t$, $\forall$ bounded $H \in \mathcal{H}$, $E^\beta[H \,|\, \mathcal{F}_t] = E^\beta[H \,|\, \mathcal{H}_t]$.

These conditions imply $\mathcal{H}_t = \mathcal{F}_t \cap \mathcal{H}$. (The converse is usually false.)

We do not assume conditional independence, but will note some of its consequences. A prime example of it is the Cox process setting.

Default valuation model $\mathcal{M}$ is *Complementary* if $\mathcal{F}_t = \mathcal{F}_t^\tau \vee \mathcal{H}_t$ for all t, where $\mathcal{F}_t^\tau := \sigma(\{\tau > s\}, s \le t)$ is the filtration generated by τ.

Let $\beta > 0$ be a numeraire. The *P^β-Conditional Survival Probability* α_t^β of default time τ is

$$\alpha_t^\beta := P^\beta[\tau > t \,|\, \mathcal{H}_t].$$

Proposition 2.2. *Let β and B be numeraires. Then $\alpha_t^B = E^\beta[1_{\tau>t} B_t/\beta_t \,|\, \mathcal{H}_t]/E^\beta[B_t/\beta_t \,|\, \mathcal{H}_t]$. Hence, $\alpha_t^B = \alpha_t^\beta$ if $B_t/\beta_t \in \mathcal{H}_t$.*

Proof. In fact, for any bounded $F \in \mathcal{F}_t$ (applied here to $F = 1_{\tau>t}$),

$$E^B[F \,|\, \mathcal{H}_t] = \frac{E^\beta[FB/\beta \,|\, \mathcal{H}_t]}{E^\beta[B/\beta \,|\, \mathcal{H}_t]} = \frac{E^\beta[FB/\beta \,|\, \mathcal{F}_t \,|\, \mathcal{H}_t]}{E^\beta[B/\beta \,|\, \mathcal{F}_t \,|\, \mathcal{H}_t]} = \frac{E^\beta[FB_t/\beta_t \,|\, \mathcal{H}_t]}{E^\beta[B_t/\beta_t \,|\, \mathcal{H}_t]}.$$

□

For the rest of this section, we denote $\alpha_t := \alpha_t^{1/\xi}$. Recall $P = P^{1/\xi}$.

Note, $0 \le \alpha_t \le 1$, $P[\tau > t] = E[\alpha_t]$, and α_t is $\mathcal{H}_t$-adapted. Also,

Proposition 2.3. *For any integrable $H \in \mathcal{H}_T$, $E[1_{\tau>T} H \,|\, \mathcal{H}_t] = E[\alpha_T H \,|\, \mathcal{H}_t]$.*[3]

Default valuation model $\mathcal{M}$ is *Positive* if for every numeraires β, a.s. $\alpha_t^\beta > 0$, all t. Proposition 2.2 easily implies that $\mathcal{M}$ is positive if a.s. $\alpha_t^\beta > 0$, all t, for *some* numeraire β (e.g., for numeraire $1/\xi$).

Assume $\mathcal{M}$ is positive. Then $P[\tau > t] = E[\alpha_t] > 0$, all t. Moreover, $\tau > 0$ a.s., because $1_{\tau>0} = \alpha_0$. Further, τ is *not* an $\mathcal{H}_t$ stopping time, unless $\tau > \overline{T}$ a.s. (In this sense, unlike standard structural models, subfiltration $\mathcal{H}_t$ does not "predict" τ.) Indeed, if $1_{\tau>t}$ is $\mathcal{H}_t$ measurable for some $t > 0$, then $\alpha_t = 1_{\tau>t}$. So, $\alpha_t > 0$ implies $\tau > t$. The same argument shows $\mathcal{F}_t^\tau \cap \mathcal{H}_t \subset \mathcal{F}_0$.

A construct related to the conditional survival property is

$$\overline{\alpha}_t^\beta := P^\beta[\tau > t \,|\, \mathcal{H}].$$

[J-R] show that if $\mathcal{M}$ is complementary, then $\mathcal{H}_t$ is β-conditionally independent if and only if $\alpha_t^\beta = \overline{\alpha}_t^\beta$ for all t.

We will not use $\overline{\alpha}_t^\beta$ in this paper. But it is worth noting some of its properties in passing. Set $\overline{\alpha}_t = \overline{\alpha}_t^{1/\xi}$.

Note, $0 \le \overline{\alpha}_t \le 1$, $P[\tau > t] = E[\overline{\alpha}_t]$, and $\overline{\alpha}_t$ is decreasing.

Note, $\overline{\alpha}_t$ is deterministic if and only if τ and $\mathcal{H}$ are P-independent; in this case $\alpha_t = \overline{\alpha}_t = P[\tau > t]$.[4]

In the infinite time horizon case, under quite general conditions, $\overline{\alpha}_\tau$ has a uniform $(0, 1)$ distribution and is independent of $\mathcal{H}$ (under P).[5]

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[3] $E[1_{\tau>T} H \,|\, \mathcal{H}_t] = E[1_{\tau>T} H \,|\, \mathcal{H}_T \,|\, \mathcal{H}_t] = E[E[1_{\tau>T} \,|\, \mathcal{H}_T] H \,|\, \mathcal{H}_t] = E[\alpha_T H \,|\, \mathcal{H}_t]$.

[4] If $\overline{\alpha}_t$ is deterministic, then $\overline{\alpha}_t = P[\tau > t]$. Hence, for bounded $H \in \mathcal{H}$, $E[1_{\tau>t} H] = E[E[1_{\tau>t} H \,|\, \mathcal{H}]] = E[\overline{\alpha}_t H] = P[\tau > t]E[H]$. The converse is clear.

[5] For example, assume that $\overline{\alpha}_t$ is strictly decreasing, and for almost all $\omega \in \Omega$, the range of $\overline{\alpha}_t(\omega)$ as a function of $t \in (0, \infty)$ is $(0, 1)$. Both claims follow once we

2.5. **Two results from [J-R] on transversal models.** A default valuation model $\mathcal{M}$ is *Transversal* if it is positive and complementary.[6]

The following key result is from [J-R]. We provide a different proof.

Theorem 2.4. *Suppose model $\mathcal{M}$ is transversal. Then for any integrable $F \in \mathcal{F}$, and all $t \le T$,*

$$E[1_{\tau>T}F \mid \mathcal{F}_t] = \frac{1_{\tau>t}}{\alpha_t} E[1_{\tau>T}F \mid \mathcal{H}_t].$$

Proof. Let X denote the RHS. By the definition of conditional expectation, we need to show that $E[XF_t] = E[1_{\tau>T}FF_t]$ for all bounded $F_t \in \mathcal{F}_t$. Because $\mathcal{M}$ is complementary, it suffices to check this when $F_t = 1_{\tau>s}H_t$, $s \le t$, $H_t \in \mathcal{H}_t$. Since $1_{\tau>T}1_{\tau>s} = 1_{\tau>T}$, we have, $E[1_{\tau>T}FF_t] = E[1_{\tau>T}FH_t]$. But, since $XF_t = XH_t$ (because $1_{\tau>s}1_{\tau>t} = 1_{\tau>t}$), and $E[X \mid \mathcal{H}_t] = E[1_{\tau>T}F \mid \mathcal{H}_t]$, we also have

$$E[XF_t] = E[XH_t] = E[E[X\,|\mathcal{H}_t]H_t] = E[E[1_{\tau>T}F\,|\mathcal{H}_t]H_t] = E[1_{\tau>T}FH_t].$$

□

Remark 2.5. A heuristic derivation of this formula is as follows:

$$E[1_{\tau>T}F \mid\ \mathcal{F}_t] = 1_{\tau>t}E[1_{\tau>T}F \mid\ \mathcal{F}_t^\tau \vee \mathcal{H}_t] = 1_{\tau>t}\frac{E[1_{\tau>T}F \mid \mathcal{H}_t]}{P[1_{\tau>t} \mid \mathcal{H}_t]}.$$

The above result is all we need from [J-R] for credit swaption valuation in the next section. But, for credit claims with recovery in section 4, we need another fact from [J-R], their Proposition 3.1 part (ii):

[JR] Statement. *Assume $\mathcal{M}$ is transversal and α_t is a $\mathcal{H}_t$ semimartingale. Let H_t be a bounded $\mathcal{H}_t$-predictable process. Then,*

$$E_t[1_{t<\tau\le T}H_{\tau\wedge T}] = -\frac{1_{\tau>t}}{\alpha_t}E[\int_t^T H_s d\alpha_s \,|\, \mathcal{H}_t], \qquad t \le T.$$

The basic idea of the proof is that, choosing a mesh $\{t_n\}$, Theorem 2.4 and Proposition 2.3 clearly imply $E_t[1_{t_n<\tau\le t_{n+1}}H_n] = 1_{\tau>t}E^\beta[(\alpha_{t_n} - \alpha_{t_{n+1}})H_{t_n} \,|\, \mathcal{H}_t]/\alpha_t$. As the mesh gets finer and finer, this formula converges to the desired formula. We will state and prove this with a stronger assumption on α_t that ensures both $\int_t^T H_s d\alpha_s$ and $1_{t<\tau\le T}H_{\tau\wedge T}$ are integrable. But, we relax the condition that H_t be bounded.

The following two results are interesting, but will not be needed.

Corollary 2.6. *Suppose model $\mathcal{M}$ is transversal. Then $1_{\tau>t}/\alpha_t$ is a $\mathcal{F}_t$-martingale.*

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show $P[\overline{\alpha}_\tau < x \,|\, \mathcal{H}] = x$, for all $x \in (0,1)$. Let $\gamma_x(\omega)$ denote the inverse of $\overline{\alpha}_t(\omega)$. Then $P[\overline{\alpha}_\tau < x \,|\, \mathcal{H}] = P[\tau > \gamma_x \,|\, \mathcal{H}] = \overline{\alpha}_{\gamma_x} = x$. (The second equality is justified by noting that γ_x is $\mathcal{H}$-measurable, hence, for any bounded Borel function $f(t,s)$ (here, for $1_{t>s}$), $E[f(\tau,\gamma_x) \,|\, \mathcal{H}](\omega) = E[f(\tau,\gamma_x(\omega)) \,|\, \mathcal{H}](\omega)$ for almost all ω.)

