Bivariate support of forward Libor and swap rates

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Abstract. Based on a certain notion of "prolific process", we find an explicit expression for the bivariate (topological) support of the solution to a particular class of 2×2 stochastic differential equations that includes those of the 3-period "lognormal" Libor and swap market models. This yields that in the lognormal swap market model (SMM), the support of the 1×1 forward Libor L_t^* equals $[l_t^*, \infty)$ for some semi-explicit $-1 \le l_t^* \le 0$, sharpening a result of Davis and Mataix-Pastor (2005) that forward Libor rates (eventually) become negative with positive probability in the lognormal SMM. We classify the instances $l_t^* < 0$, and explicitly calculate the threshold time at or before which L_t^* remains positive a.s.

1. Introduction

The swap market model (SMM) is useful for swaption pricing, as in Jamshidian (1997). Recently, Davis and Mataix-Pastor (2005) have shown that eventually forward Libor rates become negative with positive probability in the "lognormal" swap market model with constant volatilities. They use the Stroock and Varadhan's Support Theorem and solve the associated deterministic ODE control problem to arrive at this negativity result.

While to our knowledge new, this result would not surprise those who use "curve generator" software to strip discount factors from market par swap rates. Should by mishap a rate on the swap curve be much larger than previous rates, the stripped discount curve will not be decreasing, producing negative forward rates. What is interesting about the Davis and Mataix-Pastor result is that even when forward Libor rates are positive initially, they will eventually become negative (often immediately) with positive (albeit small) probability.

This paper derives an explicit formula for the joint support $\mathcal{S}(L_t, Y_t)$ of the solution (L, Y) to a certain class of two-dimensional stochastic differential equations (SDE) with deterministic volatilities ratio. This class includes the SDEs that arise from a three-period "lognormal" swap market model (SMM) and a "lognormal" Libor market model (LMM). It turns out that the joint support will be either $\mathbb{R}_+^2$, where $\mathbb{R}_+ := [0, \infty)$, or the region between two simple graphs, namely, $\mathcal{S}(L_t, Y_t) = \{(l, y) \in \mathbb{R}_+^2 : c_t l^a \le y \le C_t l^a\}$. Here, $0 < c_t \le C_t$ and a are explicitly provided in terms of L_0, Y_0 and the volatilities.

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In our application to market models, L will be the $(n+1)\times 1$ forward Libor rate, and Y can be either the $n\times 2$ forward swap rate S (SMM) or the $n\times 1$ forward-Libor rate L^* (LMM), $n\geq 1$. The formula for the support of (L,Y) will be identical in both cases.

Assuming deterministic volatilities for L and S ("lognormal SMM"), our bivariate support formula leads to the precise moment t^* at or before which $L^*_t>0$ a.s. We find an interesting range of parameters for which $t^*>0$. There are also (time-decaying volatility) cases such that $L^*_t>0$ a.s. for all $t>0$. In most cases though $\mathbb{P}\{L^*_t<0\}>0$ for all $t>0$.

For example, in the one-factor case with $S_0=L_0$ and flat volatility σ for L and S of 10, 15, or 20 percent annually, t^* equals $\frac{\log 2}{\sigma^2}$, i.e., about 69, 31, or 17 years. Forward Libor L^*_t can thus become negative only after such rather large t^*, and then with a small probability and a rather small negative support, namely, $\mathcal{S}(L^*_t)=[\frac{2-2e^{-\sigma^2 t}}{\sqrt{e^{\sigma^2 t}-1}}-1\,,\,\infty)$ for $t>t^*$.

Interesting as it is, Davis and Mataix-Pastor's finding of negative Libor rates in SMM does not in our view diminish the practical usefulness of the swap market model, either as providing approximations for fast European swaption calibration in the Libor market model, or as a fast means for Bermudan swaption pricing by Markovian approximations.

1.1. **Summary of main results.** Section 2 provides formulae for the joint support $\mathcal{S}(L_t,Y_t)$, where $L>0$ is what we call a "log-prolific" process, e.g., a geometric Brownian motion, and $Y_t=g(t,L_t)\exp(\int_0^t f(s,L_s)ds)$ for given continuous functions f and $g>0$. We basically show that $\mathcal{S}(L_t,Y_t)$ equals $\mathbb{R}^2_+$ when f is unbounded, but when f is bounded,

$$\mathcal{S}(L_t,Y_t)=\{(l,y)\in\mathbb{R}^2_+ : e^{\int_0^t \inf_l f(s,l)ds}\leq \frac{y}{g(t,l)}\leq e^{\int_0^t \sup_l f(s,l)ds}\}.$$

