Chaotic Expansion of Powers
&
Martingale Representation

Farshid Jamshidian

NIBCapital Bank

FELAB, Applied Math Dept., Univ. of Twente

http://wwwhome.math.utwente.nl/ jamshidianf/

November 2005

Contents

Two classical martingale representation results

- Multidimensional **Brownian** filtration, e.g., [D2], [E], [K-S], [Ø], [P].
- Multidimensional **Poisson** filtration, e.g., [D2], [E].

A *square-integrable* martingale M can be represented as

$$M = \sum_{n=1}^{k} \int H_n dN_n,$$

where H_n are predictable processes satisfying (with $T \leq \infty$)

$$\sum_{n=1}^{k} \mathbb{E} \int_0^T H_n^2(t) d\langle N_n \rangle_t < \infty.$$

- $\langle N_n \rangle_t = \sigma_n^2 t$ if N_n are independent **Brownian motions** with volatilities σ_n.
- $\langle N_n \rangle_t = \lambda_n t$ if N_n are independent compensated **Poisson processes** of intensities λ_n.
- Works even in infinite dimensions - simply let $k = \infty$!

Other martingale representation results

- Davis (1976): Filtration generated by a general point process. Representation as a convolution against a finite-variation random measure.
- Nualart-Schoutens (2000): Filtration generated by a Lévy process. Representation as a *series* of stochastic integral of **Teugels martingales**.
- Davis (2005): Nice, short exposition of results and ideas. Contrasts methods based on Jacod-Yor Theorem with denseness arguments in [Ø], [N-S].
- Azéma martingales are also known to satisfy the predictable representation property.
- Here: Extension of [N-S] where jump intensities are adapted to a Brownian filtration (rather than just being deterministic).

The main ideas of the approach

[N-S] ideas when the filtration is generated by a Lévy process X:

- The notion of **Teugels martingales** $X^{(n)}$ as *compensated* "power jump processes" $\sum_{s \le t} (\Delta X_s)^n$.
- The set of multivariate polynomials in X_{t_i}, $t_i \le T$, is dense in the space L^2 of square-integrable random variables.
- Any such polynomial has a chaotic representation as a finite sum of stochastic integrals of the Teugels martingales.

Suitably reformulated, the above continue to hold when all jump intensities are continuous and adapted to Brownian filtration.

- An idea of *strong orthogonalization* from Davis (2005) helps to streamline the results in the general case.

A simple example of Nualart-Schoutens result

Let $P_1, \cdots, P_k$ be independent compensated Poisson processes. Let $a_1, \cdots, a_k$ be *distinct* real numbers. Let

$$X = \sum_{i=1}^{k} a_i P_i.$$

Then, the *Teugels martingale* $X^{(n)}$ of order n is given by

$$X^{(n)} = \sum_{i=1}^{k} a_i^n P_n, \qquad n \in \mathbb{N}.$$

Assume the filtration is generated by X or equivalently by $P_1, \cdots, P_k$. Then, martingales M can represented as $M = \sum_{n=1}^{k} \int H_n dX^{(n)}$. This follows from the classical Poisson case as each P_i is a linear combination of the $X^{(j)}$ (since the Vandermonde matrix (a_i^n) is nonsingular).

Space $\mathcal{H}^2$ of square-integrable martingales

Stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$:

$$\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \quad \mathcal{F} = \mathcal{F}_T, \quad T \leq \infty.$$

L^p := space of random variables ξ such that $\mathbb{E}\,|\xi|^p < \infty$.
$\mathcal{M}$:= set of uniformly integrable martingale vanishing at $t = 0$.
$\mathcal{H}^2$:= Hilbert space of square integrable martingales:

$$\mathcal{H}^2 := \{M \in \mathcal{M} : M_T \in L^2\} = \{M \in \mathcal{M}_{\text{loc}} : [M]_T \in L^1\}.$$

If $M \in \mathcal{H}^2$, then $X^{(2)} := [X] - \langle X \rangle$ is a martingale in $\mathcal{M}$.