[6]For most our purposes, including the two [J-R] results here, the complementary assumption can be relaxed to $\mathcal{F}_t \subset \mathcal{H}_t \vee \mathcal{F}_t^\tau \vee \mathcal{G}$ for some σ-algebra $\mathcal{G}$ independent of $\mathcal{H} \vee \mathcal{F}_T^\tau$ (or more generally, conditionally independent of $\mathcal{F}_T^\tau$ given $\mathcal{H}$).

Proof. By Theorem 2.4, $E_t[\frac{1_{\tau>T}}{\alpha_T}] = \frac{1_{\tau>t}}{\alpha_t}E[\frac{1_{\tau>T}}{\alpha_T}\,|\,\mathcal{H}_t]$. But (iterating expectation given $\mathcal{H}_T$), $E[1_{\tau>T}/\alpha_T\,|\,\mathcal{H}_t] = 1$. □

Corollary 2.7. *Suppose model $\mathcal{M}$ is transversal. Given fixed t, let $F \in \mathcal{F}_t$ be integrable and satisfy $1_{\tau>t}F = F$. Then, there exists a unique $H \in \mathcal{H}_t$ such that $F = 1_{\tau>t}H$. Moreover,*

$$(*) \qquad F = \frac{1_{\tau>t}}{\alpha_t}E[F\,|\,\mathcal{H}_t].$$

Proof. Suppose $F = 1_{\tau>t}H$ for some $H \in \mathcal{H}_t$. Then, $E[F\,|\,\mathcal{H}_t] = E[1_{\tau>t}H\,|\,\mathcal{H}_t] = \alpha_t H$. Hence, $H = E[F\,|\,\mathcal{H}_t]/\alpha_t$. Uniqueness follows. To prove the existence, it suffices to establish (*). But, (*) follows immediately from the special case of Theorem 2.4 with $T = t$, using the assumption $1_{\tau>t}F = F \in \mathcal{F}_t$. □

Remark 2.8. Theorem 2.4 follows directly from (*) by applying (*) to $G := E[1_{\tau>T}F\,|\,\mathcal{F}_t]$, which clearly satisfies $1_{\tau>t}G = G \in \mathcal{F}_t$. This is how [J-R] derived Theorem 2.4, first establishing (*) by other means.

3. Valuation of survival claims and swaptions

Throughout this section, a default valuation model $\mathcal{M}$ is assumed given and fixed.

3.1. **Claim Preprices.** Let $\beta > 0$ be a numeraire. The *$\mathcal{H}^\beta$-Preprice Process* of a claim C is (the right continuous version of)

$$V_t^\beta(C) := \beta_t E^\beta[C/\beta\,|\,\mathcal{H}_t] = \beta_t E^\beta[C_t/\beta_t\,|\,\mathcal{H}_t].^7 \qquad (C \in \mathcal{C})$$

$V_t^\beta(C)/\beta_t$ is the unique $(\mathcal{H}_t, P^\beta)$ martingale with terminal value C/β.

Note, for any t, $V_t^\beta(C) = C_t$ if and only if $C_t/\beta_t \in \mathcal{H}_t$.

Note, if B is another numeraire, $dP^B|_{\mathcal{H}_t}/dP^\beta|_{\mathcal{H}_t} = (\beta_0/B_0)V_t^\beta(B)/\beta_t$.

Preprices transform under change of numeraire as follows.

Proposition 3.1. *Let β and B be numeraires. Then for any claim C,*

$$V_t^B(C) = \frac{B_t V_t^\beta(C)}{V_t^\beta(B)}.$$

Proof.

$$\frac{V_t^B(C)}{B_t} = E^B[\frac{C}{B}\,|\,\mathcal{H}_t] = \frac{E^\beta[C/\beta\,|\,\mathcal{H}_t]}{E^\beta[B/\beta\,|\,\mathcal{H}_t]} = \frac{V_t^\beta(C)}{V_t^\beta(B)}.$$

□

The following elementary result will prove quite useful.

[7] Indeed, for $T \geq t$, $E^\beta[C/\beta\,|\,\mathcal{H}_t] = E^\beta[C/\beta\,|\,\mathcal{F}_T\,|\,\mathcal{H}_t] = E^\beta[C_T/\beta_T\,|\,\mathcal{H}_t]$.

Lemma 3.2. *Let $\beta > 0$ be a numeraire, C be a claim, and $\hat{C}$ be any claim such that $\hat{C}_T = V_T^\beta(C)$ (e.g., $\hat{C} = V_T^\beta(C)\beta/\beta_T$). Then, for $t \le T$,*
(i) $V_t^\beta(\hat{C}) = V_t^\beta(C)$;
(ii) If $\hat{C}$ is a numeraire, then $\hat{C}_t = V_t^{\hat{C}}(C)$;[8]
(iii) If $\hat{C}_t/\beta_t$ is $\mathcal{H}_t$-adapted, then $\hat{C}_t = V_t^\beta(C)$.

Proof. (i): $V_T^\beta(\hat{C}) = \beta_T E^\beta[\hat{C}_T/\beta_T \,|\, \mathcal{H}_T] = \beta_T E^\beta[V_T^\beta(C)/\beta_T \,|\, \mathcal{H}_T] = V_T^\beta(C)$. It follows $V_t^\beta(C) = V_t^\beta(\hat{C})$ for $t \le T$. (ii): If $\hat{C} > 0$, then by Proposition 3.1, $V_T^{\hat{C}}(C) = V_T^\beta(C)\hat{C}_T/V_T^\beta(\hat{C}) = \hat{C}_T = V_T^{\hat{C}}(\hat{C})$. Hence, for $t \le T$, $V_t^{\hat{C}}(C) = V_t^{\hat{C}}(\hat{C}) = \hat{C}_t$. (iii): if $\hat{C}_t/\beta_t$ is $\mathcal{H}_t$- is adapted. Then, clearly, $\hat{C}_t = V_t^\beta(\hat{C})$. But, by (i) $V_t^\beta(\hat{C}) = V_t^\beta(C)$. □

Note, if $\mathcal{H}_t$ is a β-conditionally independent subfiltration, then $\hat{C}_t/\beta_t$ is $\mathcal{H}_t$-adapted.[9] Hence, (iii) above implies $\hat{C}_t = V_t^\beta(C)$.

3.2. **Survival swaption valuation in arbitrary numeraire.** A T-*Survival Claim* is a claim C with $1_{\tau>T}C = C$. We write $C \in \mathcal{S}_T$. Clearly, then for $t \le T$, $C \in \mathcal{S}_t$ and $C_t = 1_{\tau>t}C_t = C_{\tau\wedge t}$.

The first [J-R] result yields a valuation formula for T-survival claims:

Proposition 3.3. *Assume $\mathcal{M}$ is transversal. Let $\beta > 0$ be a numeraire and C be T-survival claim. Then for $t \le T$,*

$$C_t = \frac{1_{\tau>t}}{\alpha_t^\beta} V_t^\beta(C). \qquad (C \in \mathcal{S}_T\ , t \le T)$$

Proof. This follows immediately from Theorem 2.4 applied to $F = C/\beta$, but with P replaced by P^β. □

The practical significance of this result is that it reduces the pricing of a $T-$ survival claim C to calculation of preprice $V_t^\beta(C)$.

Theorem 3.4. *Assume $\mathcal{M}$ is transversal. Let $\beta > 0$ be a numeraire, A, B be two $T-$ survival claims, and C be any claim such that $C_T = (A_T - B_T)^+$. Then,*

$$C_t = \frac{1_{\tau>t}}{\alpha_t^\beta}\beta_t E^\beta[\frac{1}{\beta_T}(V_T^\beta(A) - V_T^\beta(B))^+ \,|\, \mathcal{H}_t]. \qquad (t \le T)$$

In particular, at $t = T$,

$$C_T = 1_{\tau>T}(\frac{V_T^\beta(A)}{\alpha_T^\beta} - \frac{V_T^\beta(B)}{\alpha_T^\beta})^+.$$

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[8]From these follows, $V_T^\beta(C) = \hat{C}_T = V_T^\beta(\hat{C}) = V_T^{\hat{C}}(C)$. Clearly, if $\overline{C} > 0$ is another numeraire with $\overline{C}_T = V_T^\beta(C)$, then $\overline{C}_t = \hat{C}_t$ and $P^{\overline{C}}|_{\mathcal{F}_t} = P^{\hat{C}}|_{\mathcal{F}_t}$ for $t \le T$.

[9]Indeed, the $(\mathcal{H}_t, P^\beta)$-martingale $V_t^\beta(\hat{C})/\beta_t$ will then also be a $(\mathcal{F}_t, P^\beta)$-martingale, and hence equal to martingale $\hat{C}/\beta_t$,as both have same terminal value.

Proof. Applying above proposition to C,

$$C_t = \frac{1_{\tau>t}}{\alpha_t^\beta} V_t^\beta(C) = \frac{1_{\tau>t}}{\alpha_t^\beta} \beta_t E^\beta[\frac{V_T^\beta(C)}{\beta_T} \,|\, \mathcal{H}_t].$$

It remains to show $V_T^\beta(C) = (V_T^\beta(A) - V_T^\beta(B))^+$. But, applying above proposition to A and B at $t = T$, using linearity and definition of α_T,

$$\frac{V_T^\beta(C)}{\beta_T} = E^\beta[\frac{C_T}{\beta_T} \,|\, \mathcal{H}_T] = E^\beta[\frac{1}{\beta_T}(\frac{1_{\tau>T}}{\alpha_T^\beta} V_T^\beta(A) - \frac{1_{\tau>T}}{\alpha_T^\beta} V_T^\beta(B))^+ \,|\, \mathcal{H}_T]$$

$$= E^\beta[\frac{1_{\tau>T}}{\alpha_T^\beta \beta_T}(V_T^\beta(A) - V_T^\beta(B))^+ \,|\, \mathcal{H}_T] = \frac{1}{\beta_T}(V_T^\beta(A) - V_T^\beta(B))^+.$$

□

The theorem practically reduces pricing of a T-survival swaption to valuation of an option to swap two $\mathcal{H}_t$-adapted claims. To price swaption payoff $C_T = (A_T - B_T)^+$ of two T-survival claims A and B, it constructs two claims $\widetilde{A}$ and $\widetilde{B}$ such that $\widetilde{A}_T = V_T^\beta(A)/\alpha_T^\beta$, $\widetilde{B}_T = V_T^\beta(B)/\alpha_T^\beta$, $C_T = 1_{\tau>T}(\widetilde{A}_T - \widetilde{B}_T)^+$, and $V_t^\beta(C) = V_t^\beta(\widetilde{A} - \widetilde{B})^+$.