This result has surely far less applicability than the Stroock-Varadhan's Support Theorem, and is more elementary, but it suffices for our applications, and has the advantage of yielding an explicit formula. In section 3, we apply it to the SDE system,

$$dL=L\sigma\,dW,\qquad (L_0>0)$$
$$dY=Yf(t,L)dt+Y\lambda\,dW,\qquad (Y_0>0)$$

where W is a Brownian motion, $\sigma_t>0$ and λ_t are deterministic functions, and f is a bounded continuous function. (The two-factor case, also treated, is easier, for then L_t and Y_t will be bivariately lognormally distributed under an equivalent measure, implying $\mathcal{S}(L_t,Y_t)=\mathbb{R}^2_+$.) Surprisingly, we find that if $a:=\frac{\lambda}{\sigma}$ is *not* a constant, then $\mathcal{S}(L_t,Y_t)=\mathbb{R}^2_+$, but otherwise the above formula holds with $g(t,l)=Y_0L_0^{-a}l^a\exp(\frac{1}{2}a(a-1)\int_0^t\sigma_s^2ds)$.

In Section 4, we quickly develop the facts needed about SMM and LMM, leading to their SDEs, which will be of the above form, with $f(t,l)=-\frac{\sigma_t\lambda_t l}{2+l}$ in SMM and $f(t,l)=-\frac{\sigma_t\lambda_t l}{1+l}$ in LMM. In these cases, when $a:=\frac{\lambda}{\sigma}$ is a constant, we find that if $a>0$, then

$$\mathcal{S}(L_t,Y_t)=\{(l,y)\in\mathbb{R}^2_+ : e^{-\frac{1}{2}a(1+a)\int_0^t\sigma_s^2ds}\leq\frac{L_0^a y}{Y_0 l^a}\leq e^{\frac{1}{2}a(1-a)\int_0^t\sigma_s^2ds}\},\qquad (a>0)$$

and if $a<0$, then

$$\mathcal{S}(L_t,Y_t)=\{(l,y)\in(0,\infty)^2 : e^{\frac{1}{2}a(1-a)\int_0^t\sigma_s^2ds}\leq\frac{L_0^a y}{Y_0 l^a}\leq e^{-\frac{1}{2}a(1+a)\int_0^t\sigma_s^2ds}\}.\qquad (a<0)$$

Section 5 applies this result to SMM, with Y being the forward swap rate S (so, λ is volatility of S) to see when forward Libor $L^* = \frac{(2+L)S-L}{1+L}$ can possibly become negative. We find that $L_t^* > 0$ a.s. if and only if a is constant on $[0,t]$, $0 \le a \le 1$, and $e^{\frac{1}{2}a(1+a)\int_0^t \sigma_s^2 ds} \le S_0 L_0^{-a} a^*$, where $a^* := (\frac{2}{a})^a (1-a)^{a-1}$ if $0 < a < 1$, $a^* := 2$ if $a = 1$, and $a^* := 1$ if $a = 0$.

In particular, $\mathbb{P}\{L_t^* < 0\} > 0$ for all $t > 0$ if a is not a constant, or else if $a < 0$, or $a > 1$, or $a^* S_0 \le L_0^a$. As another instance, when σ and λ are constants, $0 < a \le 1$, and $a^* S_0 > L_0^a$, it follows that $\mathbb{P}\{L_t^* < 0\} > 0$ if and only if $t > t^* := \frac{2}{a(a+1)\sigma^2} \log(S_0 L_0^{-a} a^*)$. In particular, for $a = 1$, the threshold time t^* is simply given by $t^* = \frac{1}{\sigma^2}\log(\frac{2S_0}{L_0})$.

We show that if a is not constant or is a negative constant then $\mathcal{S}(L_t^*) = [-1, \infty)$, while if a is a positive constant then $\mathcal{S}(L_t^*) = [l_t^*, \infty)$ for some well-characterized $-1 < l_t^* \le 0$. For $a = 1$, we get explicitly, $l_t^* = 0$ if $L_0 e^{\int_0^t \sigma_s^2 ds} \le 2S_0$ and $l_t^* = \frac{2 - 2\frac{S_0}{L_0} e^{-\int_0^t \sigma_s^2 ds}}{\sqrt{\frac{L_0}{S_0} e^{\int_0^t \sigma_s^2 ds} - 1}} - 1$ otherwise.

2. Prolific processes and certain bivariate support results

2.1. **Topological support.** We set $\mathbb{R}_+ := [0, \infty)$, and fix a stochastic basis $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathcal{F}, \mathbb{P})$ throughout, assuming for simplicity that $\mathcal{F}_0$ consists of the events of probability 0 or 1.

