

Measure changes for interest rate modelling: forward and swap measures

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Abstract. Forward and swap measures are defined and some of their main applications illustrated. The development is based on the construction of general numeraire measures from a state price density. Using the numeraire-invariance theorem and the fact that forward asset prices are martingales under the forward measure, we indicate how a call option is replicated when the forward asset price process is a diffusion. The standard pricing formula is derived in the deterministic-volatility case. Caps and swaptions are treated similarly, using that forward Libor and swap rates are martingales under respectively the forward and swap measures. As another application, we calculate in the continuous case the risk-neutral drift of the instantaneous forward rate and the terminal-measure drift of the Libor forward rate.

1. INTRODUCTION

Forward and swap measures are instances of general numeraire measures and are useful in interest-rate modelling and derivatives valuation. They take respectively a zero-coupon bond and an annuity as the numeraire. Accordingly, the forward price of any asset and the instantaneous and Libor forward rates are martingales under the forward measure. Likewise, the forward swap rate is a martingale under the swap measure.

Assuming deterministic forward Libor or swap rate volatility leads respectively to industry-standard Black-Scholes type pricing formulae for caplets and swaptions. The forward measure has other interesting applications in option pricing and in the Libor market model.

The forward measure was implicit in Merton's [7] extension of the Black-Scholes model to stochastic interest rates. Early development and application of the concept appeared in Jamshidian [4] and Geman [2]. The swap measure was discussed heuristically in Neuberger [8] and formalized in Jamshidian [5]. Forward and swap measures are instances of the change of numeraire, as described heuristically in Margrabe [6] and formalized in Geman, et. al. [3].

2. NUMERAIRE MEASURES

We take as given a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and a family $(A^i)_{i \in I}$ of semimartingales. Each A^i is viewed as the observable price process of a traded zero-dividend asset. We assume this family is *arbitrage-free* in that there exists a positive semimartingale $S = (S_t)$

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such that $S_- > 0$, $S_0 = 1$ and SA^i is a martingale for every $i \in I$. Such a process S is called a *state price density* (c.f. Duffie [1]), and we fix one. (In a complete market it is unique.)

For our purposes here we define an *asset* as a semimartingale C such that there exist a finite subset $J \subset I$ and bounded predictable processes δ^j satisfying $C = \sum_{j \in J} \delta^j A^j$ and $dC = \sum_{j \in J} \delta^j dA^j$. As such, an asset is the price process of a dynamic self-financing portfolio, e.g., a (static) linear combination of A^i . The δ^j are called the *deltas* or the *hedge ratios*.

It can be shown that SC is a martingale for any asset C . This implies *the law of one price*: if two assets have almost surely the same prices at some time T , then they will have identical prices at all times $t < T$.

Let N be a *numeraire*, i.e., a positive asset. For each $T > 0$ define the measure $\mathbb{P}^{N,T}$ on \mathcal{F}_T via its Radon-Nikodym derivative by

$$(2.1) \quad \frac{d\mathbb{P}^{N,T}}{d\mathbb{P}} = \frac{S_T N_T}{N_0}.$$

This is an equivalent probability measure and is called the associated *equivalent martingale measure* or *numeraire measure*. Since SN is a martingale, for any $s < T$, the restriction of $\mathbb{P}^{N,T}$ to \mathcal{F}_s equals $\mathbb{P}^{N,s}$. (In an incomplete market, $\mathbb{P}^{N,T}$ depends on the choice of S .)

By an easy and well-known consequence of the Bayes' rule, given a process C , the process SC is a \mathbb{P} -martingale on $[0, T]$ if and only if C/N is a $\mathbb{P}^{N,T}$ -martingale on $[0, T]$. In particular, this holds for all assets C , yielding for $t \leq T$ the pricing formula,

$$(2.2) \quad C_t = N_t \mathbb{E}^{\mathbb{P}^{N,T}} \left[\frac{C_T}{N_T} \mid \mathcal{F}_t \right].$$

A useful technique is the *change of numeraire*. Suppose B is another numeraire and M is a $\mathbb{P}^{N,T}$ -martingale. Note, both $F := N/B$ and MF are $\mathbb{P}^{B,T}$ -martingales. Hence, using Itô's product rule, $\int F_- dM + [M, F]$ is a $\mathbb{P}^{B,T}$ -local martingale. (Here, $[M, F]$ denotes the quadratic covariation of M and F .) Thus, dividing by F_- , so is $M + \int d[M, F]/F_-$. In particular, if F is continuous, then the $\mathbb{P}^{B,T}$ -drift of M equals $-d[M, \log F]$.

