The Duality of Optimal Exercise and Domineering Claims:

A Doob-Meyer decomposition approach to the Snell envelope

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1. Summary

Time t: $\mathbb{T} = [0, \infty)$ (set $m := \infty$), or $\mathbb{T} = [0, m]$.
$\mathcal{T} :=$ {(finite) stopping times with values in $\mathbb{T}$}.
$\mathcal{C} :=$ {***claims***}. (UI martingales), $\mathcal{C}^+ :=$ {***numeraires***}.
Let Z be a ***payoff process*** (RC, adapted, class D).

$$V_0 := \sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T.$$

1.1. Duality.

$$V_0 = \min_{Z \le C \in \mathcal{C}} C_0.$$

1.2. Additive minimax duality.

Rogers [30], Haugh-Kogan [15]:

$$V_0 = \min_{C \in \mathcal{C}} (C_0 + \mathbb{E} \sup_{t \in \mathbb{T}} (Z_t - C_t)).$$

1.3. Multiplicative minimax duality. $Z > 0$:

$$V_0 = \min_{B \in \mathcal{C}^+} \mathbb{E}(B_m \sup_{t \in \mathbb{T}} \frac{Z_t}{B_t}).$$

1.4. Nonnegative minimax duality. $Z \ge 0$:

$$V_0 = \inf_{B \in \mathcal{C}^+} \mathbb{E}(B_m \sup_{t \in \mathbb{T}} \frac{Z_t}{B_t}).$$

1.5. **Snell envelope V.** Smallest ***superclaim*** $\geq Z$:

$$V_t := \; ess. \sup_{t \leq T \in \mathcal{T}} \mathbb{E}(Z_T \,|\, \mathcal{F}_t).$$

1.6. **Bellman equation.** $\alpha \in \mathcal{F}_\tau^{\text{bounded}}$, $S \leq \tau \in \mathcal{T}$:

$$\mathbb{E}(\alpha V_\tau \,|\, \mathcal{F}_S) = ess. \sup_{\tau \leq T \in \mathcal{T}} \mathbb{E}(\alpha Z_T \,|\, \mathcal{F}_S).$$

1.7. **Canonical decomposition:** $X = X^p + X^m$**.** $X_0^p = 0$. Compensator V^p is *decreasing*, so $V \leq V^m$.

1.8. **Pathwise optimality.** Implicit in [6], Davis-Karatzas [8].

$$\sup_{t \in \mathbb{T}} (Z_t - V_t^m) = 0.$$

1.9. **Main pathwise optimality result.** $\tau \in \mathcal{T}$:

$$V_\tau^p = \sup_{\tau \leq t \in \mathbb{T}} (Z_t - V_t^m).$$

$$0 = \sup_{\tau \leq t \in \mathbb{T}} (Z_t - V_t + V_t^p - V_\tau^p).$$

1.10. **Corollary:** $\sup_{\tau \leq t \in \mathbb{T}} (Z_t - V_t) = 0$**, any** $\tau \in \mathcal{T}$**.**

1.11. **Corollary:** $1_{\{Z < V\}} dV^p = 0$**.**

1.12. **Lemma.** X is special semimartingale, $X_0 = 0$:

$$\mathcal{E}(X) = \mathcal{E}(X^p)\mathcal{E}(X^m - \sum_{s \le \cdot} \frac{\Delta X_s^p \Delta X_s^m}{1 + \Delta X_s^p}).$$

1.13. **Multiplicative decomposition of $V > 0$.**

$$V = AB,$$

where ($A > 0$ decreasing, $A_0 = 1$, B local martingale)

$$A = \mathcal{E}(\int \frac{dV^p}{V_-}),$$

$$B = V_0 \mathcal{E}(\int \frac{dV^m}{V_-} - \sum_{s \le \cdot} \frac{\Delta V_s^p \Delta V_s^m}{(V_{s-} + \Delta V_s^p) V_{s-}}).$$

1.14. **Main multiplicative pathwise formula.**

$$A_\tau = \sup_{\tau \le t \in \mathbb{T}} \frac{Z_t}{B_t}. \qquad (\tau \in \mathcal{T})$$

1.15. **Bayes' rule.** The Snell envelope is decreasing and predictable relative to numeraire measure $\mathbb{P}^B$.

Optimal stopping times

$T^* \in \mathcal{T}$ is ***optimal*** if $\mathbb{E}\, Z_{T^*} = \sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T$. Set

$$T_n := \inf\{t \in [0,m] : Z_t \geq V_t^m - \frac{1}{n}\}.$$

1.16. **Lemma.** $\lim_{n\to\infty} \mathbb{E}\, Z_{T_n} = \sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T$.

1.17. **Definition:** $T^\star := \lim_{n\to\infty} T_n$. So, $T_n \uparrow T^\star$.

1.18. **Theorem.** Set $\Lambda := \bigcap_{n=1}^\infty \{T_n < T^\star\}$. Then

$$\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T = \mathbb{E}\, Z_{T^\star} - \mathbb{E}(1_\Lambda \Delta Z_{T^\star}).$$

1.19. **Corollary.** $T^\star$ is optimal iff $\mathbb{E}(1_\Lambda \Delta Z_{T^\star}) = 0$.

1.20. **Corollary.** $T^\star$ is optimal if $\mathbb{E}(1_{T<\infty} \Delta Z_T) = 0$ for all predictable stopping stopping times T.

1.21. **Proposition.** If $T^\star$ is optimal then

$$T^\star = \inf\{t \in \mathbb{T} : Z_t = V_t^m\} = \inf\{t \in \mathbb{T} : Z_t = V_t\}.$$

$$T^* \text{ optimal} \implies T^\star \leq T^* \leq \inf\{t \in \mathbb{T} : V_t^p < 0\}.$$

1.22. **Uniform ε-optimal stopping.** $\forall\, \tau \in \mathcal{T}, \exists\, T^\varepsilon$:

$$T^\varepsilon \geq \tau, \quad \mathbb{E}(Z_{T^\varepsilon} \,|\, \mathcal{F}_\tau) \geq V_\tau - \varepsilon.$$

1.23. Following an approach of Beibel and Lerche [2]:

Perpetual American put: Let $\mathbb{T} = [0, \infty)$. Let X be a continuous local martingale, $X_0 = 0$, such that $[X]_t \to \infty$ a.s. as $t \to \infty$. Let

$$0 < \alpha < 1, \quad 0 < K < 1 + \frac{1}{\alpha}.$$

Set

$$r := \frac{\alpha}{2}\frac{d[X]}{dt}, \qquad S := e^{\int r dt}\mathcal{E}(X).$$

Then,

$$\sup_{T \in \mathcal{T}} \mathbb{E}(e^{-\int_0^T r_t dt}(K - S_T)^+) = \frac{\alpha^\alpha K^{1+\alpha}}{(1+\alpha)^{1+\alpha}}.$$

Moreover, T_* is optimal, where

$$T_* := \inf\{0 \le t < \infty : S_t = \frac{\alpha K}{1+\alpha}\}.$$

Chen-Glasserman [6] supersolution iteration

A ***supersolution*** is a superclaim $W \geq Z$. Define

$$W'_t := W^m_t + \mathbb{E}(\sup_{t \leq s \in \mathbb{T}} (Z_s - W^m_s) \,|\, \mathcal{F}_t).$$

1.24. **Prop. [6].** W' is a supersolution and $W' \leq W$.

1.25. **Bermudan convergence [6].** $\mathbb{T} = \{0, 1, \cdots, m\}$. Set $W^{(0)} := W'$ and inductively $W^{(n+1)} := W^{(n)'}$. Then $W^{(m)} = V$. More generally, $W^{(n)}_t = V_t$ for $t \geq m - n$.

1.26. **Conjecture.** $W^{(n)} \downarrow V$ in continuous time.

1.27. **Prop.** Let $t \in \mathbb{T}$. Then $W_t = V_t$ iff $W'_t = W_t$.