$$\text{Hilbert Norm:} \quad \|M\|^2 := \mathbb{E}\, M_T^2 = \mathbb{E}\,[M]_T = \mathbb{E}\,\langle M \rangle_T.$$

Stable subspaces of $\mathcal{H}^2$

For $N \in \mathcal{H}^2$, set

$$\mathcal{S}(N) := \{\int H dN : H \text{ a predictable process; } \int_0^T H^2 d\langle N\rangle \in L^1\}.$$

Then $\mathcal{S}(N)$ is a **stable subspace,** i.e., a ***closed*** subspace of $\mathcal{H}^2$ that contains $\mathcal{S}(M)$ for any $M \in \mathcal{S}(N)$, or equivalently, is closed under stopping (c.f., [P], [E]).

Let $(M_i)_{i\in I}$ be a family of martingales in $\mathcal{H}^2$. Set

$$\mathcal{S}(M_i)_{i\in I} := \text{closure of Span}(\mathcal{S}(M_i))_{i\in I}.$$

(Given a family $(\mathcal{K}_i)_{i\in I}$ of subspaces of a vector space, Span$(\mathcal{K}_i)_{i\in I}$ denotes their linear span.)

Then, $\mathcal{S}(M_i)_{i\in I}$ is the **stable subspace generated by** $(M_i)_{i\in I}$, i.e., it is the smallest stable subspace that contains all M_i.

Power brackets and Teugels martingales

For a semimartingale X, set $[X]^{(1)} := X$, $[X]^{(n+1)} = [X, [X]^{(n)}]$:

$$[X]^{(2)} := [X] = [X^c] + \sum_{s\leq\cdot} (\Delta X_s)^2;$$

$$[X]^{(n)} := \sum_{s\leq\cdot} (\Delta X_s)^n, \quad 3 \leq n \in \mathbb{N}.$$

Assume X is special and $[X]_T^{(2n)}$ is integrable, $n \geq 1$. Denote

$$\langle X \rangle^{(n)} := \text{compensator of } [X]^{(n)}, \quad n \in \mathbb{N};$$

$$X^{(n)} := [X]^{(n)} - \langle X \rangle^{(n)}.$$

The **Teugels martingales** $X^{(n)}$ are always in $\mathcal{M}$ for $n \geq 2$.

We will assume all $\langle X \rangle^{(n)}$ are continuous. Then $[X^{(n)}] = [X]^{(2n)}$, implying $X^{(n)} \in \mathcal{H}^2$.

[N-S] notion differs. Alternative notation for $[X]^{(n)}$ is $X^{[n]}$ and for $\langle X \rangle^{(n)}$ is $X^{\langle n \rangle}$.

Relationship between powers and power brackets

Let X be a special semimartingale with X_0 = 0 and $n \in \mathbb{N}$. Then

$$X^n = \sum_{i=0}^{n-1} \binom{n}{i} \int X_-^i d[X]^{(n-i)}.$$

Implies canonical decomposition $X^n = \widetilde{X^n} + \widehat{X^n}$, where

$$\widetilde{X^n} = \sum_{i=0}^{n-1} \binom{n}{i} \int X_-^i dX^{(n-i)} = X^{(n)} + \cdots + n \int X_-^{n-1} dX^{(1)} \in \mathcal{M};$$

$$\widehat{X^n} = \sum_{i=0}^{n-1} \binom{n}{i} \int X_-^i d\langle X \rangle^{(n-i)} = \langle X \rangle^{(n)} + \cdots + n \int X_-^{n-1} d\langle X \rangle^{(1)}.$$

So, $\mathcal{S}(X^{(n)})_{n=1}^k = \mathcal{S}(\widetilde{X^n})_{n=1}^k$ for any $k \leq \infty$.