Note, if $\mathcal{H}_t$ is β-conditionally independent, then $\mathcal{H}_t$-conditional expectation in the theorem can be replaced by $\mathcal{F}_t$-conditional expectation.

3.3. **Swap rate and swaption pricing in prenumeraire measure.** A claim B is a *Prenumeraire* if $E[\xi B \,|\, \mathcal{H}] > 0$ a.s. We write $B \succ 0$. Note, then, for any numeraire β, a.s. $V_t^\beta(B) > 0$, all t. Conversely, if $V_T^\beta(B) > 0$ for some numeraire β, then B is a prenumeraire.

Note, if B is a T-survival prenumeraire and $\mathcal{M}$ is transversal, then Proposition 3.3 implies that, for $t \le T$, $B_t = 1_{\tau>t} X_t$ for some $X_t > 0$.

An important consequence of Proposition 3.1 is that the V_t^β *ratio* of two claims is independent of the choice of numeraire β:

Theorem and Definition 3.5. *Let A be a claim, $B \succ 0$ be a prenumeraire, and $\beta > 0$ be a numeraire. Then the ratio*

$$S_t^{A,B} := \frac{V_t^\beta(A)}{V_t^\beta(B)} \in \mathcal{H}_t$$

is independent of numeraire β, and called the Swap Rate at time t of A with respect to B. Moreover, let $T > 0$, and $\hat{B} > 0$ be any numeraire such that $\hat{B}_T = V_T^\beta(B)$ (e.g., $\hat{B} = V_T^\beta(B)\beta/\beta_T$). Then, $S_t^{A,B}$ is a $(\mathcal{H}_t, P^{\hat{B}})$-martingale on $[0, T]$.

Proof. The independence of $S_t^{A,B}$ from β follows from Proposition 3.1. Next, by Lemma 3.2, $\hat{B}_t = V_t^{\hat{B}}(B)$ for $t \le T$. Hence, on $[0, T]$, $S_t^{A,B} = V_t^{\hat{B}}(A)/\hat{B}_t$, which is a $(\mathcal{H}_t, P^{\hat{B}})$-martingale by definition. □

The following shows that the swap rate is the only $\mathcal{H}_t$-adapted candidate for a "generalized ratio."

Proposition 3.6. *Let $t > 0$, A be a claim and $B \succ 0$ be a prenumeraire. If $A_t = S_t B_t$ for some $S_t \in \mathcal{H}_t$, then $S_t = S_t^{A,B}$.*

Proof. $V_t^\beta(A) = \beta_t E^\beta[A_t/\beta_t \,|\, \mathcal{H}_t] = \beta_t E^\beta[S_t B_t/\beta_t \,|\, \mathcal{H}_t] = S_t V_t^\beta(B)$. □

Existence of "generalized ratios" for T-survival claims follows from

Proposition 3.7. *Assume $\mathcal{M}$ is transversal. Let A be a T-survival claim and $B \succ 0$ be a T-survival prenumeraire. Then, for any $t \le T$,*

$$A_t = S_t^{A,B} B_t. \qquad (t \le T, \;\; A \in \mathcal{S}_T \ni B \succ 0)$$

Proof. Apply Proposition 3.3 to in any numeraire β to A and B, and substitute $V_t^\beta(A) = S_t^{A,B} V_t^\beta(B)$. □

We can apply our previous swaption formula in the special numeraire $\hat{B}$ to get a swaption valuation formula in terms of the swap rate:

Theorem 3.8. *Let A be a T-survival claim, $B \succ 0$ be a T-survival prenumeraire, $K \in \mathbb{R}$, C be any claim such that $C_T = (A_T - KB_T)^+$, $\beta > 0$ be a numeraire, and $\hat{B} > 0$ be any numeraire such that $\hat{B}_T = V_T^\beta(B)$ (e.g., $\hat{B} = V_T^\beta(B)\beta/\beta_T$). Assume $\mathcal{M}$ is transversal. Then*

$$C_t = B_t E^{\hat{B}}[(S_T^{A,B} - K)^+ \,|\, \mathcal{H}_t]. \quad (t \le T)$$

Proof. Applying Theorem 3.4 in numeraire $\hat{B}$,

$$C_t = \frac{1_{\tau > t}}{\alpha_t^{\hat{B}}} \hat{B}_t E^{\hat{B}}[(\frac{V_T^{\hat{B}}(A) - KV_T^{\hat{B}}(B)}{\hat{B}_T})^+ \,|\, \mathcal{H}_t].$$

By Lemma 3.2 (ii), $\hat{B}_t = V_t^{\hat{B}}(B)$ for $t \le T$. Hence by Proposition 3.3 applied to claim B (instead of C) and numeraire $\hat{B}$ (instead of β) $1_{\tau > t}\hat{B}_t/\alpha_t^{\hat{B}} = B_t$. Further, since $\hat{B}_T = V_T^{\hat{B}}(B)$, by Theorem and Definition 3.5 applied in numeraire $\hat{B}$, $(V_T^{\hat{B}}(A) - KV_T^{\hat{B}}(B))/\hat{B}_T = V_T^{\hat{B}}(A)/V_T^{\hat{B}}(B) - K = S_T^{A,B} - K$. □

The above result constitutes a principal result of this paper. We call the numeraire measure $P^{\hat{B}}$, associated to numeraire $\hat{B}$ above, the *Prenumeraire Measure* associated to prenumeraire B. Clearly its restriction $P^{\hat{B}}|_{\mathcal{F}_T}$ to $\mathcal{F}_T$ (hence also to $\mathcal{H}_T$) does not depend on the choice of numeraire $\hat{B}$ satisfying $\hat{B}_T = V_T^\beta(B)$. We can always take $\hat{B} = V_T^\beta(B)\beta/\beta_T$ if we wish.

Let us examine the dependence of $dP^{\hat{B}}|_{\mathcal{H}_t}$ on T and β. In general $dP^{\hat{B}}|_{\mathcal{H}_t} = c(V_t^\beta(\hat{B})/\beta_t)dP^\beta|_{\mathcal{H}_t} = c(V_t^\beta(B)/\beta_t)dP^\beta|_{\mathcal{H}_t}$ for $t \le T$, where $c = \beta_0/V_0^\beta(\hat{B}) = \beta_0/V_0^\beta(B)$. Hence, $dP^{\hat{B}}|_{\mathcal{H}_t}$ does not depend on T. Next, let $\beta'_t > 0$ be another numeraire. Set $\hat{B}' = V_T^{\beta'}(B)\beta'/\beta'_T$. Then,

$$dP^{\hat{B}'}|_{\mathcal{H}_t} = c'\frac{V_t^{\beta'}(B)}{\beta'_t}dP^{\beta'}|_{\mathcal{H}_t} = c'\frac{V_t^{\beta'}(B)}{\beta'_t}\frac{V_t^\beta(\beta')}{\beta_t}dP^\beta|_{\mathcal{H}_t} = \frac{c'V_t^{\beta'}(B)V_t^\beta(\beta')}{cV_t^\beta(B)\beta'_t}dP^{\hat{B}}|_{\mathcal{H}_t}.$$

Hence, if $\hat{B}$ and β are coadapted, i.e., if $\hat{B}_t/\beta_t$ is $\mathcal{H}_t$-adapted, then $P^{\hat{B}}|_{\mathcal{H}_T} = P^{\hat{B}'}|_{\mathcal{H}_T}$ (since coadaptedeness clearly implies $V_t^{\beta'}(C) = V_t^{\beta}(C)$ for all claims C). As such $P^{\hat{B}}|_{\mathcal{H}_T}$ depends only on the coadapted equivalence class of β.

Note, if $\mathcal{H}_t$ is β-conditionally independent, then $\mathcal{H}_t$-conditional expectation in the theorem can be replaced by $\mathcal{F}_t$-conditional expectation.

3.4. **Black-Scholes approximation for survival swaptions.** In this subsection, we adopt the assumptions and notation of the previous theorem. Assume further that $\mathcal{H}_t$ is a Brownian filtration, so that all $(\mathcal{H}_t, P^{\hat{B}})$-martingales are continuous. In particular, $S_t := S_t^{A,B}$ is continuous. We also assume A is a prenumeraire, so that a.s. $S_t > 0$.