1.28. **Bermudan dual stopping times.** $T^W_m = m$,

$$T^W_t := m \wedge \min\{t \leq s \leq m - 1 : W^p_{s+1} < W^p_t\}.$$

1.29. **Theorem.** Let $0 \leq n \leq m$. For $t \geq m - n$:

$$\mathbb{E}(Z_{T_t^{W^{(n)}}} \,|\, \mathcal{F}_t) = V_t.$$

Narrowing duality gap: $\mathbb{E}(Z_{T_0^{W^{(n)}}}) \leq V_0 \leq W^{(n)}_0$.

1.30. **Early exercise premium formula.**

$$V_t = \mathbb{E}(Z_m - \int_t^m dV_s^p \,|\, \mathcal{F}_t).$$

Useful: V^p decreases only in stopping region:

$$dV^p = 1_{\{Z=V\}} dV^p.$$

1.31. **Heuristics (diffusion):** $dV^p = 1_{\{Z=V\}} dZ^p$**.**

1.32. **Early exercise premium (diffusion).**

$$V_t = \mathbb{E}(Z_m - \int_t^m 1_{\{Z_s=V_s\}} dZ_s^p \,|\, \mathcal{F}_t).$$

1.33. **Pure diffusion with interest rates.**

$$V_t = \mathbb{E}(e^{-\int_t^m r_s ds} Z_m \,|\, \mathcal{F}_t)$$
$$+\mathbb{E}(\int_t^m e^{-\int_t^s r_u du} 1_{\{V_s=Z_s\}}(r_s Z_s ds - dZ_s^p) \,|\, \mathcal{F}_t).$$

1.34. **Call** $Z = (S-K)^+$**.** Set $dS^p =: (r-y)S dt$:

$$V_t = \mathbb{E}(e^{-\int_t^m r_s ds}(S_m - K)^+ \,|\, \mathcal{F}_t)$$
$$+\mathbb{E}(\int_t^m e^{-\int_t^s r_u du} 1_{\{V_s=S_s-K\}}(y_s S_s - r_s K) ds \,|\, \mathcal{F}_t).$$

Markovian jump diffusion (zero rates)

$Z = (S-K)^+$. Stock S is Markovian semimartingale.

1.35. $\nu(dt,dx)$ is the **compensator measure** of $X = \log(S)$.

1.36. **Smooth paste assumption.** $V = V(t,S)$ is C^1 and as a distribution $\frac{\partial^2 V}{\partial S^2}$ is locally integrable on $[0,m]\times(0,\infty)$.

1.37. **Apply Itô-Meyer formula to V, Itô to $S = e^X$.**

$$dV^p = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS^p + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}d[S^c]$$

$$+\int_{x\neq 0}(V(\cdot,S_-e^x) - V(\cdot,S_-) - (e^x-1)S_-\frac{\partial V}{\partial S}(\cdot,S_-))\nu(dt,dx).$$

1.38. **Continuous optimal boundary $t \mapsto S^*(t)$.** This means, $\{V_t = S_t - K\} =: \{S_t \geq S^*(t)\}$. Use $dV^p = 1_{\{Z=V\}}dV^p$ on 1.37:

$$dV^p = 1_{\{S_->S^*\}}(dS^p + \int_{x=-\infty}^{\log(\frac{S^*}{S_-})}(V(\cdot,S_-e^x) - S_-e^x + K)\nu(dt,dx)).$$

1.39. By the general formula $V_t = \mathbb{E}(Z_m - \int_t^m dV_s^p \mid \mathcal{F}_t)$ and 1.38,

$$V_t = \mathbb{E}((S_m-K)^+ - \int_t^m 1_{\{S_{s-}>S^*(s)\}}(dS_s^p +$$

$$\int_{x=-\infty}^{\log(\frac{S^*(s)}{S_{s-}})}(V(s,S_{s-}e^x) - S_{s-}e^x + K)\nu(ds,dx)) \mid S_t).$$

General Markovian jump diffusion

Let r be spot rate, y yield: $dS^p = (r-y)S\,dt$.

1.40. **Delayed exercise compensation pricing formula.**

$$V_t = \mathbb{E}(e^{-\int_t^m r_s ds}(S_m - K)^+ \mid S_t)$$
$$+\mathbb{E}(\int_t^m 1_{\{S_{s-}>S^*(s)\}} e^{-\int_t^s r_u du}((y_s S_s - r_s K)ds$$
$$-\int_{x=-\infty}^{\log(\frac{S^*(s)}{S_{s-}}}(V(s, S_{s-}e^x) - S_{s-}e^x + K)\nu(ds, dx)) \mid S_t).$$

1.41. **Delayed exercise compensation IPDE.** Define volatility σ by $d[S^c] =: \sigma^2 S_-^2 dt$, and $\tilde{\nu}(t, dx)dt := \nu(dt, dx)$. Then

$$\frac{\partial V}{\partial t} + (r-y)S_- \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_-^2 \frac{\partial^2 V}{\partial S^2} - rV$$
$$+\int_{x\neq 0}(V(t, S_- e^x) - V(t, S_-) - (e^x - 1)S_- \frac{\partial V}{\partial S}(\cdot, S_-))\tilde{\nu}(t, dx)$$
$$+1_{\{S_->S^*\}}(yS_- - rK - \int_{x=-\infty}^{\log(\frac{S^*}{S_-})}(V(t, S_- e^x) - S_- e^x + K)\tilde{\nu}(t, dx)) = 0.$$

1.42. **Chiarella-Ziogas [7] explicit solution of IPDE.** Using *Fourier inversion*, V is an infinite sum of double integrals, for constant r and finite activity exponential Lévy process S. Results in coupled integral equation for $V(S,t)$, $S^*(t)$ to solve numerically.

2. Domineering claims, Optimal stopping, Duality

2.1. **Finite versus infinite horizon.** We let $\mathbb{T}$ stand for either the interval $[0, \infty)$ (infinite horizon) or the compact interval $[0, m]$ for some fixed $0 < m < \infty$ (finite horizon).

2.2. **Stochastic basis.** We fix throughout a stochastic basis

$$(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P}).$$

We assume $\mathcal{F}_0$ is trivial.

In the infinite horizon case $\mathbb{T} = [0, \infty)$, we let $\mathcal{T}$ stand for the set of all ***finite*** **stopping times** $T < \infty$ a.s. In the finite horizon case $\mathbb{T} = [0, m]$, we let $\mathcal{T}$ denote the set of all stopping times $T \leq m$ a.s.

2.3. **Claims, numeraires, and payoff processes.**

A ***claim*** is a right-continuous, uniformly integrable martingale. We denote their set $\mathcal{C}$.

In infinite horizon, if $C = (C_t)_{t \geq 0}$ is a claim, then C_t converges as $t \to \infty$, a.s. and in L^1, to a random variable C_∞, denoted also C_m.

A ***numeraire*** is a claim B such that $B_m > 0$ a.s. We denote their set $\mathcal{C}^+$.

So **a numeraire** B **is a positive claim**: a.s. $B_t > 0$ all $t \in \mathbb{T}$.

A ***payoff process*** is a right-continuous adapted process of class D.[1]

A claim is well-known to be a payoff process.

The financial interpretation here corresponds to zero interest rates and risk premia. It is possible to generalize to the case where the *deflated* price of a claim is considered a martingale, etc.

[1] A process $X = (X_t)_{t \in \mathbb{T}}$ is of class D if the family of random variables $(X_T)_{T \in \mathcal{T}}$ is uniformly integrable. If $\mathbb{E}(\sup_{t \in \mathbb{T}} |X_t|) < \infty$, then X is class D.

2.4. **Duality of optimal stopping times and domineering claims.** The following sets the agenda.

Proposition 2.1. *Let Z be a payoff process and C be a claim. Assume $Z \leq C$ and the stopping time $T_* := \inf\{t \in \mathbb{T} : Z_t = C_t\}$ is finite a.s. Then*

$$\sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T = C_0 = \mathbb{E}\, Z_{T_*}.$$

Proof. By the right continuity of $Z - C$ we have, $Z_{T_*} = C_{T_*}$. Hence, using the optional sampling theorem twice, $\sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T \leq \sup_{T \in \mathcal{T}} \mathbb{E}\, C_T = C_0 = \mathbb{E}\, C_{T_*} = \mathbb{E}\, Z_{T_*} \leq \sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T$. □

The following gives the easy half of the aimed duality.

Proposition 2.2. *Let Z be a payoff process and $C \geq Z$ a claim. Then* $\sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T \leq C_0$.