The (topological) ***support*** $\mathcal{S}(X)$ of an n-dimensional random variable X is defined by

$$\mathcal{S}(X) := \{x \in \mathbb{R}^n : \mathbb{P}\{X \in U\} > 0 \text{ for all open subsets } U \text{ of } \mathbb{R}^n \text{ containing } x\}.$$

Note, $\mathcal{S}(X)$ is a nonempty *closed* subset of $\mathbb{R}^n$. Also, $X \in \mathcal{S}(X)$ a.s., and if $f : \mathcal{S}(X) \to \mathbb{R}^m$ is a continuous mapping, then $\mathcal{S}(f(X))$ equals the closure of $f(\mathcal{S}(X))$ in $\mathbb{R}^m$.

Lemma 2.1. *Let X and Y be two random variables and h be a continuous function on $\mathbb{R}$. Then, $\mathcal{S}(X, h(X) + Y) = \{(x, y) \in \mathbb{R}^2 : (x, y - h(x)) \in \mathcal{S}(X, Y)\}$.*

Proof. Define the homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\varphi(x, y) = (x, h(x) + y)$. Then, $\mathcal{S}(X, h(X)+Y) = \mathcal{S}(\varphi(X, Y)) = \varphi(\mathcal{S}(X, Y)) = \{(x, y) \in \mathbb{R}^2 : (x, y-h(x)) \in \mathcal{S}(X, Y)\}$. □

2.2. **Prolific processes.** We call a univariate measurable process $X = (X_t)_{t=0}^\infty$ ***prolific*** if

$$\mathbb{P}\{\sup_{s \in [0,t]} |X_s - c(s)| < \varepsilon\} > 0$$

for all $t > 0$, $\varepsilon > 0$, and continuous functions $c : [0, t] \to \mathbb{R}$ with $c(0) \in \mathcal{S}(X_0)$.[1]

Basically this means that the set of paths of X which in any bounded interval $[0, t]$ lie uniformly within an ε-band of any curve c has positive measure. Intuitively, like a Brownian motion, X can follow any conceivable continuous path, at least approximately.

Clearly, if X is a prolific process, then it remains prolific under any equivalent measure; further, $f(X)$ is then prolific for any homeomorphism $f : \mathbb{R} \to \mathbb{R}$, as is the process $(c(t) + X_t)_{t \in \mathbb{R}_+}$ for any continuous function c on $\mathbb{R}_+ := [0, \infty)$ (as is $(c(t)X_t)_{t \in \mathbb{R}_+}$ if $c > 0$).

[1]Note, for X to be prolific, it is sufficient that the defining property holds for all polynomials c (by the Weierstrass approximation theorem) or for all continuous, piecewise-linear functions c.

We call a measurable process X ***log-prolific*** if $X > 0$ and $\log(X)$ is prolific. It is easy to see that a positive process X is log-prolific if and only if $\mathbb{P}\{\sup_{s\in[0,t]} |X_s - c(s)| < \varepsilon\} > 0$ for all $t > 0$, $\varepsilon > 0$, and *positive* continuous functions c on $[0, t]$ with $c(0) \in \mathcal{S}(X_0)$.

Clearly, if X is prolific (resp. log-prolific) then $\mathcal{S}(X_t) = \mathbb{R}$ (resp. $\mathcal{S}(X_t) = \mathbb{R}_+$) for $t > 0$.

2.3. **Examples of prolific processes.** The primary example is the Brownian motion:

Proposition 2.2. *A Brownian motion W (with $W_0 = 0$) is prolific.*

Proof. Let $t > 0$, $\varepsilon > 0$, and c be a continuous function on $[0, t]$ with $c(0) = 0$. We must show $\mathbb{P}\{\sup_{s\in[0,t]} |W_s - c(s)| < \varepsilon\} > 0$. As pointed out in footnote 1, we may assume c is smooth. By Girsanov's theorem, the process $(W_s - c(s))_{s=0}^t$ is a $\mathbb{Q}$-Brownian motion on $[0, t]$, where $\mathbb{Q}$ is the equivalent measure defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp(-\frac{1}{2}\int_0^t (c'(s))^2 ds + \int_0^t c'(s) dW_s)$. It follows that (see e.g. [2], Exercise 2.8.11, p. 99),

$$\mathbb{Q}\{\sup_{s\in[0,t]} |W_s - c(s)| < \varepsilon\} = \int_t^\infty \sum_{n=-\infty}^{\infty} (4n+1) e^{-\frac{(4n+1)^2\varepsilon^2}{2u}} \frac{2\varepsilon du}{\sqrt{2\pi u^3}},$$

which is positive, since the integrand is positive. The desired result follows as $\mathbb{Q} \sim \mathbb{P}$. □