If $\log S_t$ has deterministic quadratic variation (in filtration $\mathcal{H}_t$), then, because S_t is a continuous $P^{\hat{B}}$-martingale, $\log S_t$ will follow a $P^{\hat{B}}$-Gaussian process, and hence S_T will be $P^{\hat{B}}$-lognormally distributed. Then, clearly, the above theorem provides a Black-Scholes formula for the expectation in the swaption formula. In general, $\log S_t$ will not have deterministic quadratic variation. Nevertheless, we can define an approximation to S_t with this property in the sense of least squares. Consider the Hilbert space of all square-integrable $(\mathcal{H}_t, P^{\hat{B}})$-martingales M_t with $M_0 = 0$, equipped with the inner product

$$\langle M, N\rangle = E^{\hat{B}}[M_T N_T] = E^{\hat{B}}([M,N]_T).$$

(Note, for an $\mathcal{H}_t$ Ito process $\| \int_0^{\cdot} \sigma_s dz_s\|^2 = E^{\hat{B}}[\int_0^T \sigma_s^2 ds]$.) Let π denote the orthogonal projection onto the closed subspace of all $(\mathcal{H}_t, P^{\hat{B}})$-square-integrable (Gaussian) martingales with deterministic $\mathcal{H}_t$-quadratic variation. (Note, for an Ito process, $\pi \int_0^t \sigma_s dz_s = \int_0^t E^{\hat{B}}[\sigma_s] dz_s$.) Assume $S_{t-} > 0$. Then $\mathcal{L}(S_t) := \int_0^t dS_s/S_{s-}$ is a $(P^{\hat{B}}, \mathcal{H}_t)$ local martingale. Assume it is a square integrable martingale. Set

$$v_t^2 := E^{\hat{B}}[\pi\mathcal{L}(S_t), \pi\mathcal{L}(S_t)] = \mathrm{var}^{\hat{B}}[\pi\mathcal{L}(S_t)], \qquad v_{t,T} := \sqrt{v_T^2 - v_t^2}.$$

(Note, if $\mathcal{L}(S_t) = \int_0^t \sigma_s dz_s$ is an Ito process, then, $v_t^2 = \int_0^t (E^{\hat{B}}[\sigma_s])^2 ds$.) The *Approximate Swaption Price Process* $\overline{C}_t$ is

$$\overline{C}_t := B_t(S_t N(\frac{\log S_t/K}{v_{t,T}} + \frac{v_{t,T}}{2}) - KN(\frac{\log S_t/K}{v_{t,T}} - \frac{v_{t,T}}{2})).$$

This Gaussian (deterministic volatility) approximation $C_t \approx \overline{C}_t$ is exact if (and only if) $\mathcal{L}(S_t)$, or equivalently $\log S_t$, has deterministic quadratic variation. This approximation effectively replaces $E_t^{\hat{B}}(S_T - K)^+$ by $E_t^{\hat{B}}(\mathcal{E}(\pi\mathcal{L}(S_T)) - K)^+$. Since $B_t S_t = A_t$, we can rewrite the option pricing formula as

$$\overline{C}_t = A_t N(\frac{\log S_t/K}{v_{t,T}} + \frac{v_{t,T}}{2}) - KB_t N(\frac{\log S_t/K}{v_{t,T}} - \frac{v_{t,T}}{2}).$$

For $t = 0$, a similar formula appears in [S] and [A-G].

The above Black-Scholes formula also provides the delta-hedge for the credit swaption. The swaption is replicated by being long at any time t notional $N(\frac{\log S_t/K}{v_{t,T}} + \frac{v_{t,T}}{2})$ of the protection leg and being short notional $N(\frac{\log S_t/K}{v_{t,T}} - \frac{v_{t,T}}{2})$ of the spread leg. Indeed, by comparing these two weights with the two terms in the above Black-Scholes formula, we see that the price of this replicating portfolio equals the swaption price at all times by construction. Note, both the replicating portfolio and the swaption price drop to zero after default.

Viewing the Black-Scholes formula as an approximation is also useful in other applications. For example, in the context of the deterministic volatility Libor market model, European swaption prices are often approximated by a Black-Scholes formula, which arises from approximating the stochastic forward swap rate volatility by a certain deterministic volatility function. The orthogonal projection method outlined above would result in a different such approximation. It is not as practical to implement, but at least it lends some theoretical support (in the sense of least squares) to such deterministic volatility approximations.

3.5. **Examples and Discussion.** Typical examples of T-survival claims are the two legs of a T-forward credit default swap (CDS), the spread premium leg, and the default protection leg. Each leg is a "stream" of payments, i.e., a portfolio of cashflows. But, before describing them, we record a useful result and discuss its implications.

Proposition 3.9. *Let $\beta > 0$ be a numeraire and $\widetilde{C}$ be a claim such that $\widetilde{C}_T/\beta_T \in \mathcal{H}_T$ for some $T \geq 0$.*[10] *Then*

$$V_t^\beta(1_{\tau>T}\widetilde{C}) = V_t^\beta(\alpha_T^\beta \widetilde{C}). \qquad (t \leq T)$$

Moreover, if $\mathcal{M}$ is transversal, then, setting, $\zeta_t := \beta_t/\alpha_t^\beta$,

$$\mathcal{V}_t(1_{\tau>T}\widetilde{C}) = 1_{\tau>t}\, \zeta_t E^\beta[\widetilde{C}_T/\zeta_T \,|\, \mathcal{H}_t]. \qquad (t \leq T)$$

Proof. $V_t^\beta(1_{\tau>T}\widetilde{C})/\beta_t = E^\beta[1_{\tau>T}\widetilde{C}_T/\beta_T \,|\, \mathcal{H}_t] = E^\beta[1_{\tau>T}\widetilde{C}_T/\beta_T \,|\, \mathcal{H}_T \,|\, \mathcal{H}_t]$ $= E^\beta[E^\beta[1_{\tau>T} \,|\, \mathcal{H}_T]\widetilde{C}_T/\beta_T \,|\, \mathcal{H}_t] = V_t^\beta(\alpha_T^\beta\widetilde{C})/\beta_t$. The second statement now follows also, using Proposition 3.3, and the fact that $V_t^\beta(\alpha_T^\beta\widetilde{C}) = \beta_t E^\beta[\alpha_T^\beta \widetilde{C}_T/\beta_T \,|\, \mathcal{H}_t]$. □

The above is commonly interpreted as practically reducing, at least for computational purposes, the valuation of a survival claim to that of a "non-defaultable claim" in a fictitious model with numeraire ζ_t.

[10] Any T-survival claim C can be represented in this form. Indeed, let β and B be any two (coadapted) numeraires such $B_T/\beta_T \in \mathcal{H}_T$ (e.g., $B = \beta$). Set $\widetilde{C} := (V_T^\beta(C)/\alpha_T^\beta)(B/B_T)$. Then, $\widetilde{C}_T = V_T^\beta(C)/\alpha_T^\beta$. Hence $\widetilde{C}_T/\beta_T \in \mathcal{H}_T$. Further, if C is T-survival, then by Proposition 3.4, $C = 1_{\tau>T}\widetilde{C}$.

When $\beta_t = \exp(\int_0^t r_s ds)$ and $\alpha_t = \exp(-\int_0^t \lambda_s ds)$, the above formula is none other than a formula in [L] stating

$$E_t^\beta[1_{\tau>T} e^{-\int_t^T r_s ds} \widetilde{C}_T] = 1_{\tau>t} E^\beta[e^{-\int_t^T (r_s+\lambda_s)ds} \widetilde{C}_T \mid \mathcal{H}_t].$$

Valuation of an option to exchange two survival claims similarly reduces practically to one of non-defaultable claims. For instance, suppose r_t is $\mathcal{H}_t$ adapted and $\widetilde{A}$ and $\widetilde{B}$ are claims whose prices are $\mathcal{H}_t$-adapted. The T-expiry swaption payoff $C_T = (\mathcal{V}_T(1_{\tau>a}\widetilde{A}) - \mathcal{V}_T(1_{\tau>b}\widetilde{B}))^+$ equals $1_{\tau>T}\widetilde{C}_T$, where $C_{\widetilde{T}} = \zeta_T(E^\beta[\widetilde{A}_a/\zeta_a \mid \mathcal{H}_T] - E^\beta[\widetilde{B}_b/\zeta_b \mid \mathcal{H}_T])^+$. But, this can be interpreted as an option to exchange at time T the claims $\widetilde{A}$ and $\widetilde{B}$ under the fictitious interest-rate model $\hat{r}_t := r_t + \lambda_t$. For example, when λ_t is deterministic and r_t is Gaussian, then, consistently with the previous section, a Black-Scholes formula obtains for the option to swap to survival zero-coupon bonds.

Let $T > 0$ and $\delta \in \mathcal{H}_T$ be bounded. Let $B^{T,\delta}$ denote any claim such that $B_T^{T,\delta} = 1_{\tau>T}\delta$. Their are many such claims. (For any numeraire β, $1_{\tau>T}\delta\beta/\beta_T$ is such a claim.) But, their prices and preprices agree on $[0, T]$ because they agree at time T. We think of $B^{T,\delta}$ as a claim which pays δ at time time T if default has not occurred, and zero otherwise.

Let $0 < s < T$, and $L \in \mathcal{H}_s$ be bounded. Let $A^{s,T,L}$ denote any claim with $A_T^{s,T,L} = 1_{s<\tau\leq T}L$. We think of $A^{s,T,L}$ as a claim which pays "the loss" L at time T if default has occurred between s and T, and 0 otherwise. Note, $A^{s,T,L} = B^{s,L} - B^{T,L}$, because $1_{s<\tau\leq T} = 1_{\tau>s} - 1_{\tau>T}$.

The two legs of a forward CDS can be formalized as "$\mathcal{T}$-streams", where $\mathcal{T}$ is *Payment Tenor*, meaning a sequence $\mathcal{T} = (T_1 \ldots, T_N)$, representing payment dates, such that $0 < T_1 < \ldots < T_N$.

Given a payment tenor $\mathcal{T}$ and bounded $\delta_n \in \mathcal{H}_{T_n}$, $1 \leq n \leq N$, the *Survival Annuity Stream* is the claim $B^{\delta_\mathcal{T}} := \sum_1^N B^{T_n,\delta_n}$. Clearly, $B^{\delta_\mathcal{T}}$ is a T-survival claim for any $T \leq T_1$.

In practice, the δ_n are considered daycount fractions, and assumed deterministic and approximately equal to $T_n - T_{n-1}$, with T_0 given as some "start date" before T_1. The claim $KB^{\delta_\mathcal{T}}$ then represents the spread premium leg of a forward CDS starting at T_0, where $K > 0$ represents the annualized spread premium.

Given a payment tenor $\mathcal{T}$, a "start date" $T_0 < T_1$, and bounded $L_n \in \mathcal{H}_{T_{n-1}}$, $1 \leq n \leq N$, the *Default Protection Stream* is the claim $A^{T_0,L_\mathcal{T}} := \sum_1^N A^{T_{n-1},T_n,L_n}$. Clearly $A^{T_0,L_\mathcal{T}}$ is a T-survival claim for $T \leq T_0$.

Claim $A^{T_0,L_\mathcal{T}}$ represents the protection leg of a T_0-forward CDS. L_n represents the loss that incurs, and is paid at time T_n by the seller of the protection, if default occurs between T_{n-1} and T_n. (Section 4 treats the continuous version where the insured amount is paid at any time t where default occurs.) In practice, L_n equals notional minus a deterministic recovery. Another example takes L_n a deterministic constant plus the spot Libor rate at time T_{n-1} times daycount fraction $T_n - T_{n-1}$. A floating rate defaultable bond can be hedged against default, by entering a CDS which is long this latter example, and short $KB^{\delta_{\mathcal{T}}}$, where K is the spread over Libor of the defaultable bond.