Proof. As $Z \leq C$, by the optional sampling theorem we have, $\mathbb{E} Z_T \leq \mathbb{E} C_T = C_0$ for all $T \in \mathcal{T}$. □

We say a claim C ***domineers*** a payoff process Z if

$$C \geq Z \quad \text{and} \quad C_0 = \sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T.$$

The following linear programming characterization of domineering claim is obvious from Prop. 2.2.

Proposition 2.3. *A claim C domineers Z if and only if C is a solution to the following linear-convex programming problem on the space of claims: Minimize C_0 subject to $C \geq Z$.*

The existence next of domineering claims is key to the duality and results that follow.

Theorem 2.4. *Let Z be a payoff process. Then there exists a claim that domineers Z.*

Proof. For $t \in \mathbb{T}$, set $V_t := \operatorname{ess.sup}_{t \leq T \in \mathcal{T}} \mathbb{E}(Z_T \mid \mathcal{F}_t)$. Then (V_t) has a right-continuous version V, which is a class-D supermartingale. Let $V = C - A$ be the Doob-Meyer decomposition of V, with C a claim and A an increasing predictable process, $A_0 = 0$. Then C domineers Z because $C_0 = V_0$ and $C \geq V \geq Z$. □

In general, there are infinitely many domineering claims. (If Z is a submartingale there is only one.)

We next present six corollaries of Theorem 2.4, beginning with the now obvious **duality**:

Corollary 2.5. *Let Z be a payoff process. Then*

$$\sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T = \min_{Z \le C \in \mathcal{C}} C_0.$$

Moreover, the minimum is attained at a claim $C \geq Z$ if and only if C domineers Z.

The following characterization of class D processes is an easy consequence.

Corollary 2.6. *Let X be a right-continuous adapted process. Then X is of class D if and only if $|X| \leq C$ for some claim C.*

The next result provides a fruitful pathwise characterization of the notion of domineering.

Proposition 2.7. *A claim C domineers a payoff process Z if and only if*

$$\sup_{t\in\mathbb{T}}(Z_t - C_t) = 0.$$

A numeraire B domineers Z if and only if

$$\sup_{t\in\mathbb{T}} \frac{Z_t}{B_t} = 1.$$

Proof. Assume C domineers Z. Then,

$$0 = \sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T - C_0 = \sup_{T\in\mathcal{T}} \mathbb{E}(Z_T - C_T) \le \mathbb{E} \sup_{t\in\mathbb{T}}(Z_t - C_t).$$

Since $Z \le C$, this implies $\sup_{t\in\mathbb{T}}(Z_t - C_t) = 0$ a.s. The converse is almost as easy. The statement about domineering numeraires follows from that about claim. □

2.5. **Existence of optimal stopping times.** Let Z be a payoff process. A stopping time T^* is called ***optimal*** for Z if $T^* \in \mathcal{T}$ and $\mathbb{E}\, Z_{T^*} = \sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T$.

Let C be a claim that domineers Z. Define

$$T_n := \inf\{t \in \mathbb{T} : Z_t \geq C_t - \frac{1}{n}\}, \;\; n \in \mathbb{N}.$$

Lemma 2.8. *We have $T_n \in \mathcal{T}$, all n, and*

$$\lim_{n\to\infty} \mathbb{E}\, Z_{T_n} = \sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T.$$

As (T_n) is increasing, its limit $T^\star := \lim_{n\to\infty} T_n$ exists and is a stopping time.

Corollary 2.9. *Let Z be a* ***quasi-left continuous*** *payoff process. Then the stopping time $T^\star$ defined above is optimal in the finite horizon case, and it is also optimal in infinite horizon if it is finite.*

Proof. Since Z is quasi-left continuous, $T_n \uparrow T^\star$, and $T^\star \in \mathcal{T}$, we have $Z_{T_n} \to Z_{T^\star}$ a.s. Since Z is of class D, this implies $\mathbb{E}\, Z_{T_n} \to \mathbb{E}\, Z_{T^\star}$. Hence $T^\star$ is optimal by Lemma 2.8. □

2.6. **Minimax Duality.** The additive minimax duality formula below was obtained by **Rogers** [29] and independently in the Bermudan case by **Haugh and Kogan** [15], and applied to Monte-Carlo pricing.

Corollary 2.10. ***Additive minimax duality:*** *Let Z be a payoff process. Then*

$$\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T = \min_{C\in\mathcal{C}}(C_0 + \mathbb{E}\sup_{t\in\mathbb{T}}(Z_t - C_t)).$$

The minimum is attained at any claim C that domineers Z (and at translations of C).

Proof. By the optional sampling theorem, for any claim C and stopping time $T \in \mathcal{T}$, we have

$$\mathbb{E}\, Z_T = C_0 + \mathbb{E}(Z_T - C_T) \le C_0 + \mathbb{E}\sup_{t\in\mathbb{T}}(Z_t - C_t).$$

Taking sup over T, we see "$\le$" holds. Let C be a claim that domineers Z, which exists by Theorem 2.4. By the definition of domineering claim and Prop.2.7, $\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T = C_0 = C_0 + \mathbb{E}\sup_{t\in\mathbb{T}}(Z_t - C_t)$. Equality and the second statement follow. □

The multiplicative minimax duality from [20] requires a Bayes' rule formula related to choice of numeraire:

Proposition 2.11. *Let Z be a payoff process, B be a numeraire, and $T \in \mathcal{T}$. Then*

$$\mathbb{E}\, Z_T = \mathbb{E}(B_m \frac{Z_T}{B_T}).$$

Proof. Set $\mathcal{F}_T := \{\Lambda \in \mathcal{F} : \Lambda \cap \{T \leq t\} \in \mathcal{F}_t \ \forall t\}$, as usual. Iterating expectation, then using the $\mathcal{F}_T$-measurability of $\frac{Z_T}{B_T}$,

$$\mathbb{E}(B_m \frac{Z_T}{B_T}) = \mathbb{E}(\mathbb{E}(B_m \frac{Z_T}{B_T} \,|\, \mathcal{F}_T))$$
$$= \mathbb{E}(\frac{Z_T}{B_T} \mathbb{E}(B_m \,|\, \mathcal{F}_T)) = \mathbb{E}(\frac{Z_T}{B_T} B_T) = \mathbb{E}\, Z_T,$$

where we also used and the optional sampling theorem. □

Corollary 2.12. ***Multiplicative minimax duality:*** *Let Z be a payoff process such that $Z_m > 0$ a.s. in finite horizon and $\overline{\lim}_{t\to\infty} Z_t > 0$ a.s. in infinite horizon. Then,*

$$\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T = \min_{B\in\mathcal{C}^+} \mathbb{E}(B_m \sup_{t\in\mathbb{T}} \frac{Z_t}{B_t}).$$

The minimum is attained at any numeraire B that domineers Z (and at multiples of B).

Proof. For any numeraire B, and stopping time $T \in \mathcal{T}$, we have, using Prop. 2.11,

$$\mathbb{E}\, Z_T = \mathbb{E}(B_m \frac{Z_T}{B_T}) \leq \mathbb{E}(B_m \sup_{t\in\mathbb{T}} \frac{Z_t}{B_t}).$$

Taking sup over T, we see "$\leq$" holds. Let B be a claim that domineers Z. Then B is a numeraire by the positivity assumption on Z, as $B \geq Z$. By the definition of domineering claim, martingale property of B, and second part of Prop. 2.7, $\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T = B_0 = \mathbb{E}(B_m) = \mathbb{E}(B_m \sup_{t\in\mathbb{T}} \frac{Z_t}{B_t})$. Equality and the second statement thus follow. □

Remark: A similar version holds in infinite horizon if instead of $\overline{\lim}_{t\to\infty} Z_t > 0$ we only assume $Z_t > 0$ for all $t \geq 0$. The formula then modifies to supremum over *all* positive claims B rather than over just numeraires.