A deterministic change of time shows that all continuous Gaussian local martingales are prolific and their stochastic exponentials are log-prolific:

Proposition 2.3. *Let $X = \int_0^\cdot \sigma_t dW_t$, with W a Brownian motion and $\sigma > 0$ deterministic with $[X]_t := \int_0^t \sigma_s^2 ds < \infty$ for all t. Then X is prolific and $\mathcal{E}(X) := e^{X - \frac{[X]}{2}}$ is log-prolific.*

Proof. Assume first that $[X]_\infty = \infty$. Let $t > 0$, $\varepsilon > 0$, and c be a continuous function on $[0, t]$ with $c(0) = 0$. Let τ denote the inverse of $[X]$, which exists since $[X]$ is *strictly* increasing (as $\sigma > 0$) and has image $[0, \infty)$ (as it is continuous and $[X]_\infty = \infty$). Clearly,

$$\{\sup_{s\in[0,t]} |X_s - c(s)| < \varepsilon\} = \{\sup_{u\in[0,[X]_t]} |X_{\tau(u)} - c(\tau(u))| < \varepsilon\}. \tag{2.1}$$

As is well-known, the process $(X_{\tau(u)})_{u=0}^\infty$ is a Brownian motion under the time-changed filtration $(\mathcal{F}_{\tau(u)})_{u=0}^\infty$. Therefore, by Prop. 2.2 it is prolific. Applying its prolific property to the curve $c \circ \tau : [0, [X]_t] \to \mathbb{R}$, we see that the set on the right-hand side of Eq. (2.1) has positive measure. Hence, so does the set on left-hand side, showing X is prolific.

Next assume $[X]_\infty < \infty$. Let $t > 0$. Define $\widehat{X} := \int_0^\cdot \widehat{\sigma}_s dW_s$, where $\widehat{\sigma}_s = \sigma_s$ for $s \le t$ and $\widehat{\sigma}_s = 1$ for $s > t$. Since $[\widehat{X}]_\infty = \infty$, the previous case shows that $\widehat{X}$ is prolific. Since $X = \widehat{X}$ on $[0, t]$ and t was arbitrary, X is prolific too. Finally, since $[X]$ is continuous and deterministic, it follows that $X - \frac{[X]}{2}$ is also prolific, implying $\mathcal{E}(X)$ is log-prolific. □

The following result provides more examples of prolific processes, but will not be used.

Proposition 2.4. *Let W be a Brownian motion under an equivalent measure $\mathbb{Q}$, and σ be a positive C^1 function on $\mathbb{R}$ with bounded derivative such that $\int_0^\infty \frac{dx}{\sigma(x)} = \int_{-\infty}^0 \frac{dx}{\sigma(x)} = \infty$. Then the (unique) solution X to the stochastic differential equation $dX_t = \sigma(X_t) dW_t$, $X_0 = 0$, is prolific, and if more strongly σ is bounded, then $\mathcal{E}(X)$ is log-prolific.*

Proof. The SDE has a unique solution X since σ is globally Lipschitz (as σ' is bounded). The function g defined by $g(x) := \int_0^x \frac{dy}{\sigma(y)}$ is a C^2 diffeomorphism on $\mathbb{R}$ by the assumptions. Set $f := g^{-1}$. Note, $f'(z) = \frac{1}{g'(f(z))} = \sigma(f(z))$ and $f''(z) = \sigma'(f(z))f'(z) = \sigma'(f(z))\sigma(f(z))$. Define the processes $Z := W - \frac{1}{2}\int_0^\cdot \sigma'(X_t)dt$ and $Y := f(Z)$. Then, by Itô's formula,

$$dY = \frac{1}{2}f''(Z)dt + f'(Z)dZ = \frac{1}{2}\sigma'(Y)\sigma(Y)dt + \sigma(Y)dZ$$
$$= \frac{1}{2}\sigma(Y)(\sigma'(Y) - \sigma'(X))dt + \sigma(Y)dW.$$