Exponentially decaying laws and L^2 denseness of polynomials

Define

$$L_* := \{\text{random variables } \xi : \mathbb{E}\exp(a|\xi|) < \infty \text{ for some } a > 0\}.$$

$$\mathcal{C}_* := \{\text{right continuous processes } X\text{: } X_t \in L_* \text{ for all } t \in [0,T]\}.$$

Proposition. *Let $X \in \mathcal{C}_*$. Assume $\mathbb{F}$ is generated by X. Then* Span$\{X_{t_1} \cdots X_{t_n}\}_{(t_1,\cdots,t_n)\in[0,T]^n,\, n\geq 0}$ *is dense in L^2.*

Remark: As t_i need not be distinct, powers X_t^n are included.

Remark: The above suffices for the (univariate) Lévy case. But, our main result requires a similar multivariate version.

Martingale representation in the Lévy case

Theorem. *Let X be a Lévy process. Assume $X \in \mathcal{C}_*$. Assume $\mathbb{F}$ is generated by X. Then $\mathcal{H}^2 = \mathcal{S}(X^{(n)})_{n=1}^{\infty}$.*

Remark: The above is valid more generally for a nonstationary Lévy process X, i.e., one with $\langle X \rangle^{(n)}$ deterministic and continuous, all $n \in \mathbb{N}$. For a standard Lévy process, each $\langle X \rangle^{(n)}$ is a constant times t.

Remark: In the Brownian (resp. Poisson), $X^{(n)}$ equals 0 (resp. $X^{(1)}$) for $n \geq 2$. So, in both cases, $\mathcal{H}^2 = \mathcal{S}(X^{(1)})$.

Remark: We also have the representation $\mathcal{H}^2 = \mathcal{S}(\widetilde{X^n})_{n=1}^{\infty}$.

Multivariate Version: *If $\mathbb{F}$ is generated by Lévy processes $(X_i)_{i \in I}$ in $\mathcal{C}_*$ satisfying $[X_i, X_j] = 0$, $i \neq j$, then $\mathcal{H}^2 = \mathcal{S}(X_i^{(n)})_{n \in \mathbb{N}, i \in I}$.*

Main result: Martingale representation for processes with Brownian-adapted jump intensities

Let $(B_i)_{i\in I}$ denote a family of *Brownian martingales* such that $[B_i, B_j] = 0$ for $i \neq j$. So, each B_i is a continuous martingale with deterministic quadratic variation, e.g., a Brownian motion.

Theorem. *Let* $X \in \mathcal{M} \cap \mathcal{C}_*$. *Assume* $[X, B_i] = 0$ *for all* i, *and* $\langle X \rangle^{(n)}$ *is continuous and adapted to the filtration generated by* $(B_i)_{i\in I}$ *for all* n. *Assume* $\mathbb{F}$ *is generated by* X *and* $(B_i)_{i\in I}$. *Then,* $\mathcal{H}^2 = \mathcal{S}(X^{(n)}, B_i)_{n\in\mathbb{N},\, i\in \mathrm{I}}$.

Generalizes to a semimartingale X, provided X and its compensator $\langle X \rangle^{(1)}$ are both in $\mathcal{C}_*$.

Remark: Lévy case follows by setting $B_i = 0$, all i.
Remark: We also have $\mathcal{H}^2 = \mathcal{S}(\widetilde{X^n}, B_i)_{n\in\mathbb{N}, i\in I}$.
Jump-diffusion Example: $X_t = \sum_{i=1}^{C_t} \xi_i$, where C is a **Cox process** and $\xi_i \in L_*$ are iid.

Multivariate version: If $(X_j)_{j\in J}$ is like X above and satisfy $[X_j, X_k] = 0$, $j \neq k$, then $\mathcal{H}^2 = \mathcal{S}(X_j^{(n)}, B_i)_{n\in\mathbb{N}, j\in J, i\in I}$.

Basic idea of the proof = Denseness + Expansion

By the L^2**-denseness of polynomials** it suffices to show:

- Lévy case: products of form $X_{t_1} \cdots X_{t_k}$ can be represented by the $X^{(n)}$, $n \leq k$.
- General case: $X_{t_1} \cdots X_{t_k} B_{i_1}(s_1) \cdots B_{i_l}(s_l)$ can be represented by the $X^{(n)}$, B_i.