The above positive multiplicative minimax duality extends itself to the nonnegative case, the only difference being that the "min" gets replaced by an "inf":

Corollary 2.13. ***Nonnegative minimax duality***: *Let $Z \geq 0$ be a payoff process. Then*

$$\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T = \inf_{B\in\mathcal{C}^+} \mathbb{E}(B_m \sup_{t\in\mathbb{T}} \frac{Z_t}{B_t}).$$

Proof. As in proof of Cor 2.12, "$\leq$" holds. To show "$\geq$", let $\varepsilon > 0$. Let C be a claim that domineers Z. Then $C \geq Z \geq 0$. Set $B^\varepsilon = C + \varepsilon$. Clearly $B^\varepsilon \in \mathcal{C}^+$ domineers the payoff process $Z + \varepsilon$. Hence, by the second part of Cor 2.12 applied to $Z + \varepsilon$, we have

$$\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T + \varepsilon = \mathbb{E}(B^\varepsilon_m \sup_{t\in\mathbb{T}} \frac{Z_t + \varepsilon}{B^\varepsilon_t})$$

$$\geq \mathbb{E}(B^\varepsilon_m \sup_{t\in\mathbb{T}} \frac{Z_t}{B^\varepsilon_t}) \geq \inf_{B\in\mathcal{C}^+} \mathbb{E}(B_m \sup_{t\in\mathbb{T}} \frac{Z_t}{B_t}).$$

Since $\varepsilon > 0$ was arbitrary, it follows that "$\geq$" holds. □

Remark: The "inf" above is attained at certain "semipositive options" defined in [20].

3. The Snell envelope and its Doob-Meyer decomposition

A ***Superclaim*** is a right-continuous class-D supermartingale.

The **compensator** of a special semimartingale X is denoted X^p. Set $X^m := X - X^p$. So, X^p is the unique predictable finite-variation process such that $X_0^p = 0$ and X^m is a local martingale. As such, $X = X^p + X^m$ is the **canonical decomposition** of X.

The **Doob-Meyer decomposition** theorem states that a superclaim X has such a decomposition with X^m a claim and X^p decreasing, predictable, and integrable (and in infinite horizon, $\lim_{t\to\infty} X_t^p \in L^1$).

The **Snell Envelope** process V of a payoff process Z is defined, for each t a.s. by

$$V_t := \text{ess.} \sup_{t \leq T \in \mathcal{T}} \mathbb{E}(Z_T \mid \mathcal{F}_t).$$

3.1. **The Snell envelope is a superclaim.** We prove the following general version of the well-known supermartingale property of the Snell envelope.

Theorem 3.1. *Let Z be a payoff process. Then its Snell envelope V has a unique right-continuous version, denoted also V, which is a superclaim.*

Moreover, V is the smallest superclaim satisfying $V \geq Z$, i.e., if V' is another, then $V \leq V'$.

In addition, for any two stopping times $S \leq \tau \in \mathcal{T}$,

$$\mathbb{E}(V_\tau \,|\, \mathcal{F}_S) = \text{ess.} \sup_{\tau \leq T \in \mathcal{T}} \mathbb{E}(Z_T \,|\, \mathcal{F}_S). \tag{3.1}$$

The idea is to prove a generalization of Eq. 3.1 and use it thrice. The right continuity of Z implies a very small delay in optimal exercise causes little loss. This basically reduces Eq. 3.1 to the finite Bermudan case, where it is well-known by backward induction.

Remark. Even when Z is continuous, the Snell envelope V may have jumps. An example is American option on a stock with continuous price but a volatility that jumps at a totally inaccessible time T, causing V to jump at T.

3.2. **Domineering claims at τ and implications.** Let Z be a payoff process and $\tau \in \mathcal{T}$ a stopping time.

We say a claim C ***domineers*** Z ***at*** τ if $C_\tau = V_\tau$ and $C \geq Z$ on or after τ, i.e., on the stochastic interval $[[\tau, \infty[[$ (or $[[\tau, m]]$ if finite horizon). By the optional sampling theorem then $C \geq V$ on or after τ.

The pathwise characterization in Prop 2.7 generalizes:

Proposition 3.2. *Let C be a claim and $\tau \in \mathcal{T}$. Then C domineers Z at τ if and only if*

$$\sup_{\tau \leq t \in \mathbb{T}} (Z_t - C_t) = 0.$$

For any integrable random variable ξ, denote by $\mathbb{E}(\xi \,|\, \mathbb{F})$ the unique claim whose value at $t \in \mathbb{T}$ equals $\mathbb{E}(\xi \,|\, \mathcal{F}_t)$ a.s. The fact that V^m domineers Z easily implies

Proposition 3.3. *Let $\tau \in \mathcal{T}$ be s stopping time. Then the claim $V^m + \mathbb{E}(V^p_\tau \,|\, \mathbb{F})$ domineers Z at τ.*

The following is an interesting pathwise formula for the compensator V^p of the Snell envelope.

Corollary 3.4. *Let V be the Snell envelope of Z. Then,*

$$V_t^p = \sup_{t \leq s \in \mathbb{T}} (Z_s - V_s^m).$$

Proof. Let $t \in \mathbb{T}$. By propositions 3.3 and 3.4, $0 = \sup_{t \leq s \in \mathbb{T}} (Z_s - C_s) = \sup_{t \leq s \in \mathbb{T}} (Z_s - V_s^m - V_t^p)$, as $C_s = V_s^m + V_t^p$ for $s \geq t$. □

Another useful consequence is that given $\tau \in \mathcal{T}$, there is a sequence $T_n \geq \tau$ such that $\mathbb{E}(Z_{T_n} \,|\, \mathcal{F}_\tau) \to V_\tau$ a.s. *uniformly* over Ω:

Corollary 3.5. ***Uniform ε-optimal stopping:*** *Let τ be a stopping time in $\mathcal{T}$ and $\varepsilon > 0$. Then there exists a stopping time $T^\varepsilon \in \mathcal{T}$ such that $T^\varepsilon \geq \tau$ and*

$$\mathbb{E}(Z_{T^\varepsilon} \,|\, \mathcal{F}_\tau) \geq V_\tau - \varepsilon.$$

3.3. **Chen-Glasserman supersolution iteration to the Snell envelope.** Let Z be a payoff process.

Following [6], we call a superclaim $W \geq Z$ a ***super-solution***, and define

$$W'_t := W^m_t + \mathbb{E}(\sup_{t \leq s \in \mathbb{T}} (Z_s - W^m_s) \,|\, \mathcal{F}_t).$$

As in [6], W' is then an "improved" supersolution:

Proposition 3.6. *Let W be a supersolution. Then W' is a supersolution and $W' \leq W$.*

In particular, $V' = V$, since V is the smallest supersolution. More strongly, Prop. 3.3 easily implies that V is the unique fix point of the mapping $W \mapsto W'$:

Proposition 3.7. *Let W be a supersolution, $t \in \mathbb{T}$. Then $W_t = V_t$ if and only if $W'_t = W_t$.*

Conjecture: Let W^1 be a supersolution. Define inductively the decreasing sequence $W^{n+1} := W^{n'}$. Then $W^n_t \downarrow V_t$ a.s. all t and $W^n \to W$ in a suitable topology.

3.4. **Doob-Meyer decomposition in arbitrary numeraire.** Given any numeraire B, let $\mathbb{P}^B$ denote the numeraire measure defined by $\frac{d\mathbb{P}^B}{d\mathbb{P}} := \frac{B_m}{B_0}$, and $\mathbb{E}^B$ denote its expectation operator. If Z is a payoff process, then by the Bayes' rule, $\frac{Z}{B}$ is a $\mathbb{P}^B$-payoff process.

Proposition 3.8. *Let V be a superclaim and B a numeraire. Then there is a unique claim C and an increasing predictable process A, $A_0 = 0$, such that*

$$V = C - AB.$$

Also, $A = -\int \frac{dV^p}{B_-}$ *and* $C = V^m + \int A_- dB + [A, B]$.

Proposition 3.9. *Let Z be a payoff process, B be numeraire, and $V = C - AB$ be the above decomposition of Snell envelope V of Z. Let $\tau \in \mathcal{T}$. Then the claim $C - \mathbb{E}(B_m A_\tau \,|\, \mathbb{F})$ domineers Z at τ.*

Proposition 3.10. ***Early exercise premium:*** *In finite horizon, with notation as in Prop 3.9,*

$$\begin{aligned} V_t - \mathbb{E}(Z_m \,|\, \mathcal{F}_t) &= \mathbb{E}(B_m(A_m - A_t) \,|\, \mathcal{F}_t) \\ &= B_t \, \mathbb{E}^B(A_m - A_t \,|\, \mathcal{F}_t). \end{aligned}$$

4. The multiplicative Doob-Meyer-Itô decomposition

Recall, $X = X^p + X^m$ denotes the canonical decomposition, with X^p the compensator.