As an SDE for Y with random coefficients, the above has a unique solution, and since X and $f(Z)$ are both solutions, it follows that $X = f(Z)$. Therefore, to show X is prolific, it suffices to show Z is prolific. The process $M := \mathcal{E}(\frac{1}{2}\int_0^\cdot \sigma'(X_t)dW_t)$ is a $\mathbb{Q}$-martingale since σ' is bounded. Let $T > 0$ and define the measure $\mathbb{P}'$ by $\frac{d\mathbb{P}'}{d\mathbb{Q}} = M_T$. By Girsanov's Theorem, Z is a $\mathbb{P}'$-Brownian motion on $[0,T]$. Prop. 2.2 implies Z is prolific, as T was arbitrary. Hence, $X = f(Z)$ is prolific. Finally, if σ is bounded, then $X - \frac{[X]}{2} = \int \sigma(X)dW'$, with $W' := W - \frac{1}{2}\int \sigma(X)dt$ a $\mathbb{Q}'$-Brownian motion on $[0,T]$ where $\frac{d\mathbb{Q}'}{d\mathbb{Q}} := \mathcal{E}(\frac{1}{2}\int \sigma(X)dW)_T$. But then the first part implies that $X - \frac{[X]}{2}$ is prolific, showing $\mathcal{E}(X)$ is log-prolific. □

Remark: The above result generalizes to when $\sigma = \sigma(t,x)$ is C^1 in both t and x. The proof is similar: one defines $g(t,x) := \int_0^x \frac{dy}{\sigma(t,y)}$ and $f(t,z)$ as the inverse of g in x, and shows similarly that $X_t = f(t, Z_t)$, where now $Z_t := W_t - \int_0^t (\frac{1}{2}\sigma_x(s,X_s)ds + \int_0^{X_s} \frac{\sigma_t(s,y)}{\sigma^2(s,y)}dy)ds$.

A Poisson process is not prolific, but its sum with a Brownian motion certainly is.

We conjecture that infinite activity Lévy processes are prolific, and the sum of a prolific continuous semimartingale with a purely discontinuous semimartingale is prolific.

2.4. **Bivariate support results: bounded case.** This case leads to our most interesting applications where the joint support is a region in $\mathbb{R}_+^2$ between two similar graphs.

In this subsection, given $t > 0$ and a bounded, continuous function $f = f(s,x)$ on $[0,t] \times \mathbb{R}$ or on $[0,t] \times \mathbb{R}_+$, we define the continuous functions

$$f_*(s) := \inf_x f(s,x), \qquad F_*(t) := \int_0^t f_*(s)ds;$$
$$f^*(s) := \sup_x f(s,x), \qquad F^*(t) := \int_0^t f^*(s)ds.$$

Lemma 2.5. *Let $t > 0$, and f be a bounded continuous function on $[0,t] \times \mathbb{R}$. Then, for any $x_0, x \in \mathbb{R}$ and $F_*(t) < y < F^*(t)$, there exists a continuous function c on $[0,t]$ such that $c(0) = x_0$, $c(t) = x$, and $\int_0^t f(s,c(s))ds = y$.*

Proof. Let $\mathcal{C}$ denote the Banach space of all continuous functions on $[0,t]$, with norm $\|c\| = \sup_{s\in[0,t]} |c(s)|$. Define the function F on $\mathcal{C}$ by $F(c) := \int_0^t f(s,c(s))ds$. Let $\mathcal{C}' := \{c \in \mathcal{C} : c(0) = x_0,\ c(t) = x\}$. We must show that $F(\mathcal{C}')$ contains the interval $(F_*(t), F^*(t))$.

We first show that F is continuous and $\mathcal{C}'$ is connected, implying $F(\mathcal{C}')$ is connected. Let $c \in \mathcal{C}$ and $\varepsilon > 0$. Then, using the uniform continuity of f over compact sets, one easily sees that there exists a $\delta > 0$ such that $|f(s,z) - f(s,c(s))| < \frac{\varepsilon}{t}$ for any $s \in [0,t]$ and $z \in \mathbb{R}$ satisfying $|z - c(s)| < \delta$. It follows that $|F(\hat{c}) - F(c)| < \varepsilon$ whenever $\|\hat{c} - c\| < \delta$. Hence, F is continuous. Next, $\mathcal{C}'$ is a (closed) affine subspace of $\mathcal{C}$, hence connected. (Indeed, $\mathcal{C}' = c' + \mathcal{C}_0$ for any $c' \in \mathcal{C}'$, where $\mathcal{C}_0$ is the closed subspace $\{c \in \mathcal{C} : c(0) = c(t) = 0\}$.)

Since as just shown, $F(\mathcal{C}')$ is connected, i.e., is an interval, it suffices to show that for every $\varepsilon > 0$, there exist $c^*, c_* \in \mathcal{C}'$ such that $|F(c^*) - F^*(t)| < \varepsilon$ and $|F(c_*) - F_*(t)| < \varepsilon$. We will only show the of existence c^* as the existence of c_* is similar.