3. THE FORWARD MEASURE

For a fixed maturity T , let us assume there exists an asset P^T such that $P^T_T = 1$. Such an asset, called the *T -maturity zero-coupon bond*, is necessarily positive and unique on $[0, T]$ by the law of one price. Its associated numeraire measure on \mathcal{F}_T is called the *T -forward measure* and denoted \mathbb{P}^T . Its expectation operator is denoted \mathbb{E}^T . Since $P^T_T = 1$, by Eq. (2.1)

$$(3.1) \quad \frac{d\mathbb{P}^T}{d\mathbb{P}} = \frac{S_T}{P^T_0}.$$

By Eq. (2.2), the *T -forward price* C/P^T of any asset C is a \mathbb{P}^T -martingale on $[0, T]$ and

$$(3.2) \quad C_t = P^T_t \mathbb{E}^T [C_T \mid \mathcal{F}_t].$$

A European option's price thus equals the discounted expected value of its payoff.

Another important property is that *forward interest rates* are martingales under the forward measure. The simple (or Libor) T -forward rate $L^{T,\varepsilon}$ of length $\varepsilon > 0$ is defined by

$$L^{T,\varepsilon}_t := \frac{P^{T-\varepsilon}_t - P^T_t}{\varepsilon P^T_t}.$$

Assuming that $P^{T-\varepsilon}$ is an asset, $P^{T-\varepsilon}/P^T$, and with it, $L^{T,\varepsilon}$ is a \mathbb{P}^T -martingale.

As ε approaches zero, $L_t^{T,\varepsilon}$ approaches the instantaneous forward rate f_t^T defined by

$$f_t^T := -\frac{\partial \log(P_t^T)}{\partial T}.$$

As such, in the limit, the instantaneous forward rate process f^T is also a \mathbb{P}^T -martingale.

4. OPTION PRICING AND HEDGING

Consider a T -expiry option on an asset A (e.g., a stock or a bond) with time T -payoff $g(A_T)$, where $g(x)$ is a Borel function of linear growth. For example $g(x) = \max(x - K, 0)$ for a call option struck at K . We wish to construct an asset C satisfying $C_T = g(A_T)$. From Eq. (3.2) we know that the only possible candidate is $C = P^T F$, where

$$(4.1) \quad F_t := \mathbb{E}^T[g(A_T) | \mathcal{F}_t].$$

This works if we assume the process $X := A/P^T$ is continuous and $F_t = f(t, X_t)$ for some $C^{1,2}$ function $f(t, x)$. The desired deltas (hedge ratios) are then simply given by

$$(4.2) \quad \delta_t^A := \frac{\partial f}{\partial x}(X_t, t), \quad \delta^P := F - \delta^A X.$$

Indeed, $C := P^T F = \delta^A A + \delta^P P^T$ obviously. Moreover, since both X and F are \mathbb{P}^T -martingales, Itô's formula implies $dF = \delta^A dX$. An application of Itô's product rule known as the *numeraire-invariance theorem* (c.f. [1]) then shows that $dC = \delta^A dA + \delta^P dP^T$.

The above Markovian assumption that $F_t = f(t, X_t)$ for some function $f(t, x)$ is generally satisfied when X is a positive diffusion, specifically when $d[X]_t = X_t^2 \sigma^2(t, X_t) dt$ for some positive continuous bounded function $\sigma(t, x)$. (Here $[X]$ denotes the quadratic variation of X .) This is equivalent to $dX_t = X_t \sigma(t, X_t) dW_t$ for some \mathbb{P}^T -Brownian motion W . In this case, $f(t, x)$ is basically obtained as $\mathbb{E}^T[g(X_T) | X_t = x]$. By Itô's formula, $f(t, x)$ satisfies

$$(4.3) \quad \frac{\partial f}{\partial t} + \frac{1}{2} x^2 \sigma^2(t, x) \frac{\partial^2 f}{\partial x^2} = 0. \quad (f(x, T) = g(x))$$