We show this using an **inductive chaotic expansion of products** $X_{t_1} \cdots X_{t_n}$.

Basically, we show that *such a product is a finite sum of products of form* $A_T M_T$*, where* $M \in \mathbb{R} + \mathcal{S}(X^{(n)})_{n=1}^{\infty}$*, and* A *is an iterated (multiple) integral of the* $\langle X \rangle^{(n)}$. Using this,

- The Lévy case follows immediately as A_T is deterministic (because $\langle X \rangle^{(n)}$ are so).
- General case: A is Brownian adapted, since $\langle X \rangle^{(n)}$ are so by assumption. Hence, by Brownian representation, $A_T = \sum \int_0^T H_i dB_i$, for some predictable H_i. Thus,

$$A_T M_T = \int_0^T A dM + \sum_i \int_0^T M_{i-} H_i dB_i.$$

(Above we used $[B_i, M] = 0$, which follows from $[B_i, X^{(n)}] = 0$: for $n = 1$ by assumption $[B_i, X] = 0$, and for $n \geq 2$ because B_i are continuous while $X^{(n)}$ are purely discontinuous.)

Example: inductive chaotic expansion of powers

$$X^2 = \langle X\rangle + X^{(2)} + 2\int X_- dX, \quad X \in \mathcal{M}_{\text{loc}}.$$

For $X \in \mathcal{M}_{\text{loc}}$, $X^3 = \widehat{X^3} + \widetilde{X^3}$, where $\widetilde{X^3} = X^{(3)} + 3\int X_- dX^{(2)} + 3\int X_-^2\, dX$,

$$\widehat{X^3} = \langle X\rangle^{(3)} + 3\int X_- d\langle X\rangle = \langle X\rangle^{(3)} + 3\langle X\rangle X - 3\int \langle X\rangle dX.$$

Let X be a semimartingale, $X_0 = 0$. Assume all $\langle X\rangle^{(n)}$ are continuous. We claim X^n *is a sum of forms* AM*, where* $M \in \mathbb{R} + \mathcal{S}(X^{(i)})_{i=1}^n$*, and* A *is an iterated integral of* $\langle X\rangle^{(j)}$*,* $j \le n$*.*

Indeed, recall $X^n = \widetilde{X^n} + \widehat{X^n}$, where $\widehat{X^n} = \sum_{i=0}^{n-1} \binom{n}{i} \int X_-^i\, d\langle X\rangle^{(n-i)}$, and by a similar formula, $\widetilde{X^n} \in \mathcal{S}(X^{(i)})_{i=1}^n$. By induction X^i is a sum of form AM, with $M \in \mathcal{S}(X^{(j)})_{j=1}^i$ and A an iterated integral of $\langle X\rangle^{(j)}$, $j \le i$. Then, $\int X_-^i\, d\langle X\rangle^{(n-i)}$ is a sum of the form $\int AM_- d\langle X\rangle^{(n-i)}$. Integrating by parts,

$$\int AM_- d\langle X\rangle^{(n-i)} = M\int Ad\langle X\rangle^{(n-i)} - \int (\int Ad\langle X\rangle^{(n-i)}) dM.$$

But, $\int Ad\langle X\rangle^{(n-i)}$ is an iterated integral of the $\langle X\rangle^{(j)}$ as A is so. The claim follows.

Explicit chaotic expansion of powers

Set $\mathbb{N}_n^p := \{I = (i_1, \cdots, i_p) \in \mathbb{N}^p : i_1 + \cdots + i_p = n\}, \ 1 \leq p \leq n \in \mathbb{N}$.