Lemma 4.1. *Let X be a special semimartingale with $X_0 = 0$. Assume $1 + \Delta X^p \neq 0$ everywhere. Then $\sum_{s\leq \cdot} \frac{\Delta X^p_s \Delta X^m_s}{1+\Delta X^p_s}$ is a finite variation local martingale (the countable sums being absolutely convergent), and*

$$\mathcal{E}(X) = \mathcal{E}(X^p)\mathcal{E}(X^m - \sum_{s\leq \cdot} \frac{\Delta X^p_s \Delta X^m_s}{1 + \Delta X^p_s}).$$

This gives the multiplicative decomposition of the stochastic exponential $\mathcal{E}(X)$. The term $\mathcal{E}(X^p)$ is predictable and of finite variation as X^p is, and the second term is a local martingale as $X^m - \sum_{s\leq \cdot} \frac{\Delta X^p_s \Delta X^m_s}{1+\Delta X^p_s}$ is.

Theorem 4.2. *Let Y be a special semimartingale with $Y > 0$ and $Y_- > 0$. Then $Y_- + \Delta Y^p > 0$ and there exits a unique decomposition $Y = AM$, where M is a local martingale and $A > 0$ is a predictable, finite-variation process with $A_0 = 1$. Moreover,*

$$A = \mathcal{E}(\int \frac{dY^p}{Y_-}),$$

$$M = Y_0 \mathcal{E}(\int \frac{dY^m}{Y_-} - \sum_{s \leq \cdot} \frac{\Delta Y_s^p \Delta Y_s^m}{(Y_{s-} + \Delta Y_s^p) Y_{s-}}).$$

Remark. The multiplicative formula in Theorem II.8.21 of [17] reads $Y = A'M'$, where

$$A' = 1/\mathcal{E}(-\int \frac{dY^p}{Y_- + \Delta Y^p}), \quad M' = Y_0 \mathcal{E}(\int \frac{dY^m}{Y_- + \Delta Y^p}).$$

We have $A' = A$ and $M' = M$.

Remark. A right-continuous local supermartingale Y always has left limits. If additionally $Y > 0$, then $Y_- > 0$ and the multiplicative compensator $A = \mathcal{E}(\int \frac{dY^p}{Y_-})$ is decreasing as Y^p is so.

Remark. The multiplicative decomposition is numeraire-invariant in the sense that if $Y = AM$ is the multiplicative decomposition of Y and B is any numeraire, then $\frac{Y}{B} = A\frac{M}{B}$ is that of $\frac{Y}{B}$ under numeraire measure $\mathbb{P}^B$, as $\frac{M}{B}$ is a $\mathbb{P}^B$-local martingale.

The following provides a pathwise formula for the multiplicative compensator of the Snell envelope.

Corollary 4.3. *Let Z be a positive payoff process and $V = AB$ be the multiplicative decomposition of its Snell envelope V. Assume B is a numeraire. Then,*

$$A_t = \sup_{t \leq s \in \mathbb{T}} \frac{Z_s}{B_s}.$$

Proof. By the Bayes' rule, $A = \frac{V}{B}$ is the Snell envelope under $\mathbb{P}^B$ of the $\mathbb{P}^B$-payoff process $\frac{Z}{B}$. Under any measure, $A^p = A - 1$ and $A^m = 1$. So, Cor. 3.5 applied to the $\mathbb{P}^B$-payoff process $\frac{Z}{B}$ in measure $\mathbb{P}^B$ yields $A_t - 1 = \sup_{t \leq s \in \mathbb{T}} (\frac{Z_s}{B_s} - 1)$. □

Corollary 4.4. *Let Y be a positive continuous semimartingale. Then $Y = Y_0 \, e^{\int \frac{dY^p}{Y}} \, \mathcal{E}(\int \frac{dY^m}{Y})$. Moreover, $\mathcal{E}(\int \frac{dY^m}{Y})$ is a claim if $\mathbb{E} \exp(\frac{1}{2} \int_0^m \frac{d[Y_t]}{Y_t^2}) < \infty$.*

5. Optimal stopping times

Recall, a finite stopping time T^* is called ***optimal*** if

$$\mathbb{E}\, Z_{T^*} = \sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T \ .$$

Proposition 5.1. *A stopping time T^* is optimal for a payoff process Z if and only if $Z_{T^*} = C_{T^*}$ for some, hence all, claims C that domineer Z.*

Since $Z \leq V \leq V^m$, applying to $C := V^m$ yields

Corollary 5.2. *If T^* is optimal for Z then $V_{T^*} = Z_{T^*}$ where V is the Snell envelope of Z.*

Prop. 5.1 easily implies that domineering claims are unique in the "continuation region":

Corollary 5.3. *Let C be a claim that domineer Z, and V be the Snell envelope of Z. Assume T^* is an optimal stopping time. Then $C = V$ on the stochastic interval $[[0, T^*]] := \{(t, \omega)\} : t \leq T^*(\omega)\}$.*

In particular, $V^p = 0$ and $C = V^m$ on $[[0, T^*]]$.

5.1. **Existence and the shortest stopping time.** The following is another easy consequence.

Corollary 5.4. *A payoff process Z has an optimal stopping time if and only if for some, hence all, domineering claims C, the stopping time*

$$T_* := \inf\{t \in \mathbb{T} : Z_t = C_t\}$$

is finite. In this case, T_* is optimal and independent of choice of C, and $T_* \le T^*$ for all optimal stopping times T^*, and (with V denoting the Snell envelope),

$$T_* = \inf\{t \in \mathbb{T} : Z_t = V_t\}.$$

Returning to the stopping $T^\star$ in Sec. 2.2, recall,

$$T_n := \inf\{t \in \mathbb{T} : Z_t \ge C_t - \frac{1}{n}\}, \quad T^\star := \lim_{n\to\infty} T_n,$$

given a domineering claim C. Recall $T_n \in \mathcal{T}$, and, in the finite horizon case, $T^\star \in \mathcal{T}$.

Theorem 5.5. *Let Z be a cádlág payoff process and T_n and $T^\star$ be as above. Assume $T^\star \in \mathcal{T}$. Then,*

$$\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T = \mathbb{E}\, Z_{T^\star} - \mathbb{E}(1_\Lambda \Delta Z_{T^\star}),$$

where $\Lambda := \bigcap_{n=1}^{\infty}\{T_n < T^\star\}$. Moreover, $T^\star$ is optimal if and only if $\mathbb{E}(1_\Lambda \Delta Z_{T^\star}) = 0$, and in this case $T^\star = T_*$, with T_* as in Cor. 5.4.

A consequence is a weaker while similar condition than quasi-left continuity which is sufficient for existence:

Corollary 5.6. *Assume the payoff process Z has left limits, $T^\star \in \mathcal{T}$, and $\mathbb{E}(1_{T<\infty}\Delta Z_T) = 0$ for all predictable stopping stopping times T. Then the stopping time $T^\star$ is optimal.*

Proof. Define the predictable stopping time $T^\star_\Lambda$ by $T^\star_\Lambda(\omega) = T^\star(\omega)$ if $\omega \in \Lambda$ and $T^\star_\Lambda(\omega) = \infty$ otherwise. Then, $1_\Lambda \Delta Z_{T^\star} = 1_{T^\star_\Lambda<\infty}\Delta Z_{T^\star_\Lambda}$. The result now follows from Theorem 5.5 by the assumption. □

5.2. **The longest optimal stopping time.** The optimal stopping time is normally unique, i.e., equals the shortest one $T_* = \inf\{t \in \mathbb{T} : Z_t = V_t\}$ in Cor. 5.4.

In general, define

$$T_0 := \inf\{t \in \mathbb{T} : V_t^p < 0\}.$$

(Actually, $T_0 = \inf\{t \in \mathbb{T} : V_t < C_t\}$ for any domineering claim C.) Clearly, $1_{T_0<\infty} V_{T_0-}^p = 0$.

