On the Combinatorics of Iterated Stochastic Integrals

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Version: 10 April 2008. The paper is available at http://wwwhome.math.utwente.nl/~ jamshidianf.

1. Introduction

1.1. Known results. (For *continuous* semimartingales)

- Relation between the powers X^n and iterated integrals

$$X^{(n)} = \int \cdots \int dX \cdots dX$$

of a continuous semimartingale X with $X_0 = 0$.

- Itô's (1951) formula for $X^{(n)}$ for a Brownian motion.
- Revuz and Yor (1991) generalization to continuous case:

$$X^{(n)} = \sum_{i,j \geq 0;\, i+2j=n} \frac{(-1)^j}{i!j!2^j} X^i [X]^j.$$

- Coefficients above are those of *Hermite Polynomials*.
- The stochastic exponential as sum of iterated integrals:

$$\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{(n)}.$$

1.2. **This paper.** We derive

- A formula for $X^{(n)}$ for general semimartingales.
- It uses a recursion, e.g., in the continuous case:

$$nX^{(n)} = XX^{(n-1)} - [X]X^{(n-2)}.$$

- Derive a formula for X^n, e.g., in the continuous case,

$$X^n = \sum_{i,j \geq 0;\ i+2j=n} \frac{n!}{j!2^j} X^{(i)}[X]^j.$$

- Prove $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{(n)}$ in general, with the sum absolutely convergent a.s.
- Uses a new exponential formula $\mathcal{E}(X)$.
- Uses a result for the iterated integral of sum.

- Show, when $X_t = \sum_{s \le t} \Delta X_s$,

$$X_t^{(n)} = \sum_{s_1 < \cdots < s_n \le t} \Delta X_{s_1} \cdots \Delta X_{s_n}.$$

- Show, for a counting (e.g., Poisson or Cox) process N:

$$N^{(n)} = 1_{N \ge n} \binom{N}{n};$$

and

$$N^n = \sum_{i=1}^{n} c_{n,i} N^{(i)},$$

where,

$$c_{n,i} := \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^n.$$

$c_{n,i}/i!$ are the *Stirling numbers of second kind.*

- An application to chaotic martingale representation.

2. Results for general semimartingales

Below, $X = (X_t)_{t=0}^{\infty}$ is a semimartingale with $X_0 = 0$.

2.1. **Inductive definition of iterated integral:**

$$X^{(0)} := 1, \qquad X^{(n)} = \int X_{-}^{(n-1)} dX.$$

Thus,

$$X^{(1)} = X,$$
$$X^{(2)} = \int X_{-} dX = \int \int^{-} dX dX,$$
$$X^{(3)} = \int X_{-}^{(2)} dX = \int \int^{-} \int^{-} dX dX dX.$$

2.2. **Definition of power jump processes $X^{[n]}$:**

$$X^{[1]} := X, \qquad X^{[n]} = [X^{[n-1]}, X].$$

Thus,

$$X^{[2]} := [X] = [X]^c + \sum_{s \leq \cdot} (\Delta X_s)^2,$$

$$X_t^{[n]} = \sum_{s \leq t} (\Delta X_s)^n \ \text{ for } n \geq 3.$$

2.3. **Recursion formula:**

$$nX^{(n)} = \sum_{i=1}^{n} (-1)^{i-1} X^{[i]} X^{(n-i)}.$$

Proof: Induction, Itô's product rule, properties of the bracket.

2.4. **Formula for $X^{(n)}$:**

$$X^{(n)} = \sum_{i_1,\cdots,i_n \geq 0;\ i_1+2i_2+\cdots+ni_n=n} \frac{(-1)^{i_2+i_4+\cdots+i_{2[n/2]}}}{i_1!\cdots i_n! 2^{i_2}\cdots n^{i_n}} X^{i_1}(X^{[2]})^{i_2}\cdots(X^{[n]})^{i_n}.$$

Proof: Substitute for each $X^{(n-i)}$ in the recursion 2.3 by induction, and rearrange using index manipulation.

2.5. **Continuous case.** Then, $X^{(n)} = 0$ for $n \geq 3$, so

$$X^{(n)} = \sum_{i,j \geq 0;\, i+2j=n} \frac{(-1)^j}{i!j!2^j} X^i [X]^j.$$

2.6. **Examples,** $n \leq 5$**:**

$$2X^{(2)} = X^2 - [X].$$

$$6X^{(3)} = X^3 - 3[X]X + 2X^{[3]}.$$

$$4!X^{(4)} = X^4 - 6[X]X^2 + 3[X]^2 + 8XX^{[3]} - 6X^{[4]}.$$

$$5!X^{(5)} = X^5 - 10X^3[X] + 20X^2X^{[3]} + 15X[X]^2 - 30XX^{[4]} - 20[X]X^{[3]} + 4!X^{[5]}.$$

$6!X^{[6]}$ contains the term $-120X[X]X^{[3]}$, etc.

2.7. **Stochastic exponential:** $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{(n)}$.

The continuous case simply follows using 2.5:

$$\mathcal{E}(X) = e^{X-[X]/2} = \sum_{i=0}^{\infty} \frac{X^i}{i!} \sum_{j=0}^{\infty} \frac{(-1)^j [X]^j}{2^j j!} = \sum_{n=0}^{\infty} X^{(n)}.$$

For the general case, we first show:
If $|\Delta X| < 1$, *then* $\sum_{k=1}^{\infty} |X^{[k]}/k| < \infty$, *and*

$$\mathcal{E}(X) = \exp\Big(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} X^{[k]}\Big).$$

This and the formula 2.5 for $X^{(n)}$ are employed to show

$$\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{(n)}$$

when $|\Delta X| < 1$, with the series absolutely convergent.