Let $\varepsilon > 0$. The continuity of f and f^* implies that for any $s \in [0,t]$ there is a number x_s^* and an open set U_s containing s such that $|f^*(u) - f(s, x_s^*)| < \frac{\varepsilon}{2t}$ for all $u \in U_s$. Since $[0,t]$ is compact, it is covered by a finite number of such sets $U_{s_1}, \cdots U_{s_n}$. This implies that there exists a piecewise constant function p on $[0,t]$ (taking values in the finite set $\{f(s_i, x_{s_i}^*),\ i = 1, \cdots n\}$) such that $|f(s,p(s)) - f^*(s)| < \frac{\varepsilon}{2t}$ for all $s \in [0,t]$. Clearly then, $|\int_0^t f(s,p(s))ds - F^*(t)| < \frac{\varepsilon}{2}$. So, it suffices to show that there exists a $c^* \in \mathcal{C}'$ such that $\int_0^t |f(s,p(s)) - f(s,c^*(s))|ds < \frac{\varepsilon}{2}$. To this end, say p is constant on the intervals $(t_{i-1}, t_i]$ with values p_i, where $0 = t_0 < t_1 \cdots < t_n = t$. Set $m := \max(|x_0|, |x|, |p_1|, \cdots, |p_n|)$, $C := \sup_{[0,t]\times[-m,m]} |f|$, and choose any $0 < \delta < \frac{\varepsilon}{8(n+1)C}$. Shrink $\delta > 0$ if necessary so that $2\delta < t_i - t_{i-1}$ for all i. Then the continuous piecewise-linear function c^* whose graphs joins the points $(0, x_0)$, (δ, p_1), $(t_1 - \delta, p_1)$, $(t_1 + \delta, p_2)$, $(t_2 - \delta, p_2)$, $\cdots$, $(t_{n-1} + \delta, p_n)$, $(t - \delta, p_n)$, (t, x) has the requisite properties. Indeed, set $g(s) := |f(s,p(s)) - f(s,c^*(s))|$. Then, by construction, $g = 0$ on the intervals $[t_{i-1}+\delta, t_i-\delta]$, and on the rest of $[0,t]$, which consists of the $n+1$ intervals $[0,\delta]$, $[t-\delta]$, and $[t_i-\delta, t_i+\delta]$, $i = 1, \cdots, n-1$, we have $g \leq 2C$ because the graphs of both c^* and p lie in $[0,t] \times [-m,m]$. Therefore, $\int_{t_i-\delta}^{t_i+\delta} g(s)ds \leq 4C\delta < \frac{\varepsilon}{2(n+1)}$. As there are $n+1$ such intervals, we see that $\int_0^t g(s)ds < \frac{\varepsilon}{2}$, as desired. □

Theorem 2.6. *Let X be a prolific process, $t > 0$, and f be a bounded continuous function on $[0,t] \times \mathbb{R}$. Then*

$$\mathcal{S}(X_t, \int_0^t f(s, X_s)ds) = \mathbb{R} \times [F_*(t), F^*(t)].$$

Proof. . Set $Y_t := \int_0^t f(s, X_s)ds$. Obviously $F_*(t) \leq Y_t \leq F^*(t)$, implying the inclusion "$\subset$". To show the "$\supset$", let $x \in \mathbb{R}$ and $F_*(t) < y < F^*(s)$. Since the support is a closed set, it suffices to show that $(x,y) \in \mathcal{S}(X_t, Y_t)$, i.e., that for all $\varepsilon > 0$, $\mathbb{P}\Lambda > 0$, where $\Lambda := \{|X_t - x| < \varepsilon\} \cap \{|Y_t - y| < \varepsilon\}$. Let $\varepsilon > 0$. Choose any $x_0 \in \mathcal{S}(X_0)$. By Lemma 2.5 there exists a continuous function c on $[0,t]$ such that $c(0) = x_0$, $c(t) = x$, and $\int_0^t f(s, c(s))ds = y$. The uniform continuity of f over compact sets easily implies that there exists a $\delta > 0$ such that $|f(s,z) - f(s,c(s))| < \frac{\varepsilon}{t}$ whenever $|z - c(s)| < \delta$. Set $\Gamma := \{\sup_{s\in[0,t]} |X_s - c(s)| < \min(\varepsilon, \delta)\}$. Since X is prolific, $\mathbb{P}\,\Gamma > 0$. Hence, it suffices to show $\Gamma \subset \Lambda$. But, on Γ, we have

$$|Y_t - y| = |\int_0^t f(s, X_s)ds - \int_0^t f(s, c(s))ds| \leq \int_0^t |f(s, X_s) - f(s, c(s))|ds < \varepsilon.$$

Hence, $\Gamma \subset \{|Y_t - y| < \varepsilon\}$. Also, $\Gamma \subset \{|X_t - x| < \varepsilon\}$ since $c(t) = x$. Thus $\Gamma \subset \Lambda$. □

We will use the following consequence which is formulated for log-prolific processes.