Let X be a semimartingale with $X_0 = 0$ and $n \in \mathbb{N}$. Then

$$X^n = \sum_{p=1}^{n} \sum_{I \in \mathbb{N}_n^p} \frac{n!}{i_1! \cdots i_p!} \int \int^- \cdots \int^- d[X]^{(i_1)} \cdots d[X]^{(i_{p-1})} d[X]^{(i_p)}.$$

Let M, $A_1, \cdots, A_n$ be semimartingales, $A_i(0) = 0$. Assume A_i are continuous and of finite variation. Then

$$\int \cdots \int M_- dA_1 \cdots dA_n = \sum_{p=0}^{n} \sum_{0=i_0<i_1<\cdots<i_p\leq n} (-1)^p (\int A_{i_0,i_1} \cdots A_{i_{p-1},i_p} \, dM) \, A_{i_p,n},$$

where $A_{i,i} = 1$, $A_{i,j} := \int \cdots \int dA_{i+1} \cdots dA_j$, $0 \leq i < j \leq n$.

Example: $n = 12 + 1$, $p = 4$, and $(i_1, i_2, i_3, i_4) = (2, 6, 7, 10)$. Then the corresponding term is

$$\int (\int A_1 dA_2 (\int \int \int A_3 dA_4 dA_5 dA_6) A_7 \int \int A_8 dA_9 dA_{10}) \, dM \ (\int \int A_{11} dA_{12} dA_{13}).$$

Strong orthogonalization

If $M, N \in \mathcal{H}^2$ are strongly orthogonal, i.e, $\langle M, N \rangle = 0$, then $\mathcal{S}(M) \perp \mathcal{S}(M)$, hence $\mathcal{S}(M) + \mathcal{S}(N)$ is closed, so equals $\mathcal{S}(M, N)$. More generally, given a sequence $(N_i)_{i \in \mathbb{N}}$ of mutually strongly orthogonal martingales in $\mathcal{H}^2$, we have

$$\mathcal{S}(N_i)_{i \in \mathbb{N}} = \bigoplus_{i=1}^{\infty} \mathcal{S}(N_i) := \{ \sum_{i=1}^{\infty} X_i : X_i \in \mathcal{S}(N_i); \ \sum_{i=1}^{\infty} \|X_i\|^2 < \infty \}.$$

Given any sequence $(M_i)_{i \in \mathbb{N}}$ of martingales in $\mathcal{H}^2$, define its *strong orthogonalization* $(N_i)_{i \in \mathbb{N}}$ by $N_i =$ orthogonal projection of M_i on orthogonal complement of $\mathcal{S}(M_j)_{j=1}^{i-1}$, $\ N_1 := M_1$.
Then, $\mathcal{S}(M_i)_{i=1}^{k} = \bigoplus_{i=1}^{k} \mathcal{S}(N_i)$, for all $k \leq \infty$.
Davis (2005): If sequence (M_i) is dense then $\mathcal{H}^2 = \bigoplus_{i=1}^{\infty} \mathcal{S}(N_i)$.

Proof: Then $\mathcal{S}(M_i)_{i=1}^{\infty}$ is dense in $\mathcal{H}^2$. Hence, $\mathcal{H}^2 = \mathcal{S}(M_i)_{i=1}^{\infty} = \bigoplus_{i=1}^{\infty} \mathcal{S}(N_i)$.

A technical issue of integrability: spaces $\mathcal{H}^*$, $\mathcal{A}^*$

Set $L^* := \bigcap_{n=1}^{\infty} L^n = \{$random variables of finite moments$\}$;

$$\mathcal{H}^\star := \{M \in \mathcal{H}^2 : M_T \in L^*\} = \{M \in \mathcal{M}_{\text{loc}} : [M]_T \in L^*\};$$

The equality follows from Burkholder-Davis-Gundy inequalities.

$M, N \in \mathcal{H}^\star \Rightarrow \int M_- dN \in \mathcal{H}^\star$ by BDG, Doob's maximal, Schwartz inequalities.

If $M \in \mathcal{H}^\star$, then $[M]_T^{(n)}, \langle M \rangle_T^{(n)} \in L^*$, and $M^{(n)} \in \mathcal{H}^\star$, all n.

Set $\mathcal{H}^* := \{M \in \mathcal{H}^\star : \langle M \rangle^{(n)}$ is continuous for all $n \geq 2\}$.

If $M, N \in \mathcal{H}^*$, then $\int M_- dN, M^{(n)} \in \mathcal{H}^*$, all n.