Also, $T_0 \geq T^*$ for any optimal stopping T^* because $V^p = 0$ on $[[0, T^*]]$ by Cor. 5.3. So, if T_0 is optimal, then it is the longest optimal stopping time.

In discrete time, T_0 is always optimal. In continuous time, we can say the following.

Proposition 5.7. *The stopping time T_0 is optimal if and only if T_0 is finite and $V_{T_0}^p = 0$.*

6. The perpetual American put

The perpetual American put problem can be solved by an explicit construction of a "domineering martingale" M which is a geometric Brownian motion, a direct construction as in [2] that bypasses the Snell envelope and its Doob-Meyer decomposition. As a geometric Brownian motion is not uniformly integrable in infinite horizon, the optional sampling theorem does not apply directly to M. Still, for certain range of parameters, it turns out that the stopping time $T_* := \inf\{0 \le t < \infty : Z_t = M_t\}$ is optimal and $\mathbb{E}\, Z_{T_*} = M_0$.

Proposition 6.1. *Let Z be a payoff process and $M \ge Z$ be a positive local martingale. Assume $T_* := \inf\{0 \le t < \infty : Z_t = M_t\}$ is finite a.s. and the stopped local martingale $M^{T_*} := M_{\cdot \wedge T^*}$ is a claim. Then T_* is optimal and $\mathbb{E}\, Z_{T_*} = M_0$.*

Proof. A standard argument gives $\mathbb{E}\, M_T \le M_0$ for all finite stopping times T. This, followed by using that M^{T_*} is a claim, and that by right continuity $Z_{T_*} = M_{T_*}$, gives

$$\begin{aligned} \sup_{0\le T<\infty} \mathbb{E}\, Z_T \le \sup_{0\le T<\infty} \mathbb{E}\, M_T \le M_0 = \mathbb{E}\, M^{T_*}_\infty \\ = \mathbb{E}\, M_{T_*} = \mathbb{E}\, Z_{T_*} \le \sup_{0\le T<\infty} \mathbb{E}\, Z_T. \end{aligned}$$

□

Theorem 6.2. *Let X be a continuous local martingale with $X_0 = 0$ such that $[X]_t \to \infty$ a.s. as $t \to \infty$. Let $0 < \alpha < 1$ and $0 < K < 1 + \frac{1}{\alpha}$. Define the (bounded) payoff process*

$$Z := (Ke^{-\frac{1}{2}\alpha[X]} - \mathcal{E}(X))^+.$$

Then, $\sup_{0 \le T < \infty} \mathbb{E}\, Z_T = \frac{\alpha^\alpha K^{1+\alpha}}{(1+\alpha)^{1+\alpha}}$. In fact, define the positive local martingale

$$M := \frac{\alpha^\alpha K^{1+\alpha}}{(1+\alpha)^{1+\alpha}} \mathcal{E}(-\alpha X).$$

Then, $Z \le M$, the stopping time $T_* := \inf\{0 \le t < \infty : Z_t = M_t\}$ is finite a.s., and the stopped process M^{T_*} is a claim. Therefore by Prop. 6.1,

$$\sup_{0 \le T < \infty} \mathbb{E}\, Z_T = \mathbb{E}\, Z_{T_*} = M_0 = \frac{\alpha^\alpha K^{1+\alpha}}{(1+\alpha)^{1+\alpha}}.$$

Moreover, M^{T_*} is bounded, and

$$T_* = \inf\{0 \le t < \infty : e^{\frac{1}{2}\alpha[X]_t} \mathcal{E}(X)_t = \frac{\alpha K}{1+\alpha}\}.$$

Remark: The above result was based for the most part on Beibel and Lerche [2], where X is assumed to be a Brownian motion.

Remark: For the perpetual American put, $r := \frac{\alpha}{2}\frac{d[X]}{dt}$ is interpreted as the short rate, and $S = e^{\int r dt}\mathcal{E}(X)$ as the stock price (zero-dividend here). Note then, $Z = e^{-\int r dt}(K - S)^+$.

Remark: If $\alpha > 1$ and $K > 1 + \frac{1}{\alpha}$, we still have T_* is finite, $Z \leq M$, and $Z_{T_*} = M_{T_*}$. But, we cannot conclude T_* is optimal, for in this case M^{T_*} is not a claim because $\mathbb{E}M_{T_*} < M_0$.

7. Iterative construction of Bermudan Snell envelope and optimal exercise

In this section we study the **finite Bermudan** case with $\mathbb{T} = \{0, 1, \cdots, m\}$, and define a stopping time at each iteration of the m-step construction of the Snell envelope in [6], so that they converge in m-steps to the optimal stopping time, similarly to [26].

In the Bermudan case, a (super) claim is a (super) martingale, a payoff process is an adapted integrable process, and a predictable process is a previsible process. The canonical decomposition $X = X^p + X^m$ of a payoff process X is characterized inductively by

$$X^m_{t+1} - X^m_t = X_{t+1} - \mathbb{E}(X_{t+1} \,|\, \mathcal{F}_t), \quad X^m_0 = X_0; \tag{7.1}$$

$$X^p_{t+1} - X^p_t = \mathbb{E}(X_{t+1} - X_t \,|\, \mathcal{F}_t), \qquad X^p_0 = 0. \tag{7.2}$$

Let Z be a payoff process. Following [6], we call a supermartingale $W \geq Z$ a ***supersolution***, and set

$$W'_t := W^m_t + \mathbb{E}\Big(\sup_{t \leq s \leq m} (Z_s - W^m_s) \,|\, \mathcal{F}_t\Big).$$

Note, $W'_m = Z_m$. Prop 7.1 and Cor. 7.3 are from [6].

Proposition 7.1. *Let W be a supersolution. Then W' is also a supersolution and $W' \leq W$.*

Lemma 7.2. *Let V be the Snell envelope and W be any supersolution. Let $0 \leq t \leq m-1$, and assume $W_s = V_s$ for $s > t$. Then $W'_s = V_s$ for $s \geq t$.*

The lemma and an induction immediately yields

Corollary 7.3. *Let $W^{(0)}$ be a supersolution with $W^{(0)}_m = Z_m$. Set inductively $W^{(n+1)} := W^{(n)\prime}$. Then $W^{(n)}_t = V_t$ for $t \geq m-n$.*

Since V^p is previsible, T_0 is a stopping time, where

$$T_0 := m \wedge \min\{0 \leq t \leq m-1 : V^p_{t+1} < 0\}.$$

Proposition 7.4. *The stopping time T_0 above and $T_* := \min\{0 \leq t \leq m : Z_t = V_t\}$ are optimal.*

Moreover, $T_* \leq T^* \leq T_0$ for all optimal stopping times T^*. Furthermore, $V^p_{T_0} = 0$.

For any supersolution W, set $T^W_m := m$, and for $0 \leq t \leq m-1$, define the stopping time

$$T^W_t := m \wedge \min\{t \leq s \leq m-1 : W^p_{s+1} < W^p_t\}.$$

Proposition 7.5. *Let* $0 \le t \le m$*. Then*

$$\mathbb{E}(Z_{T_t^V} \,|\, \mathcal{F}_t) = V_t.$$

Lemma 7.6. *Let* $0 \le t \le m$ *and* W *be a supersolution such that* $W_s = V_s$ *for* $t \ge s$*. Then* $W_s^p - W_t^p = V_s^p - V_t^p$ *for all* $s \ge t$*, and hence* $T_t^W = T_t^V$*.*

The next result complements the iteration of [6] in Cor. 7.3. We obtain, $\mathbb{E}(Z_{T_0^{W^{(n)}}}) \le V_0 \le W_0^{(n)}$ for all $0 \le n \le m$, with the duality gap narrowing as n increases:

Theorem 7.7. *Let* $W^{(0)}$ *be a supersolution with* $W_m^{(0)} = Z_m$*. Define inductively,* $W^{(n+1)} := W^{(n)\prime}$*. Then,* $\mathbb{E}(Z_{T_t^{W^{(n)}}} \,|\, \mathcal{F}_t) = V_t$ *for* $t \ge m - n$ *and* $0 \le n \le m$*.*

Proof. Let $0 \le n \le m$. By Cor. 7.3 we have $W_t^{(n)} = V_t$ for $t \ge m - n$. Hence by Lemma 7.6, $T_t^{W^{(n)}} = T_t^V$ for $t \ge m - n$. Therefore by Prop. 7.5, $\mathbb{E}(Z_{T_t^{W^{(n)}}} \,|\, \mathcal{F}_t) = V_t$ for $t \ge m - n$. □

Remark: As in [6], Prop 7.1 and Cor. 7.3 continue to hold if W' were defined **multiplicatively** as $W_t' := \mathbb{E}(B_m \sup_{t \le s \le m} \frac{Z_s}{B_s} \,|\, \mathcal{F}_t) = B_t \mathbb{E}^B(\sup_{t \le s \le m} \frac{Z_s}{B_s} \,|\, \mathcal{F}_t)$, where $W = AB$ is the multiplicative decomposition of W. So will Prop. 7.5 and Theorem 7.7, with $T_t^W := m \wedge \min\{t \le s \le m-1 : A_{s+1} < A_t\}$.