Corollary 2.7. *Let L be a log-prolific process. Let $f = f(t,l)$ and $g = g(t,l)$ be two functions of $t \geq 0$ and $l > 0$. Assume f is continuous and for each t is bounded on $[0,t] \times (0,\infty)$, and g is continuous in l and positive. Then, for any $t > 0$,*

$$\mathcal{S}(L_t,\, g(t,L_t)\, e^{\int_0^t f(s,L_s)ds}) = \text{closure}\,\{(l,k) \in (0,\infty)^2 : e^{F_*(t)} \leq \frac{k}{g(t,l)} \leq e^{F^*(t)}\}.$$

Proof. By considering the continuous mapping $\varphi(x,y) = (e^x, e^y)$ on $\mathbb{R}^2$, it suffices to show

$$\mathcal{S}(X_t, h(t,X_t) + Y_t) = \{(x,y) \in \mathbb{R}^2 : F_*(t) \leq y - h(t,x) \leq F^*(t)\},$$

where $X := \log(L)$, $h(x) = \log(g(t,e^x))$, and $Y_t := \int_0^t f(s,L_s)ds$. For $g = 1$, i.e., $h = 0$, this follows from Theorem 2.6 applied to the function $\hat{f}(s,x) := f(s,e^x)$ (since $Y_t = \int_0^t \hat{f}(s,X_s)ds$), and for a general g it then follows from Lemma 2.1. □

2.5. **Bivariate support results: unbounded case.** This case is less interesting; it will be applied to one-factor market models with "generic" time-dependent volatilities.

Theorem 2.8. *Let $t > 0$, and f be a continuous function on $[0,t] \times \mathbb{R}$ such that*

$$\inf_{c \in \mathcal{C}} \int_a^b f(s,c(s))ds = -\infty, \qquad \sup_{c \in \mathcal{C}} \int_a^b f(s,c(s))ds = \infty$$

for some $0 < a < b < t$, where $\mathcal{C}$ is the space of all continuous functions on $[0,t]$. Then:

(*a*) *For any x_0, $x, y \in \mathbb{R}$, there exists a continuous function c on $[0,t]$ such that $c(0) = x_0$, $c(t) = x$, and $\int_0^t f(s,c(s))ds = y$.*

(*b*) *For any càdlàg prolific process X, we have, $\mathcal{S}(X_t, \int_0^t f(s,X_s)ds) = \mathbb{R}^2$.*

Proof. (a): Define the function F on $\mathcal{C}$ by $F(c) = \int_0^t f(s,c(s))ds$, and let $\mathcal{C}' := \{c \in \mathcal{C} : c(0) = x_0,\ c(t) = x\}$. We must show $F(\mathcal{C}') = \mathbb{R}$. As in the proof of Lemma 2.5, F is continuous (under the sup norm) and $\mathcal{C}'$ is connected. Hence, $F(\mathcal{C}')$ is an interval. Therefore, it suffices to show that for any $k \in \mathbb{R}$, there exist $c^*, c_* \in \mathcal{C}'$ such that $F(c^*) > k$ and $F(c_*) < k$. We will only show the existence of c^* as the existence of c_* is similar.

Let $k \in \mathbb{R}$. The assumption on f implies that there exists a $c \in \mathcal{C}$ such that

$$\int_0^a f(s,x_0)ds + \int_a^b f(s,c(s))ds + \int_b^t f(s,x)ds > k + 1. \tag{2.2}$$

Set $m := \max(|x_0|, |x|, |c(a)|, |c(b)|)$, $C := \sup_{[0,t]\times[-m,m]} |f|$, and choose $0 < \alpha < \min(\frac{1}{4C}, a)$ and $0 < \beta < \min(\frac{1}{4C}, t-b)$. Let $c^* \in \mathcal{C}'$ be the function that equals x_0 on $[0, a-\alpha]$, is affine-linear on $[a-\alpha, a]$, equals c on $[a,b]$, is affine-linear on $[b, b+\beta]$, and equals x on $[b+\beta, t]$. Then, in view of Eq. (2.2) and these definitions, we have,

$$|F(c^*) - k - 1| < \int_{a-\alpha}^a |f(s,x_0) - f(s,c^*(s))|\, ds + \int_b^{b+\beta} |f(s,x_0) - f(s,c^*(s))|\, ds \leq 2C\alpha + 2C\beta < 1.$$

Hence, $F(c^*) > k$, completing the proof of part (a).