5. CLOSED-FORM SOLUTIONS

The classical case assumes as in [7] that the forward price volatility $\sigma(t, X_t)$ is deterministic, i.e., independent of X . Then X is a \mathbb{P}^T log-Gaussian martingale, and hence conditioned on time t , $X_T = A_T$ is \mathbb{P}^T -lognormally distributed with mean X_t and log-variance $\int_t^T \sigma^2(s) ds$. As such, for say a call option with payoff function $g(x) = \max(x - K, 0)$, (4.1) readily yields

$$(5.1) \quad C(t) = K P_t^T \text{C}_{\text{BS}}\left(\frac{A_t}{P_t^T K}, \int_t^T \sigma^2(s) ds\right),$$

where, denoting the standard normal distribution function by $N(\cdot)$,

$$(5.2) \quad \text{C}_{\text{BS}}(x, v) := x N\left(\frac{\log(x)}{\sqrt{v}} + \frac{\sqrt{v}}{2}\right) - N\left(\frac{\log(x)}{\sqrt{v}} - \frac{\sqrt{v}}{2}\right).$$

Specific examples are the *Vasicek* or more general *Gaussian interest-rate models* (see eqf11-027) where the deterministic (forward) zero-coupon bond price volatilities are determined endogenously in terms of mean reversion and other model parameters. For zero-coupon bond options, Eq. (4.1) can be computed in the *Cox-Ingersoll-Ross model* (see eqf11-025) and the *quadratic Gaussian model* (see eqf11-028) in terms of the noncentral chi-squared distribution

function. This is derived by showing that the spot interest rate $r_T := f_T^T$ is noncentral chi-squared distributed under the forward measure \mathbb{P}^T (e.g., [4]).

6. CAP PRICING

A cap is a portfolio of consecutive caplets, and a caplet of maturity T , length ε , and strike rate K is an option with payoff $\varepsilon \max(L_{T-\varepsilon}^{T,\varepsilon} - K, 0)$ at time T . A caplet is actually equivalent to a zero-coupon bond put option, so a bond option model as in the previous section is applicable. But more directly, by the pricing formula (3.2) the caplet price C_t is given by

$$C_t = \varepsilon P_t^T \mathbb{E}^T[\max(L_{T-\varepsilon}^{T,\varepsilon} - K, 0) | \mathcal{F}_t].$$

By Section 3, the forward Libor process $L^{T,\varepsilon}$ is a martingale under the forward measure \mathbb{P}^T . Hence, if its volatility $\sigma(t)$ is deterministic, it is log-Gaussian and we get as in Section 5,

$$(6.1) \quad C_t = \varepsilon K P_t^T C_{\text{BS}}\left(\frac{L_t^{T,\varepsilon}}{K}, \int_t^T \sigma^2(t) dt\right).$$

7. THE FORWARD MEASURE BY CHANGING THE RISK-NEUTRAL MEASURE

Let $r_t := f_t^t$ denote the spot interest rate. One often starts with the “money market” asset $\exp(\int_0^t r_t dt)$ as the numeraire and utilizes its equivalent martingale measure, often called the *risk-neutral measure* and denoted \mathbb{Q} . Accordingly, $\exp(-\int_0^t r_t dt)C$ is a \mathbb{Q} -martingale for any asset C . One can then change the numeraire to the T -maturity bond P^T and obtain the T -forward measure \mathbb{P}^T by the formula

$$(7.1) \quad \frac{d\mathbb{P}^T}{d\mathbb{Q}} = \frac{1}{P_0^T} \exp\left(-\int_0^T r_t dt\right).$$

Since the forward rate process f^T is a \mathbb{P}^T -martingale, it follows from Section 2 that when P^T is continuous, the \mathbb{Q} -drift of f^T equals $-d[\log P^T, f^T]$.

8. LIBOR MARKET MODEL SDE IN THE FORWARD MEASURE

Consider a sequence of dates $0 < T_1 \cdots < T_{n+1}$, e.g., equidistant semiannually. Given “daycount fractions” $\varepsilon_i \approx T_{i+1} - T_i$, the forward Libor rates L_t^i are defined as in Section 3 by

$$\varepsilon_i L_t^i = \frac{P_t^{T_i}}{P_t^{T_{i+1}}} - 1. \quad (t \leq T_i, i = 1, \dots, n).$$

Evidently, L^i is a $\mathbb{P}^{T_{i+1}}$ -martingale on $[0, T_i]$.