The above Proposition provides the pricing of these claims:

Corollary 3.10. *Let $\beta > 0$ be a numeraire such that β_t is $\mathcal{H}_t$-adapted, $\mathcal{T}$ be a payment tenor, and $T_0 < T_1$. Then*

$$V_t^{\beta}(B^{\delta_{\mathcal{T}}}) = \beta_t E^{\beta}[\sum_{n=1}^{N} \alpha^{\beta}_{T_n} \delta_{T_n}/\beta_{T_n} \,|\, \mathcal{H}_t]; \qquad (t \leq T_1)$$

$$V_t^{\beta}(A^{T_0,L_{\mathcal{T}}}) = \beta_t E^{\beta}[\sum_{n=1}^{N} (\alpha^{\beta}_{T_{n-1}} - \alpha^{\beta}_{T_n}) L_{T_n}/\beta_{T_n} \,|\, \mathcal{H}_t]. \qquad (t \leq T_0)$$

When $\mathcal{M}$ is transversal, these formulae together with Theorem 3.4 or Theorem 3.8 lead to a valuation formula for a CDS swaption, and the previous section provides a Black-Scholes approximation. Again the valuation of CDS swaption valuation practically reduces to valuation of a option to swap two $\mathcal{H}_t$ claims, namely, the two claims

$$\widetilde{B}^{\delta_{\mathcal{T}}} := \beta E^{\beta}[\sum_{n=1}^{N} \alpha^{\beta}_{T_n} \delta_{T_n}/\beta_{T_n} \,|\, \mathcal{H}_T]/\alpha_T \in \mathcal{C} \cap \mathcal{H};$$

$$\widetilde{A}^{T_0,L_{\mathcal{T}}} := \beta E^{\beta}[\sum_{n=1}^{N} (\alpha^{\beta}_{T_{n-1}} - \alpha^{\beta}_{T_n}) L_{T_n}/\beta_{T_n} \,|\, \mathcal{H}_T]/\alpha_T \in \mathcal{C} \cap \mathcal{H}.$$

Indeed, for expiry $T \leq T_0$ and $K \in \mathbb{R}$, the swaption payoff satisfies

$$(A_T^{T_0,L_{\mathcal{T}}} - KB_T^{\delta_{\mathcal{T}}})^+ = 1_{\tau > T}(\widetilde{A}_T^{T_0,L_{\mathcal{T}}} - K\widetilde{B}_T^{\delta_{\mathcal{T}}})^+.$$

Further, by Theorem 3.4, the swaption price for $t \leq T$ is

$$\mathcal{V}_t((A_T^{T_0,L_{\mathcal{T}}} - KB_T^{\delta_{\mathcal{T}}})^+) = 1_{\tau > t}\beta_t E[(\widetilde{A}_T^{T_0,L_{\mathcal{T}}} - K\widetilde{B}_T^{\delta_{\mathcal{T}}})^+/\beta_T \,|\, \mathcal{H}_t]/\alpha_t.$$

4. Valuation of credit claims with recovery

Throughout this section, a default valuation model $\mathcal{M}$ is fixed.

4.1. **Recovery processes.** Let C be a claim. A process R_t is a *Recovery Process of C on $[0,T]$* if $1_{\tau \leq T} C_{\tau \wedge T} = 1_{\tau \leq T} R_{\tau \wedge T}$, or equivalently, if $C_{\tau(\omega)}(\omega) = R_{\tau(\omega)}(\omega)$ whenever $\tau(\omega) \leq T$.[11] Theorem 2.1 yields at once

Theorem 4.1. *Let C be a claim, R_t be a recovery process of C on $[0,T]$, and $\beta > 0$ be a numeraire. Then for $t \leq T$,*

$$1_{\tau > t} C_t = \mathcal{V}_t(1_{t < \tau \leq T} \beta R_{\tau \wedge T}/\beta_{\tau \wedge T} + 1_{\tau > T} C).$$

[11]This implies that for any $s \leq T$, R_t is a recovery process of C on $[0,s]$. Use $1_{\tau \leq s} = 1_{\tau \leq s} 1_{\tau \leq T}$, and that for any process X_t, $1_{\tau \leq s} X_{\tau \wedge s} = 1_{\tau \leq s} X_{\tau \wedge T}$.

Or equivalently,

$$1_{\tau>t}C_t = \beta_t E_t^\beta[1_{t<\tau\leq T}\frac{R_{\tau\wedge T}}{\beta_{\tau\wedge T}} + 1_{\tau>T}\frac{C_T}{\beta_T}].$$

Price C_t itself is a recovery process for claim C. But, the above representation is then a simple tautology. Next subsections will give it substance by imposing predictability and other requirements on R_t.

Corollary 4.2. *Let $s \leq T$, C be a s-survival claim, R_t be a recovery process of C on $[0,T]$, and $\beta > 0$ be a numeraire. Then for $t \leq s$,*

$$C_t = \mathcal{V}_t(1_{s<\tau\leq T}\beta R_{\tau\wedge T}/\beta_{\tau\wedge T} + 1_{\tau>T}C). \qquad (t \leq s)$$

Proof. Because both sides divided by β_t are P^β martingales on $[0,s]$, it suffices to show the above for $t = s$. But, this then follows from the theorem because C is t-claim for $t \leq s$, hence, $1_{\tau>t}C_t = C_t$. □

Given suitable R_t, we can always construct (infinitely) many different claims having R_t as a recovery process. Just "invest" R_t in any numeraire when default occurs, and hold it until the terminal date:

Proposition 4.3. *Let $\beta > 0$ be a numeraire, and R_t be a $\mathcal{F}_t$-progressively measurable process such that $1_{\tau\leq T}R_{\tau\wedge T}/\beta_{\tau\wedge T}$ is P^β integrable.*[12] *Then $C^{R_t,\beta} := 1_{\tau\leq T}\beta R_{\tau\wedge T}/\beta_{\tau\wedge T}$ is a claim, $C^{R_t,\beta}_{\tau\wedge T} = 1_{\tau\leq T}R_{\tau\wedge T}$, and R_t is a recovery process of $C^{R_t,\beta} + C$ on $[0,T]$ for any T-survival claim C. Moreover, for $t \leq T$*

$$C_t^{R_t,\beta} = 1_{\tau\leq t}\beta_t\frac{R_{\tau\wedge t}}{\beta_{\tau\wedge t}} + \mathcal{V}_t(1_{t<\tau\leq T}\beta\frac{R_{\tau\wedge T}}{\beta_{\tau\wedge T}}).$$

Proof. As R_t is progressively measurable, $C^{R,\beta}/\beta$ is $\mathcal{F}_{\tau\wedge T}$-measurable. Therefore, $C^{R,\beta}/\beta = E[C^{R,\beta}/\beta \,|\, \mathcal{F}_{\tau\wedge T}] = E_{\tau\wedge T}[C^{R,\beta}/\beta] = C^{R,\beta}_{\tau\wedge T}/\beta_{\tau\wedge T}$. Hence, $C^{R_t,\beta}_{\tau\wedge T} = 1_{\tau\leq T}R_{\tau\wedge T}$. Since C is T-survival, $1_{\tau\leq T}C_{\tau\wedge T} = 0$. Hence, $1_{\tau\leq T}\mathcal{V}_{\tau\wedge T}(C^{R_t,\beta}+C) = 1_{\tau\leq T}C^{R_t,\beta}_{\tau\wedge T} = 1_{\tau\leq T}R_{\tau\wedge T}$. Finally, $1_{\tau\leq T} = 1_{\tau\leq t}+1_{t<\tau\leq T}$. Hence, $C_t^{R_t,\beta} = \mathcal{V}_t(1_{\tau\leq t}\beta R_{\tau\wedge T}/\beta_{\tau\wedge T})+\mathcal{V}_t(1_{t<\tau\leq T}\beta R_{\tau\wedge T}/\beta_{\tau\wedge T})$. But, since $1_{\tau\leq t}\beta R_{\tau\wedge T}/\beta_{\tau\wedge T} = 1_{\tau\leq t}\beta R_{\tau\wedge t}/\beta_{\tau\wedge t}$ is $\mathcal{F}_t$-measurable, $\mathcal{V}_t(1_{\tau\leq t}\beta R_{\tau\wedge T}/\beta_{\tau\wedge T}) = 1_{\tau\leq t}\beta_t R_{\tau\wedge t}/\beta_{\tau\wedge t}$, giving the last formula. □

4.2. **Pricing formula for claims with recovery.** Let β be a numeraire. Let $\mathcal{G}_t$ denote either $\mathcal{F}_t$ or $\mathcal{H}_t$. We say a process X_t belongs to $\mathcal{H}^2(\mathcal{G}_t, P^\beta)$ if it is a $(\mathcal{G}_t, P^\beta)$-special semimartingale and, in its unique decomposition $X_t = N_t + A_t$, N_t is a $(\mathcal{G}_t, P^\beta)$ square-integrable martingale, and A_t is $\mathcal{G}_t$ predictable process of P^β-square integrable total variation, i.e, if $E^\beta[\int_0^T d[N,N]_s] + E^\beta[(\int_0^T |H_s||dA_s|)^2] < \infty$. (See [P], section IV.2.) Process X_t will then be P^β-integrable.

[12]Let $\mathcal{G}_t$ denote either $\mathcal{F}_t$ or $\mathcal{H}_t$. Recall, a $\mathcal{G}_t$ progressively measurable process (p.m.p.) is $\mathcal{G}_t$-adapted. A $\mathcal{H}_t$ p.m.p. is a priori a $\mathcal{F}_t$ p.m.p. Any right continuous $\mathcal{G}_t$-adapted process or $\mathcal{G}_t$-predictable (e.g., left continuous) process is a $\mathcal{G}_t$ p.m.p. A $\mathcal{H}_t$-predictable process is a priori $\mathcal{F}_t$-predictable.