The argument further utilizes the following:

$$e^{\sum_{k=1}^{\infty} x_k} = \prod_{k=1}^{\infty} e^{x_k} = \prod_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{x_k^i}{i!}$$

$$= \sum_{n=0}^{\infty} \Big(\sum_{i_1,\cdots,i_n \geq 0;\, i_1+2i_2+\cdots n i_n = n} \frac{x_1^{i_1} \cdots x_n^{i_n}}{i_1! \cdots i_n!} \Big),$$

with $x_k = (-1)^{k-1} X^{[k]}/k$.

The general case follows easily from this case $|\Delta X| < 1$ and the results of Section 3 (by decomposing X into its part with jump size less than 1 and its complement which is the sum of jumps of absolute size ≥ 1).

2.8. **The powers X^n.** "Inverting" 2.5 yields,

$$X^n = \sum_{i_1,\cdots,i_n \geq 0;\; i_1+2i_2+\cdots+ni_n=n} \frac{(-1)^{n-i_1} n!}{i_2! \cdots i_n! 2^{i_2} \cdots n^{i_n}} X^{(i_1)} (X^{[2]})^{i_2} \cdots (X^{[n]})^{i_n}.$$

For the case $|\Delta X| < 1$, we can also derive this using

$$\sum_{j=0}^{\infty} \lambda^j X^{(j)} = \mathcal{E}(\lambda X) = \exp\Big(\sum_{k=1}^{\infty} (-1)^{k-1} \lambda^k X^{[k]}/k\Big).$$

2.9. **The continuous case.** Simplifies to

$$X^n = \sum_{i,j \geq 0,\; i+2j=n} \frac{n!}{j! 2^j} X^{(i)} [X]^j.$$

2.10. **Examples, $n \leq 6$, continuous case.**

$$X^2 = 2X^{(2)} + [X].$$
$$X^3 = 6X^{(3)} + 3[X]X.$$
$$X^4 = 24X^{(4)} + 12[X]X^{(2)} + 3[X]^2.$$
$$X^5 = 120X^{(5)} + 60[X]X^{(3)} + 15[X]^2X.$$
$$X^6 = 720X^{(6)} + 360[X]X^{(4)} + 90[X]^2X^{(2)} + 15[X]^3.$$

2.11. **An application.** Let X be a Brownian motion. Then, for $T > 0$, we get a *chaotic representation* of X_T^n:

$$X_T^n = n! \sum_{i,j\geq 0,\ i+2j=n} \frac{T^j}{j!2^j} X_T^{(i)}.$$

3. Related results

To complete the proof of $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{(n)}$ in Sec. 2.7 for the general case, we use the following two results.

3.1. **Sum of jump processes.** *Suppose X has finite variation and $X_t = \sum_{s \le t} \Delta X_s$. Then*

$$X_t^{(n)} = \sum_{s_1 < \cdots < s_n \le t} \Delta X_{s_1} \cdots \Delta X_{s_n}.$$

Moreover,

$$\mathcal{E}(X) = \prod_{s \le \cdot} (1 + \Delta X_s) = \sum_{n=0}^{\infty} X^{(n)},$$

with $\sum_{n=0}^{\infty} |X^{(n)}| \le \exp(\sum_{s \le \cdot} |\Delta X_s|) < \infty$.

3.2. **Iterated integrals of a sum.** *Let X and Y be semimartingales with $X_0 = Y_0 = 0$ and $[X,Y] = 0$. Then*

$$(X+Y)^{(n)} = \sum_{i=0}^{n} X^{(i)} Y^{(n-i)}.$$

(This easily implies that if $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{(n)}$ and $\mathcal{E}(Y) = \sum_{n=0}^{\infty} Y^{(n)}$, then $\mathcal{E}(X+Y) = \sum_{n=0}^{\infty} (X+Y)^{(n)}$.)

Another version is: *Let X and Y be continuous semimartingales with $X_0 = Y_0 = 0$. Then,*

$$(X+Y)^{(n)} = \sum_{i,j,k\geq 0;\, i+j+2k=n} \frac{(-1)^k}{k!} X^{(i)} Y^{(j)} [X,Y]^k.$$

4. The case of a counting process

We call a semimartingale N with $N_0 = 0$ a *counting process* if $[N] = N$.
Equivalently, N is the sum of its jumps all which equal 1.

Examples are Poisson , or more generally, Cox processes.

If N is a counting process, then for $\lambda, a \in \mathbb{R}$ we have,

$$(1+\lambda)^N = \sum_{n=0}^{\infty} \lambda^n N^{(n)};$$

$$e^{aN} = \sum_{n=0}^{\infty} (e^a - 1)^n N^{(n)};$$

Further, for $n \in \mathbb{N}$, *we have,*

$$N^{(n)} = 1_{N \geq n} \binom{N}{n};$$

$$N^n = \sum_{i=1}^{n} c_{n,i} N^{(i)},$$

where for $n, i = 0, 1, 2 \cdots,$

$$c_{n,i} := \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^n.$$

The numbers $c_{n,i}/i!$ are the *Stirling numbers of the second kind,* i.e., the number of partitions of $\{1, \cdots, n\}$ into i nonempty subsets.

4.1. **An application.** For a Poisson process N with intensity λ, and $T > 0$, we get the *chaotic representation*

$$N_T^n = \sum_{i=0}^{n} a_{n,i,T} (N - \lambda t)_T^{(i)},$$

where $a_{n,i,T}$ are constants given by

$$a_{n,i,T} := \sum_{k=0}^{n-i} \sum_{j=1}^{k+i} (-1)^{k+i-j} \frac{(k+i)!j^n}{(k+i-j)!j!k!} \lambda^k T^k.$$

A similar chaotic expansion holds for Cox processes.

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