In some applications such as valuation by Monte-Carlo simulation, it is necessary to determine the dynamics of all the forward Libor processes L^i under the same measure. One appropriate measure is the *spot-Libor measure*, a simple-compounding analog of the risk-neutral measure that takes as numeraire a “rolling zero-coupon bond” (c.f. [5]). Another convenient measure is the *terminal measure*, i.e., the T_{n+1} -forward measure $\mathbb{P}^{T_{n+1}}$. Let W^1, \dots, W^n be $\mathbb{P}^{T_{n+1}}$ -Brownian motions with correlations ρ^{ij} , that is, $d[W^i, W^j]_t = \rho_t^{ij} dt$. Assume

$$(8.1) \quad dL_t^i = \mu_t^i dt + \sigma_t^i dW_t^i$$

for some predictable processes μ^i and σ^i . For example, in the deterministic-volatility Libor market model, $\sigma_t^i = \sigma_i(t)L_t^i$ for some deterministic functions $\sigma_i(t)$. Since L^i is a $\mathbb{P}^{T_{i+1}}$ -martingale, it follows from Section 2 that $\mu^i dt = -d[L^i, \log F]$, where

$$F := \frac{P^{T_{i+1}}}{P^{T_{n+1}}} = \prod_{j=i+1}^n (1 + \varepsilon_j L^j).$$

Therefore the drift of the forward Libor rate in the terminal measure is given by

$$(8.2) \quad \mu^i dt = - \sum_{j=i+1}^n \frac{\varepsilon_j d[L^i, L^j]}{1 + \varepsilon_j L^j} = - \sum_{j=i+1}^n \frac{\varepsilon_j \sigma^i \sigma^j \rho^{ij}}{1 + \varepsilon_j L^j} dt.$$

9. THE SWAP MEASURE

Let T_i and ε_i be as in the previous section. For each $1 \leq i < j$, the swap measure \mathbb{P}^{ij} is defined on $\mathcal{F}_{T_{i+1}}$ as the equivalent martingale measure associated to the *annuity numeraire*

$$A^{ij} := \varepsilon_i P^{T_{i+1}} + \dots + \varepsilon_{j-1} P^{T_j}.$$

The forward swap rate S_t^{ij} with start date T_i and end date T_j is defined for $t \leq T_i$ by

$$S_t^{ij} := \frac{P_t^{T_i} - P_t^{T_j}}{A_t^{ij}}.$$

It follows from Section 2 that S^{ij} is a martingale under the swap measure \mathbb{P}^{ij} .

The main application of the swap measure is to European swaptions, i.e., options to enter an interest-rate swap at a fixed strike rate. Specifically, a payer swaption with start date T_i , end date T_j , expiration $T \leq T_i$ and strike rate K has the payoff C_T at time T given by

$$(9.1) \quad C_T = A_T^{ij} \max(S_T^{ij} - K, 0).$$

(When $j = i + 1$, a payer swaption is just a caplet). Arguments similar to those in Section 4 show that the swaption is replicable under general diffusion assumptions, e.g., when S^{ij} has deterministic volatility or more generally when it is a diffusion process under the swap measure \mathbb{P}^{ij} . The swaption price process C is then uniquely characterized by C/A^{ij} being a \mathbb{P}^{ij} martingale, implying by (9.1) that

$$(9.2) \quad C_t = A_t^{ij} \mathbb{E}^{\mathbb{P}^{ij}}[\max(S_T^{ij} - K, 0) | \mathcal{F}_t].$$

When S^{ij} has a deterministic volatility $\sigma_{ij}(t)$ (i.e., $d[S^{ij}]_t = \sigma_{ij}^2(t)(S_t^{ij})^2 dt$), this yields

$$(9.3) \quad C_t = K A_t^{ij} \text{C}_{\text{BS}}\left(\frac{S_t^{ij}}{K}, \int_t^T \sigma_{ij}^2(t) dt\right).$$

The market uses this formula to quote swaptions, namely a constant volatility σ_{ij} is quoted from which one computes the swaption price by (9.3). Receiver swaptions are treated similarly.

The valuation of Bermudan options is more complex. Here, the swaption can be exercised at any time T_i, \dots, T_{j-1} . One approach, known as the *coterminal swap market model*, assumes that the forward swap rates $S^{ij}, \dots, S^{j-1,j}$ all have deterministic volatilities. Due to (9.3), the model is then automatically calibrated to all the European swaptions with start dates T_i, \dots, T_{j-1} and the same end date T_j , thus ruling out obvious arbitrage opportunities.

Constructs similar to the swap measure have been applied to credit default swaptions.

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