A process H_t is then said to $(\mathcal{H}^2(\mathcal{G}_t, P^\beta), X_t)$-integrable if it is $\mathcal{G}_t$ predictable and $E^\beta[\int_0^T H_s^2 d[N,N]_s] + E^\beta[(\int_0^T |H_s||dA_s|)^2] < \infty$. Then, the $\mathcal{G}_t$-stochastic integral process $\int_0^t H_s dX_s$ also belongs to $\mathcal{H}^2(\mathcal{G}_t, P^\beta)$, hence $\int_t^T H_s dX_s$ will be P^β-integrable for all $t \leq T$. Clearly, all bounded $\mathcal{G}_t$ predictable processes are $(\mathcal{H}^2(\mathcal{G}_t, P^\beta), X_t)$-integrable.

A numeraire β is $\mathcal{M}$-*Integrable* if (i) α_t^β belongs to $\mathcal{H}^2(\mathcal{H}_t, P^\beta)$, and (ii) the default process $1_{\tau\leq t}$ belongs to $\mathcal{H}^2(\mathcal{F}_t, P^\beta)$. We then say a process H_t is $(\mathcal{M}, \beta)$-*Integrable*, if H_t is a $\mathcal{H}_t$-predictable process that is both $(\mathcal{H}^2(\mathcal{H}_t, P^\beta), \alpha_t^\beta)$-integrable and $(\mathcal{H}^2(\mathcal{F}_t, P^\beta), 1_{\tau\leq t})$-integrable. Clearly, all bounded $\mathcal{H}_t$-predictable processes are $(\mathcal{M}, \beta)$-integrable.

Our version of [JR] statement alluded to in section 2.5 is as follows.

Lemma 4.4. *Let $\beta > 0$ be a $\mathcal{M}$-integrable numeraire. Let process H_t be $(\mathcal{M}, \beta)$-integrable. Then, $1_{t<\tau\leq T} H_{\tau\wedge T}$ and $\int_t^T H_s d\alpha_s^\beta$ are P^β-integrable for $t < T$. Moreover, if $\mathcal{M}$ is transversal, then for $t \leq s \leq T$,*

$$E_t^\beta[1_{s<\tau\leq T} H_{\tau\wedge T}] = \frac{1_{\tau>t}}{\alpha_t} E^\beta[-\int_s^T H_u d\alpha_u^\beta \,|\, \mathcal{H}_t].$$

Proof. Since H_t is $(\mathcal{H}^2(\mathcal{H}_t, P^\beta), \alpha_t^\beta)$-integrable, $\int_s^T H_s d\alpha_u^\beta$ is P^β-integrable. Since H_t is $(\mathcal{H}^2(\mathcal{F}_t, P^\beta), 1_{\tau\leq t})$-integrable, $1_{s<\tau\leq T} H_{\tau\wedge T} = \int_s^T H_u d1_{\tau\leq u}$ is P^β-integrable. Next, assume $\mathcal{M}$ is transversal. For positive integer N, set $t_n = T/N$, $n = 0, \ldots, N$. Then, the sequences of processes $X_T^n = 1_{\tau\leq T} \sum_0^{N-1} 1_{t_n<\tau\leq t_{n+1}} H_{t_n}$ converges in $\mathcal{H}^2(\mathcal{F}_t, P^\beta)$ (as function of T) to process $1_{\tau\leq T} H_{\tau\wedge T}$. Hence, $E_t^\beta[1_{s<\tau\leq T} H_{\tau\wedge T}] = \lim E_t^\beta[X_T^n - X_s^n]$. On the other hand, the sequence of processes $Y_T^n = 1_{t_n\leq T} \sum_{n=0}^{N-1} (\alpha_{t_n} - \alpha_{t_{n+1}}) H_{t_n}$ converge in $\mathcal{H}^2(\mathcal{H}_t, P^\beta)$ to $-\int_0^T H_u d\alpha_u^\beta$. Hence, $E^\beta[-\int_s^T H_u d\alpha_u^\beta \mid \mathcal{H}_t] = \lim E^\beta[Y_T^n - Y_s^n \,|\, \mathcal{H}_t]$. The desired result thus follows if we show $E_t^\beta[X_T^n - X_s^n] = 1_{\tau>t} E^\beta[Y_T^n - Y_s^n \,|\, \mathcal{H}_t]/\alpha_t$. But indeed, using Theorem 2.4, Proposition 2.3, and the fact that $1_{t_n<\tau\leq t_{n+1}} = 1_{\tau>t_n} - 1_{\tau>t_{n+1}}$, we have $E_t^\beta[1_{t_n<\tau\leq t_{n+1}} H_n] = 1_{\tau>t} E^\beta[(\alpha_{t_n} - \alpha_{t_{n+1}}) H_{t_n} \,|\, \mathcal{H}_t]/\alpha_t$. □

We now reformulate this result in terms of claims.

Theorem 4.5. *Let $\beta > 0$ be a $\mathcal{M}$-integrable numeraire, and R_t be a process such that R_t/β_{t_-} is $(\mathcal{M}, \beta)$-integrable (in particular R_t/β_{t_-} is $\mathcal{H}_t$-predictable). Assume $\mathcal{M}$ is transversal. Then, the variates $1_{\tau\leq T} \beta R_{\tau\wedge T}/\beta_{\tau\wedge T}$ and $\beta \int_t^T R_s d\alpha_s^\beta / \beta_{s_-}$ are claims, and for $t \leq T$,*

$$\mathcal{V}_t(1_{\tau\leq T} \beta \frac{R_{\tau\wedge T}}{\beta_{\tau\wedge T}}) = 1_{\tau\leq t} \beta_t \frac{R_{\tau\wedge t}}{\beta_{\tau\wedge t}} - \frac{1_{\tau>t}}{\alpha_t^\beta} V_t^\beta(\beta \int_t^T \frac{R_s}{\beta_{s_-}} d\alpha_s^\beta),$$

and for $t \leq s \leq T$,

$$\mathcal{V}_t(1_{s<\tau\leq T} \beta \frac{R_{\tau\wedge T}}{\beta_{\tau\wedge T}}) = \frac{1_{\tau>t}}{\alpha_t^\beta} V_t^\beta(-\beta \int_s^T \frac{R_u}{\beta_{u_-}} d\alpha_u^\beta).$$

Proof. The second formula follows from the Lemma applied to $H_t = R_t/\beta_t$. The first formula follows from the second formula with $s = t$. Indeed, using $1_{\tau\le T} = 1_{\tau\le t} + 1_{t<\tau\le T}$, we see from the second formula that it suffices to show $\mathcal{V}_t(1_{\tau\le t}\beta R_{\tau\wedge T}/\beta_{\tau\wedge T}) = 1_{\tau\le t}\beta_t R_{\tau\wedge t}/\beta_{\tau\wedge t}$. But, this is the case because $1_{\tau\le t}R_{\tau\wedge T}/\beta_{\tau\wedge T}$ is $\mathcal{F}_t$-measurable. □

With $\beta_t = \exp(\int_0^t r_s ds)$ and $\alpha_t^\beta = \exp(-\int_0^t h_s ds)$, the above equation is none other than a formula in [L] stating

$$E_t[1_{s<\tau\le T}e^{-\int_t^\tau r_u du}R_\tau] = 1_{\tau>t}E[\int_s^T e^{-\int_t^u (r_a+h_a)da}h_u R_u du \,|\, \mathcal{H}_t].$$

Combining the above with Theorem 4.1, we get the following "abstract" version of essentially the same result.

Corollary 4.6. *Let C be a claim, R_t be a recovery process of C on $[0,T]$, and $\beta > 0$ be a $\mathcal{M}$-integrable numeraire. Assume $\mathcal{M}$ is transversal, and R_t/β_{t_-} is $(\mathcal{M},\beta)$-integrable. Then, for $t \le T$,*

$$1_{\tau>t}C_t = \frac{1_{\tau>t}}{\alpha_t^\beta}V_t^\beta(1_{\tau>T}C - \beta\int_t^T \frac{R_s}{\beta_{s_-}}d\alpha_s^\beta).$$

If further C is s-survival for some $s \le T$, then, for $t \le s$,

$$C_t = \frac{1_{\tau>t}}{\alpha_t^\beta}V_t^\beta(1_{\tau>T}C - \beta\int_s^T \frac{R_u}{\beta_{u_-}}d\alpha_u^\beta).$$

Proof. The first (resp. second) formula follows from directly from Theorem 4.1 (resp. Corollary 4.2) and Theorem 4.5. □

4.3. **Construction of claims with given predefault recovery ratio and promised payoff.** This section reformulates and extends a result of [D-S] on fractional recovery of predefault value to general numeraires β and semimartingale conditional survival probability α_t^β. It concerns the "predefault recovery ratio process" ϕ_t which equals the recovery at time of default divided by the claim price just before default. The previous section constructed a claim with a given promised payoff and a given recovery process R_t in case of default, and presented a formula for its price. Given instead a candidate recovery ratio ϕ_t and a promised payoff C, this section constructs a recovery process R_t such that ϕ_t is the recovery ratio of the claim with recovery R_t and the given promised payoff C. Moreover, it provides an alternative representation for the price of the constructed claim.