8. Financial interpretation and heuristics

The interpretations below view the Snell envelope V_t as time-t price of the *current* American option (issued at time t) on a given payoff process Z.

8.1. **The multiplicative decomposition.** Let us interpret the multiplicative decomposition of the Snell envelope, expressed in the form $AV = C$, where A is an increasing predictable process with $A_0 = 1$ as and C is a claim (that domineers Z). Relying on the right continuity of Z, we approximate the American option by a Bermudan option that can be exercised at one of only finitely many dates $0 \leq t_1 < \cdots < t_n$, receiving payoff Z_{t_i}.

Start with V_0 capital, and with it at time $t = 0$ buy $A_{t_1} = 1$ share of this Bermudan option. At time t_1 sell the option and with the proceeds buy A_{t_2} units of the Bermudan option with exercise dates $t_2, \cdots, t_n$. At time t_2 sell all A_{t_2} units of this Bermudan option and with the proceeds buy A_{t_3} units of Bermudan option with exercise dates $t_3, \cdots, t_n$. Continue this trading strategy at time t_3 and subsequent times t_i until t_n.

Clearly this is a *self-financing* trading strategy. Therefore its price (portfolio value) is a claim, which we denote by C. By its definition C_{t_i} equals the net value of Bermudan options sold at time t_i. Since A_{t_i} units of Bermudan options with exercise date $t_i, \cdots, t_n$ are sold, each unit of which is priced at V_{t_i}, we conclude that $C_{t_i} = A_{t_i} V_{t_i}$.

Moreover, the number of share (units) A is previsible by construction. It is also increasing. Indeed, if it is not optimal to exercise at t_i, then $A_{t_{i+1}} = A_{t_i}$ because the sold Bermudan options at time t_i are identical to the ones next bought at t_i. And otherwise, if it is optimal to exercise at t_i, then $A_{t_{i+1}} > A_{t_i}$ because unit price of the sold options is $S_{t_i} - K$, which is by optimality greater than unit price $\mathbb{E}(V_{t_{i+1}} \,|\, \mathcal{F}_t)$ of bought options.

8.2. **Additive decompositions and domineering claims.** Given a numeraire B, let us interpret the invariant Doob-Meyer decomposition $V = C - AB$ of Prop. 3.7, where C is a domineering claim and A is an increasing and predictable process with $A_0 = 0$.

Discretize as above, and at time 0 buy one unit of the Bermudan option with exercise dates $t_1, \cdots, t_n$. At time t_1, sell it and with proceeds buy one unit of the Bermudan option with exercise dates $t_2, \cdots, t_n$ and a_1 units of the numeraire B. (Clearly $a_1 \geq 0$.) At time t_2, sell the Bermudan options bought at t_1, and with proceeds buy one unit of the Bermudan option with exercise dates $t_3, \cdots, t_n$ and $a_2 \geq 0$ units of the numeraire B. Continue this self-financing trading strategy at t_3 and subsequent times t_i until t_n.

Define the claim C to be the price process of this self-financing portfolio. Set $A_{t_1} = 0$, and for $i \geq 2$, define $A_{t_i} = a_1 + \cdots + a_{i-1}$. Clearly, A is previsible and increasing. By construction, A_{t_i} is the number of shares of the numeraire B held in the portfolio at time t_i before any transaction (that is the number of shares brought forward from time t_{i-1}).

The value C_{t_i} of the portfolio, looked at time t_i just before any transaction, equals the value of one unit of the Bermudan option carried forward, i.e., V_{t_i}, plus the value of the A_{t_i} units of numeraire B brought forward, i.e. $A_{t_i} B_{t_i}$. Therefore, $C = V + AB$.

8.3. **Invariance of the multiplicative decomposition.** As the above shows, investments in the additive decomposition $V = C - AB$ are rolled over both the numeraire B and (one unit of) the newly issued American option V (in the case of $V = C - A$, the reference numeraire is implicitly $B = 1$), whereas in the multiplicative case all investment is rolled over V. As such, in contrast to the additive decomposition, *the multiplicative decomposition does not rest on any reference numeraire for its existence.*

8.4. **Snell envelope formula in pure diffusion.** In general $dV^p = 1_{Z=V} dV^p$. For diffusion, one may argue $dV^p = 1_{\{Z=V\}} dZ^p$. By Prop. 3.11 then

$$V_t = \mathbb{E}(Z_m - \int_t^m 1_{\{Z_s = V_s\}} dZ^p_s \,|\, \mathcal{F}_t).$$

For an American call under zero interest rates,

$$V_t = \mathbb{E}((S_m - K)^+ - \int_t^m 1_{\{V_s = S_s - K\}} dS^p_s \,|\, \mathcal{F}_t).$$

More generally, under stochastic interest rates r,

$$V_t = \mathbb{E}(e^{-\int_t^m r_s ds}(S_m - K)^+ \,|\, \mathcal{F}_t) \tag{8.1}$$

$$+\mathbb{E}(\int_t^m e^{-\int_t^s r_u du} 1_{\{V_s = S_s - K\}} (y_s S_s - r_s K) ds \,|\, \mathcal{F}_t).$$

Here, y is dividend yield process: $dS^p =: (r - y) S dt$.

More generally, for an **American swaption** with payout $Z = (S - K)^+$ with Z and K continuous semimartingales,

$$V_t = \mathbb{E}(e^{-\int_t^m r_s ds}(S_m - K_m)^+ \,|\, \mathcal{F}_t)$$

$$+\mathbb{E}(\int_t^m e^{-\int_t^s r_u du} 1_{\{V_s = S_s - K_s\}} (r_s(S_s - K_s) ds - dS^p_s + dK^p_s) \,|\, \mathcal{F}_t).$$

8.5. **Doob-Meyer decomposition by inhomogeneous PDE.** Following [19] (see also [24, 5, 16]), we formulate the problem in terms inhomogeneous PDE in the 1-factor diffusion case.

In this setting, asset prices P are functions $P_t = P(t, X_t)$ of time t and a state variable X that follows an Itô process $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$, with W a Brownian motion under the risk-neutral measure $\mathbb{P}$. Relaxing previous restriction of zero rates, the instantaneous interest rate process r is similarly of the form $r_t = r(t, X_t)$. The price process P of a derivative with a continuous dividend $c_t = c(t, X_t)$ satisfies $dP^p = (rP - c)dt$.

Applying Itô's formula, we see that $LP + c = 0$, where

$$Lf = \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} - rf, \quad f = f(t, x).$$

Consider an American call struck at K on a stock or bond with price process $S_t = S(t, X_t) > 0$. As above, $rS - \frac{dS^p}{dt}$ is interpreted as continuous dividend (or coupon). So, $y = y(t, X) := r - \frac{1}{S}\frac{dS^p}{dt}$ is the dividend yield. Thus, $dS^p = (r - y)Sdt$ and $LS + yS = 0$.