Set $\mathcal{A}^* := \mathcal{A}^{*+} \ominus \mathcal{A}^{*+}$, where

$$\mathcal{A}^{*+} := \{\text{increasing continuous semimartingales } A : A_T \in L^*\}.$$

If $M \in \mathcal{H}^*$, then $\langle M \rangle^{(n)} \in \mathcal{A}^*$ for all n.

If $A, B \in \mathcal{A}$ then $AB \in \mathcal{A}^*$. If $M \in \mathcal{H}^*$ then $\int A dM \in \mathcal{H}^*$.

More on denseness of polynomials in L^2 and $\mathcal{H}^2$

Using Schwartz inequality, L_* *is a linear space*. (It is a subspace of L^*.)

If $\mathcal{F}$ is generated by $\xi_1, \cdots, \xi_n \in L_*$, then the linear span $\mathrm{Span}\{\xi_{i_1} \cdots \xi_{i_m}\}_{1 \le i_1, \cdots, i_m \le n,\, m \ge 0}$ *is a dense subspace of L^2.*

Proof uses denseness of $\{f(\xi_1, \cdots, \xi_n) : f \in C_0^\infty(\mathbb{R}^n)\}$ in L^2 and a Paley-Weiner argument for holomorphic extension to $\mathbb{C}^n$ of the Fourier transform of a distribution.

When $\mathcal{F}$ is generated by a countable $\xi_i \in L_*$, a similar result follows by the L^2-version of martingale convergence theorem. This in turn implies the result for processes by right continuity.

Clearly, the mapping $\xi \rightsquigarrow (\overline{\mathbb{E}}(\xi \,|\, \mathbb{F}), \mathbb{E}\,\xi)$ is an isometry of L^2 onto $\mathcal{H}^2 \times \mathbb{R}$, where

$$\overline{\mathbb{E}}(\xi \,|\, \mathbb{F}) := (\mathbb{E}(\xi \,|\, \mathcal{F}_t))_{t \in [0,T]} - \mathbb{E}\,\xi \in \mathcal{H}^2.$$

Assume $\mathbb{F}$ is generated by $X_i \in \mathcal{C}_*$, $i = 1, 2, \cdots$. Then subspace $\mathrm{Span}\{\overline{\mathbb{E}}(X_{i_1}(t_1) \cdots X_{i_n}(t_n) \,|\, \mathbb{F})\}_{(t_1, \cdots, t_n) \in [0,T]^n,\, (i_1, \cdots, i_n) \in \mathbb{N}^n,\, n \in \mathbb{N}}$ *is dense in $\mathcal{H}^2$.*

References

[D]. Davis, M.H.A.: Martingale Representation and All That. *Advances in Control, Communication Networks, and Transportation Systems: In Honor of Pravin Varaiya*, E.H. Abed (Ed.), Systems and Control: Foundations and Applications Series, Birkhauser (2005).

[D2]. Davis, M.H.A.: The representation of martingales of jump processes. *SIAM J. Control* & *Optimization* 14, 623-638, (1976).

[D3]. Davis, M.H.A.: Martingales of Wiener and Poisson processes. *J. Lon. Math. Soc.* (2), 13, 336-338, (1976).

[E]. Elliott, R.J.: *Stochastic Calculus and Applications*. Springer, (1982).

[J-S]. Jacod, J. and Shiryaev, A.N.: *Limit Theorems for Stochastic Processes*. Springer Verlag, Berlin (1987).

[K-S]. Karatzas, I. and Shreve, E: *Brownian Motion and Stocahstic Calculus*. Springer-Verlag, second edition (1991).

[N-S]. Nualart, D. and Schoutens, W.: Chaotic and predictable representations for Lévy processes. *Stochastic Processes and their Applications* 90, 109-122, (2000).

[Ø]. Øksendal, B: S*tochastic Differential Equations*. Springer-Verlag, 6th ed. (2004).

[P]. Protter, P.: *Stochastic Integration and Differential Equations*. Springer-Verlag, Berlin (1992).