Theorem 4.7. . *Let $\beta > 0$ be a $\mathcal{M}$-integrable numeraire. Assume $\mathcal{M}$ is transversal, and $\alpha_{t_-} > 0$. Let R_t be a process such that R_t/β_{t_-} is $(\mathcal{M},\beta)$-integrable. Let ϕ_t be a $\mathcal{H}_t$-predictable process such that ϕ_t/α_{t-}^β is $(\mathcal{H}^2(\mathcal{H}_t, P^\beta), \alpha_t^\beta)$-integrable. Set*

$$\varphi_t := \mathcal{E}(\int_0^t \phi_s d\alpha_s^\beta/\alpha_{s_-}^\beta).$$

Assume $\varphi_T > 0$. Let C be a claim. Set

$$R_t := \phi_t \varphi_{t_-} V^\beta_{t_-}(1_{\tau>T} C/\varphi_T)/\alpha^\beta_{t_-}.$$

Assume the $\mathcal{H}_t$-martingale $V^\beta_t(1_{\tau>T}C/\varphi_T)/\beta_t$ is continuous. Then for $t \leq T$

$$\mathcal{V}_{t_-}(1_{\tau>T}C + 1_{\tau\leq T}\beta R_{\tau\wedge T}/\beta_{\tau\wedge T})\phi_t = R_t;$$

and, for $t \leq s \leq T$,

$$\mathcal{V}_t(1_{\tau>T}C + 1_{s<\tau\leq T}\beta R_{\tau\wedge T}/\beta_{\tau\wedge T}) = \mathcal{V}_t(1_{\tau>T}\varphi_s C/\varphi_T).$$

Proof. The first formula follows from the second. Indeed, using the second formula with $s = t$ and Proposition 3.3 we get

$$1_{\tau>t}\mathcal{V}_t(1_{\tau>T}C + 1_{\tau\leq T}\beta R_{\tau\wedge T}/\beta_{\tau\wedge T}) = 1_{\tau>t}\varphi_t V^\beta_t(1_{\tau>T}C/\varphi_T)/\alpha_t.$$

Since, $1_{\tau>t_-} = 1$, the second formula follows from the definition R_t. As for the first formula, in view of Theorem 4.5, it suffices to show

$$V^\beta_t(1_{\tau>T}C - \beta\int_s^T \frac{R_u}{\beta_{u_-}} d\alpha^\beta_u) = V^\beta_t(1_{\tau>T}C/\varphi_s\varphi_T).$$

By the defining property of stochastic exponential,

$$d\varphi_t = \varphi_{t_-}\phi_t d\alpha^\beta_t/\alpha^\beta_{t_-}.$$

Integrating from s to T gives

$$\varphi_T = \varphi_s + \int_s^T \varphi_{u_-}\phi_u d\alpha^\beta_u/\alpha^\beta_{u_-}.$$

Hence

$$1_{\tau>T}C\frac{\varphi_s}{\varphi_T} = 1_{\tau>T}C - \frac{1_{\tau>T}C}{\varphi_T}\int_s^T \varphi_{u_-}\phi_u \frac{d\alpha^\beta_u}{\alpha^\beta_{u_-}}.$$

Therefore it remains to show

$$V^\beta_t(\beta\int_s^T \frac{R_u}{\beta_{u_-}} d\alpha^\beta_u) = V^\beta_t(\frac{1_{\tau>T}C}{\varphi_T}\int_s^T \varphi_{u_-}\phi_u \frac{d\alpha^\beta_u}{\alpha^\beta_{u_-}}).$$

It suffices to prove this for $t = s$ as both sides divided by β_t are $\mathcal{H}_t$-martingales. Using the definition R_t and the continuity assumption, we are thus reduced to show

$$V^\beta_t(\beta\int_t^T V^\beta_u(\frac{1_{\tau>T}C}{\varphi_T})\frac{\phi_u\varphi_{u_-}}{\beta_u\alpha^\beta_{u_-}} d\alpha^\beta_u) = V^\beta_t(\frac{1_{\tau>T}C}{\varphi_T}\int_t^T \varphi_{u_-}\phi_u \frac{d\alpha^\beta_u}{\alpha^\beta_{u_-}}).$$

But,

$$E^\beta[\int_t^T V^\beta_u(\frac{1_{\tau>T}C}{\varphi_T})\frac{\phi_u\varphi_{u_-}}{\beta_u\alpha^\beta_{u_-}} d\alpha^\beta_u \,|\, \mathcal{H}_t] = E^\beta[\int_t^T \phi_u\varphi_{u_-}E^\beta[\frac{1_{\tau>T}C}{\varphi_T\beta}\,|\,\mathcal{H}_u]\frac{d\alpha^\beta_u}{\alpha^\beta_{u_-}}\,|\,\mathcal{H}_t]$$

$$= E^\beta[E^\beta[\int_t^T \phi_u\varphi_{u_-}\frac{1_{\tau>T}C}{\varphi_T\beta}\frac{d\alpha^\beta_u}{\alpha^\beta_{u_-}}\,|\,\mathcal{H}_u]\,|\,\mathcal{H}_t] = E^\beta[\int_t^T \phi_u\varphi_{u_-}\frac{1_{\tau>T}C}{\varphi_T\beta}\frac{d\alpha^\beta_u}{\alpha^\beta_{u_-}}\,|\,\mathcal{H}_t].$$

$\square$

The first formula in the theorem is interpreted as saying that the given process ϕ_t is the predefault recovery ratio of the constructed claim $A := 1_{\tau>T}C + 1_{\tau\le T}\beta R_{\tau\wedge T}/\beta_{\tau\wedge T}$, i.e., the recovery divided by price before default. In precise terms, since R_t is by construction a recovery process of A on $[0,T]$, this formula implies $1_{\tau\le T}A_{\tau\wedge T} = 1_{\tau\le T}\phi_{\tau\wedge T}A_{\tau_-\wedge T}$, or equivalently, $A_{\tau(\omega)}(\omega) = \phi_{\tau(\omega)}(\omega)\, A_{\tau(\omega)_-}(\omega)$ whenever $\tau(\omega)\le T$.

By Proposition 3.3, we can rewrite the second formula as

$$\mathcal{V}_t(1_{\tau>T}C + 1_{s<\tau\le T}\beta\frac{R_{\tau\wedge T}}{\beta_{\tau\wedge T}}) = \frac{1_{\tau>t}}{\alpha_t^\beta}V_t^\beta(1_{\tau>T}\frac{\varphi_s}{\varphi_T}C).$$

At $t=s$, we can express this formula in a more symmetric and intuitive form, provided $C/\beta\in\mathcal{H}_T$. For then $V_t^\beta(1_{\tau>T}C/\varphi_T) = V_t^\beta(\alpha_T^\beta C/\varphi_T)$, hence, setting $\alpha'_t = \alpha_t^\beta/\varphi_t$, we get

$$\mathcal{V}_t(1_{\tau>T}C + 1_{t<\tau\le T}\beta\frac{R_{\tau\wedge T}}{\beta_{\tau\wedge T}}) = \frac{1_{\tau>t}}{\alpha'_t}V_t^\beta(\alpha'_T C).$$

One can interpret the right hand side as what price of survival claim $1_{\tau>T}C$ would have been if the conditional survival probability of τ were α'_t instead of α_t^β. So, the result basically says that, provided default has not occurred, the price of claim A with given promised payoff and predefault recovery ratio ϕ_t equals the price of a claim with the same promised payoff but zero recovery, provided the latter is calculated with respect to the fictitiously increased conditional survival probability α'_t.

When α_t^β is continuous and of finite variation, $\int_0^t \phi_s d\alpha_s^\beta/\alpha_{s_-}^\beta$ is so too, and hence in the definition of φ_t we can replace the stochastic exponential with ordinary exponential, and further express α'_t in terms of the "predefault loss ratio" $1-\phi_t$:

$$\alpha'_t = \alpha_t^\beta \exp(-\int_0^t \phi_s d\alpha_s^\beta/\alpha_s^\beta) = \exp(\int_0^t (1-\phi_s)d\alpha_s^\beta/\alpha_s^\beta).$$

In the absolutely continuous case, the hazard rate $-d\alpha'_t/\alpha'_t$ of α'_t is therefore the loss ratio $1-\phi_t$ times the hazard rate $-d\alpha_t^\beta/\alpha_t^\beta$ of α_t^β. This is basically the case treated in [D-S].

4.4. **Recovery ratio representation of a claim price.** Suppose R_t is a recovery process on $[0,T]$ for a claim C which satisfies $C_{t_-}>0$. Then, we may think of $\phi_t = R_t/C_{t_-}$, as a predefault recovery ratio, for, clearly, $1_{\tau\le T}C_{\tau\wedge T} = 1_{\tau\le T}\phi_{\tau\wedge T}C_{\tau_-\wedge T}$. Theorems 4.5 and 4.6 gave a representation of the claim price in terms R_t. The following result provides one in terms of R_t/C_{t_-}.

Not surprisingly, the final formula is identical as in the previous theorem. However, it is not possible to directly deduce it from the previous result. For there is some the circularity here, trying to write C_t in terms of C_{t_-}. This complication entails a different approach. We will assume α_t^β is continuous and of finite variation for this purpose.

Theorem 4.8. *Let C be a claim, R_t be a recovery process of C on $[0,T]$, and $\beta > 0$ be a $\mathcal{M}$-integrable numeraire. Assume $\mathcal{M}$ is transversal, and R_t/β_{t_-} is $(\mathcal{M},\beta)$-integrable, $C_{t_-} > 0$, and α_t^β is continuous and of finite variation. Set*

$$\varphi_t := \exp\left(\int_0^t \frac{R_s}{C_{s_-}} \frac{d\alpha_s^\beta}{\alpha_s^\beta}\right).$$

Assume also the $\mathcal{H}_t$-martingale $N_t := V_t^\beta(1_{\tau>T}C - \beta\int_0^T R_s/\beta_{s_-} d\alpha_s^\beta)/\beta_t$ is square-integrable and $E[\int_0^T d[N_t,N_t]/\varphi_{t_-}^2] < \infty$. Then for $t \leq T$,

$$1_{\tau>t}C_t = \frac{1_{\tau>t}}{\alpha_t^\beta}\varphi_t V_t^\beta\left(\frac{1_{\tau>T}C}{\varphi_T}\right).$$

If further C is s-survival for some $s \leq T$, then, for $t \leq s$,

$$C_t = \frac{1_{\tau>t}}{\alpha_t^\beta} V_t^\beta\left(1_{\tau>T}\frac{\varphi_s}{\varphi_T}C\right).$$

Proof. The first statement for general claim C follows from the second by applying the second at $s = t$ to t-claim $1_{\tau>t}C$. As for the second statement, by corollary 4.6, it suffices to show

$$V_t^\beta\left(1_{\tau>T}C - \beta\int_s^T \frac{R_s}{\beta_{s_-}} d\alpha_s^\beta\right) = V_t^\beta(1_{\tau>T}C\varphi_s/\varphi_T),$$

As and both sides divided by β_t are $(\mathcal{H}_t, P^\beta)$-martingale, it is enough to show this at $t = s$, i.e, that $M_t = V_t^\beta(1_{\tau>T}C/\varphi_T)/\beta_t$, where

$$M_t := \frac{1}{\beta_t\varphi_t} V_t^\beta\left(1_{\tau>T}C - \beta\int_t^T \frac{R_s}{\beta_{s_-}} d\alpha_s^\beta\right).$$