Assume there is an "optimal exercise boundary" $x^*(t)$, above which the American call is optimally exercised, and below which it is kept. As American call price $V_t = V(t, X_t)$ equals $S_t - K$ at or above $x^*(t)$, it follows $LV = LS - LK = -(yS - rK) \leq 0$ on the (t, x) region $x > x^*(t)$. Below $x^*(t)$, the American call is not exercised and behaves like any asset with zero continuous dividend. So, $LV = 0$ on the (t, x) region $x < x^*(t)$. Assuming a smooth fit condition, we conclude

$$LV + 1_{x>x^*(t)}(yS - rK) = 0, \qquad V(m, x) = (S(m, x) - K)^+.$$

By the Feynman-Kac formula the solution V to this *inhomogeneous* parabolic PDE is

$$V_t = \mathbb{E}(e^{-\int_t^m r_s ds}(S_m - K)^+ \,|\, X_t) \tag{8.2}$$
$$+\mathbb{E}(\int_t^m e^{-\int_t^s r_u du} 1_{\{X_s > x^*(s)\}}(y_s S_s - r_s K) ds \,|\, X_t).$$

Eq. (8.2) evaluated at $X_t = x^*(t)$ yields an integral equation for the optimal boundary $x^*(t)$, as the left hand side becomes $S(t, x^*(t)) - K$. Eq. (8.2) also yields the Doob-Meyer decomposition of $e^{-\int r\,dt}V$, for it can be rewritten as

$$e^{-\int r\,dt}V = M - A, \tag{8.3}$$

where M is the martingale

$$M_t = \mathbb{E}(e^{-\int_0^m r_s ds}(S_m - K)^+$$
$$+ \int_0^m e^{-\int_0^s r_u du} 1_{\{X_s > x^*(s)\}}(y_s S_s - r_s K) ds \,|\, X_t),$$

and A is the increasing continuous process,

$$A_t = \int_0^t e^{-\int_0^s r_u du} 1_{\{X_s > x^*(s)\}}(y_s S_s - r_s K) ds.$$

8.6. **Extension to Markovian jump-diffusion.** Recently [7] have extended the inhomogeneous PDE approach of [19] to an inhomogeneous IPDE for the case of exponential Lévy underlyers. In addition to instantaneously compensating for dividend and penalizing for interest on strike as in pure diffusion, the infinitesimal expectation to jump down at the next instant from the stopping region into the continuation region is penalized too, because then the option price depreciates less than if it had been exercised. Here, we propose a general formulation based on the Ito-Meyer formula, assuming smooth fit. Under zero-interest rates,

$$V_t = \mathbb{E}(Z_m - \int_t^m dV_s^p \,|\, \mathcal{F}_t). \tag{8.4}$$

To calculate dV^p, we assume that $Z = (S - K)^+$ for some positive special semimartingale $S > 0$ constant K. We further assume $V_t = V(t, S_t)$ for some C^1 function $V(t, S)$ such that as a distribution $\frac{\partial^2 V}{\partial S^2}$ is locally integrable on $[0, m] \times (0, \infty)$. Then V is a difference of two convex functions in S. As such, Ito-Meyer formula is applicable (V is "almost" C^2 in S here). Applying it, then taking compensator,

$$dV^p = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS^p + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}d[S^c]$$

$$+ \int_{y \neq 0} (V(\cdot, S_- + y) - V(\cdot, S_-) - y\frac{\partial V}{\partial S}(\cdot, S_-))\hat{\nu}(dt, dy),$$

where $\hat{\nu} = \hat{\nu}(dt, dy)$ is the compensator measure of S (the jump size intensity). But, one usually models the dynamics of $X := \log(S)$. Let $\nu = \nu(dt, dx)$ denote its compensator measure. Interestingly, the jump integral transforms multiplicatively:

$$dV^p = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS^p + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}d[S^c]$$

$$+ \int_{x \neq 0} (V(\cdot, S_- e^x) - V(\cdot, S_-) - (e^x - 1)S_- \frac{\partial V}{\partial S}(\cdot, S_-))\nu(dt, dx).$$

We are interested in this formula in the stopping region $\{V = S - K\}$, as it is zero outside it. We assume a continuous optimal boundary curve $t \mapsto S^*(t)$, meaning $\{V_t = S_t - K\} = \{S_t \geq S^*(t)\}$. So, $dV^p = 1_{\{S \geq S^*\}} dV^p$. We further assume dS^p, $d[S^c]$, $\nu(dt, dx)$, and hence dV^p are dt-absolutely continuous. Then $dV^p = 1_{\{S_- > S^*\}} dV^p$. On the region $\{S_- > S^*\}$, we have $V(\cdot, S_-) = S_- - K$, so, $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial S^2} = 0$ and $\frac{\partial V}{\partial S} = 1$ (evaluated at (t, S_{t-})). Thus

$$dV^p = 1_{\{S_- > S^*\}}(dS^p + \int_{x \neq 0} (V(\cdot, S_- e^x) - S_- e^x + K)\nu(dt, dx)).$$

Since $V(t, S_{t-}e^x) - S_{t-}e^x + K = 0$ for $S_{t-}e^x \geq S^*(t)$, also

$$dV^p = 1_{\{S_- > S^*\}}(dS^p + \int_{x=-\infty}^{\log(\frac{S^*}{S_-})} (V(\cdot, S_- e^x) - S_- e^x + K)\nu(dt, dx)).$$

(Note $\log(\frac{S^*}{S_-}) < 0$ here.) Plugging into Eq. 8.4, we get,

$$V_t = \mathbb{E}((S_m - K)^+ - \int_t^m 1_{\{S_{s-} > S^*(s)\}}(dS_s^p +$$

$$\int_{x=-\infty}^{\log(\frac{S^*(s)}{S_{s-}})} (V(s, S_{s-}e^x) - S_{s-}e^x + K)\nu(ds, dx)) \mid S_t).$$

For a nonzero spot rate process r, we have similarly

$$V_t = \mathbb{E}(e^{-\int_t^m r_s ds}(S_m - K)^+) \mid S_t) \tag{8.5}$$

$$+\mathbb{E}(\int_t^m 1_{\{S_{s-} > S^*(s)\}} e^{-\int_t^s r_u du}((y_s S_s - r_s K)ds$$

$$- \int_{x=-\infty}^{\log(\frac{S^*(s)}{S_{s-}})} (V(s, S_{s-}e^x) - S_{s-}e^x + K)\nu(ds, dx)) \mid S_t).$$

Here, y is dividend yield: $dS^p = (r - y)S_-$. Define volatility σ by $d[S^c] =: \sigma^2 S_-^2 dt$, and $\tilde{\nu}(t, dx)dt := \nu(dt, dx)$. Then we have

$$\frac{\partial V}{\partial t} + (r - y)S_- \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_-^2 \frac{\partial^2 V}{\partial S^2} - rV$$

$$+ \int_{x \neq 0} (V(t, S_- e^x) - V(t, S_-) - (e^x - 1)S_- \frac{\partial V}{\partial S}(\cdot, S_-))\tilde{\nu}(t, dx)$$

$$+1_{\{S_- > S^*\}}(yS_- - rK - \int_{x=-\infty}^{\log(\frac{S^*}{S_-})} (V(t, S_- e^x) - S_- e^x + K)\tilde{\nu}(t, dx)) = 0.$$

Chiarella-Ziogas [7] solve this **IPDE** Using *Fourier inversion.*

8.7. **An American deal no one should buy.** We conclude with the observation that no reasonable investment value can be attached to an "American option" whose underlying payoff process has no optimal stopping time. For definiteness assume the underlying payoff process is $Z_t = (S_t - 1)^+$ for $t < m$ and $Z_m = (S_m - 2)^+$ with S a martingale representing the stock price. It has no optimal stopping time and $V_0 = \mathbb{E}\,(S_m - 1)^+$. So, this contract has the same "fair" price as that of the m-maturity European call struck at 1. The buyer of this contract must exercise early for the payoff drops at m. But, if he does so, he receives less than what the European contract is worth.

Therefore, an investor is strictly better off with the European option at the same cost than this contract. This suggests that such contracts are not proper financial options. As in [20], one can view an option as a pair (T, O), with random expiry $T \in \mathcal{T}$ and random payoff $O \in \mathcal{F}_T$. For a reward process Z that does have an optimal stopping time T^*, the American option can then properly be defined as the pair (T^*, Z_{T^*}).

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