As both sides are equal at T, and the RHS is a $(\mathcal{H}_t, P^\beta)$–martingale, it is enough to show M_t is a $(\mathcal{H}_t, P^\beta)$-martingale. We claim, $dN_t = \varphi_{t_-}dM_t$. This implies M_t is a local martingale, and our regularity assumption ensures it will be a (square-integrable) martingale. To show the claim, note, again by corollary 4.6, that $C_t = 1_{\tau>t}M_t\varphi_t\beta_t/\alpha_t^\beta$. Hence, $M_{t-}\varphi_{t_-} = \alpha_{t_-}^\beta C_{t_-}/\beta_{t_-}$. Next, note, by definitions of N_t and M_t,

$$N_t + \int_0^t \frac{R_s}{\beta_{s_-}} d\alpha_s^\beta = \varphi_t M_t.$$

Process φ_t is of finite variation and continuous, because α_t^β is so. So, $[\varphi, M]_t = 0$. Hence, taking differential and applying the product rule,

$$dN_t + R_t/\beta_{t_-} d\alpha_t^\beta = M_{t_-}d\varphi_t + \varphi_{t_-}dM_t.$$

As φ_t is also a stochastic exponential here, $d\varphi_t = \varphi_{t_-}(R_t/C_{t_-})d\alpha_t^\beta/\alpha_{t_-}^\beta$. Hence, $M_{t_-}d\varphi_t = M_{t_-}\varphi_{t_-}(R_t/C_{t_-})d\alpha_t^\beta/\alpha_{t_-}^\beta = R_t/\beta_{t_-}d\alpha_t^\beta$. Therefore, $dN_t = \varphi_{t_-}dM_t$, as claimed. $\square$

5. Coadapted claims and coadapted change of numeraire

Throughout this section, a default valuation model $\mathcal{M}$ is fixed.

5.1. **β-Coadapted claims.** Let $\beta > 0$ be a numeraire. A claim C is *β-Coadapted* if C_t/β_t is $\mathcal{H}_t$-adapted.

Proposition 5.1. *A claim C is β-coadapted if and only if $V_t^\beta(C) = C_t$.*

Proof. If C is β-coadpated, then $V_t^\beta(C) = \beta_t E_t^\beta[C_t/\beta_t \,|\, \mathcal{H}_t] = C_t$. The converse is more obvious because $V_t^\beta(C)/\beta_t$ is $\mathcal{H}_t$-adapted. □

From this and Proposition 3.1 we conclude

Proposition 5.2. *Suppose $\mathcal{M}$ is transversal, and C is β-coadapted. Then $\mathcal{V}_t(\alpha_T^\beta C) = V_t^\beta(\alpha_T^\beta C)$, and $\mathcal{V}_t(1_{\tau>T}C) = 1_{\tau>t}\mathcal{V}_t(\alpha_T^\beta C)/\alpha_t^\beta$.*

Because every $\mathcal{H}_t$-martingale is of the form $V_t^\beta(C)/\beta_t$ for some unique claim C satisfying $C/\beta \in \mathcal{H}$, we get the following two results.

Proposition 5.3. *If $\mathcal{H}_t$ is β-conditionally independent, then a claim C is β-coadapted if (and only if) $C/\beta \in \mathcal{H}$.*

Proposition 5.4. *$\mathcal{H}_t$ is a β-conditionally-independent subfiltration of $\mathcal{F}_t$ if and only if all claims C satisfying $C/\beta \in \mathcal{H}$ are β-coadapted.*

A β-coadapted claim C is priced by the operator $\mathcal{V}_t$ in the same way that it is β-relative valued by the operator V_t^β. By its definition, $C/\beta \in \mathcal{H}$, and $E[C/\beta \,|\, \mathcal{H}_t]$ equals $E_t[C/\beta]$. This interchange of $E[\cdot \,|\, \mathcal{H}_t]$ with $E_t[\cdot]$ is automatic when $\mathcal{H}_t$ is a β-conditionally independent. It was shown by [L] to hold in the Cox-process setting. The expectations that appear in [D-S] and [S] are all $\mathcal{F}_t$-expectations too.

5.2. **Coadapted change of numeraire.** Two numeraires β and B are *$\mathcal{H}_t$-Coadapted* if B_t/β_t is $\mathcal{H}_t$-adapted. Evidently, coadaptability is an equivalence relation on $\mathcal{C}^+$.

By Proposition 2.2 if β and B are coadapted then a.s. $\alpha_t^B = \alpha_t^\beta$, all t. The relevance of coadaptability as a concept is further underlined by the following equivalent characterizations.

Proposition 5.5. *Let β and B to numeraires. The following conditions are equivalent. (i) β and B are $\mathcal{H}_t$-coadapted. (ii) B_t/β_t is a $(P^\beta, \mathcal{H}_t)$-martingale. (iii) $B_t = V_t^\beta(B)$. (iv) $V_t^B(C) = V_t^\beta(C)$ for all claims C . (v) $dP^B{}_{|\mathcal{H}_t}/dP^\beta{}_{|\mathcal{H}_t} = (\beta_0/B_0)B_t/\beta_t$, all t.*

Proof. That (ii) implies (i) is obvious. Conversely, we already know B_t/β_t is a $(P^\beta, \mathcal{F}_t)$ martingale. If it is also $\mathcal{H}_t$-adapted, it must also be a $(P^\beta, \mathcal{H}_t)$ martingale. (ii) and (iii) are equivalent because $V_t^\beta(B) = \beta_t E^\beta[B_t/\beta_t \,|\, \mathcal{H}_t]$. Clearly, (iv) (with $C = B$) implies (iii). The converse holds because $V_t^B(C) = V_t^\beta(C)B_t/V_t^\beta(B)$. (iii) and (v) are equivalent because, in general, $dP^B{}_{|\mathcal{H}_t}/dP^\beta{}_{|\mathcal{H}_t} = (\beta_0/B_0)V_t^\beta(B)/\beta_t$. □

Note also, if β and B are coadapted numeraires, then a claim C is β-coadapted if and only if C is B-coadapted. Conditional independence is invariant under coadapted change of numeraire too:

Proposition 5.6. *Let $B > 0$ and $\beta > 0$ be $\mathcal{H}_t$-coadapted numeraires. Assume $\mathcal{H}_t$ is β-conditionally-independent. Then $\mathcal{H}_t$ is B-conditionally-independent.*

Proof. Let C be any claim such that $C/B \in \mathcal{H}$. Then $C/\beta \in \mathcal{H}$. Hence, C_t/β_t is $\mathcal{H}_t$ adapted. So, C_t/B_t is also $\mathcal{H}_t$-adapted. $\square$

The following provides a transformation formula for α_t^β under a general change of numeraire. We actually give a more general result.

Proposition 5.7. *Let β and B be numeraires and $F \in \mathcal{F}_t$ be bounded. Then, $E^B[F|\mathcal{H}_t]V_t^B(F\beta)V_t^\beta(B) = E^\beta[F|\mathcal{H}_t]V_t^\beta(FB)V_t^B(\beta)$. Moreover, if $F \geq 0$ and $E[F|\mathcal{H}_t] > 0$ for all t, then $E^B[F\beta_t/B_t|\mathcal{H}_t] > 0$ a.s., and*

$$E^B[F|\mathcal{H}_t] = E^\beta[F|\mathcal{H}_t]\ \frac{E^\beta[FB_t/\beta_t|\mathcal{H}_t]E^B[\beta_t/B_t|\mathcal{H}_t]}{E^B[F\beta_t/B_t|\mathcal{H}_t]E^\beta[B_t/\beta_t|\mathcal{H}_t]}.$$

In particular, when $\mathcal{M}$ is positive, this applied to $F = 1_{\tau>t}$ gives

$$\alpha_t^B = \alpha_t^\beta \frac{V_t^\beta(1_{\tau>t}B)V_t^B(\beta)}{V_t^B(1_{\tau>t}\beta)V_t^\beta(B)} = \ \alpha_t^\beta\ \frac{E^\beta[1_{\tau>t}B_t/\beta_t|\mathcal{H}_t]E^B[\beta_t/B_t|\mathcal{H}_t]}{E^B[1_{\tau>t}\beta_t/B_t|\mathcal{H}_t]E^\beta[B_t/\beta_t|\mathcal{H}_t]}.$$

Proof. Let C denote the claim $C = FB$. By definition, $V_t^B(C) = B_tE^B[F|\mathcal{H}_t]$. Hence by Proposition 3.1,

$$E^B[F|\mathcal{H}_t] = \frac{V_t^\beta(FB)}{V_t^\beta(B)}.$$

Reversing the role of β and B, by symmetry, we conclude also

$$E^\beta[F|\mathcal{H}_t] = \frac{V_t^B(F\beta)}{V_t^B(\beta)}.$$

The first equation follows. Using The definition the V_t operator, we can rewrite the first equation as

$$E^B[F|\mathcal{H}_t]E^B[F\beta/B|\mathcal{H}_t]E^\beta[B/\beta|\mathcal{H}_t] = E^\beta[F|\mathcal{H}_t]\ E^\beta[FB/\beta|\mathcal{H}_t]E^B[\beta/B|\mathcal{H}_t].$$

But, note in the above equation we can substitute

$$E^\beta[FB/\beta|\mathcal{H}_t] = E^\beta[FB/\beta|\mathcal{F}_t|\mathcal{H}_t] = E^\beta[FE^\beta[B/\beta|\mathcal{F}_t]|\mathcal{H}_t] = E^\beta[FB_t/\beta_t|\mathcal{H}_t].$$

Substituting similar expressions for other terms in the equation, we get

$$E^B[F|\mathcal{H}_t]E^B[F\beta_t/B_t|\mathcal{H}_t]E^\beta[B_t/\beta_t|\mathcal{H}_t] = E^\beta[F|\mathcal{H}_t]\ E^\beta[FB_t/\beta_t|\mathcal{H}_t]E^B[\beta_t/B_t|\mathcal{H}_t].$$

The positivity assumptions on F implies $V_t^B(F\beta) > 0$ The second equation thus follows by dividing. $\square$

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