(As an aside, by a slight change in the proof, part (a) is valid also when $a = 0$ or $b = t$.)

(b): Since X is càdlàg, X has bounded paths on $[0,t]$. Therefore, as f is continuous, for almost all $\omega \in \Omega$ the function $s \mapsto f(s, X_s(\omega))$ is bounded, hence integrable, on $[0,t]$. As such, $Y_t := \int_0^t f(s, X_s)ds$ is well-defined. Let $\varepsilon > 0$ and $(x, y) \in \mathbb{R}^2$. It suffices to show $\mathbb{P}\Lambda > 0$, where $\Lambda := \{|X_t - x| < \varepsilon\} \cap \{|Y_t - y| < \varepsilon\}$. By part (a) there exists a continuous function c on $[0,t]$ such that $c(0) \in \mathcal{S}(X_0)$, $c(t) = x$, and $\int_0^t f(s, c(s))ds = y$. The uniform continuity of f over compact sets easily implies that there exists a $\delta > 0$ such that $|f(s,z) - f(s, c(s))| < \frac{\varepsilon}{t}$ whenever $|z - c(s)| < \delta$. Set $\Gamma := \{\sup_{s\in[0,t]} |X_s - c(s)| < \min(\varepsilon, \delta)\}$. Since X is prolific, $\mathbb{P}\,\Gamma > 0$. Hence, it suffices to show $\Gamma \subset \Lambda$. But, on Γ, we have

$$|Y_t - y| = |\int_0^t f(s, X_s)ds - \int_0^t f(s, c(s))ds| \leq \int_0^t |f(s, X_s) - f(s, c(s))|ds < \varepsilon.$$

Hence, $\Gamma \subset \{|Y_t - y| < \varepsilon\}$. Also, $\Gamma \subset \{|X_t - x| < \varepsilon\}$ since $c(t) = x$. Thus $\Gamma \subset \Lambda$. □

Remark: The condition on the continuous function f in Theorem 2.8 is obviously satisfied if $\inf_{x\in\mathbb{R}} \int_a^b f(s,x)ds = -\infty$ and $\sup_{x\in\mathbb{R}} \int_a^b f(s,x)ds = \infty$, or, using an argument like that for Lemma 2.5, simply if $\inf_{x\in\mathbb{R}} f(s,x) = -\infty$ and $\sup_{x\in\mathbb{R}} f(s,x) = \infty$ for all $s \in [a,b]$.

Corollary 2.9. *Let L be a càdlàg log-prolific process. Let $f = f(t,l)$ and $g = g(t,l)$ be two functions of $t \geq 0$ and $l > 0$. Assume that f is continuous and for each t is bounded on $[0,t] \times (0,\infty)$, and g is continuous in l and positive. Let α be a continuous function on $\mathbb{R}_+$. Then, for any $t > 0$ such that α does not vanish identically on $[0,t]$, we have*

$$\mathcal{S}(L_t, g(t, L_t)e^{\int_0^t (f(s,L_s)+\alpha(s)\log(L_s))ds}) = \mathbb{R}_+^2.$$

Proof. Set $\hat{f}(s,x) := f(s, e^x) + \alpha(s)x$ and $h(x) := \log(g(t, e^x))$. It suffices to show that $\mathcal{S}(X_t, Y_t) = \mathbb{R}^2$, where $X := \log(L)$ and $Y_t := h(t, X_t) + \int_0^t \hat{f}(s, X_s)ds$. By Lemma 2.1, we may clearly assume $g = 1$, i.e., $h = 0$. As such, by Theorem 2.8 (b), it suffices to show that $\inf_{c\in\mathcal{C}} \int_0^u \hat{f}(s, c(s))ds = -\infty$ and $\sup_{c\in\mathcal{C}} \int_a^b \hat{f}(s, c(s))ds = \infty$ for some $0 < a < b < t$. We only show the latter, as the former is similar. Obviously, it is enough to show $\sup_{x\in\mathbb{R}} \int_a^b \hat{f}(s,x)ds = \infty$ for some $0 < a < b < t$. Since f is bounded $[0,t] \times (0,\infty)$, this follows once we show $\sup_{x\in\mathbb{R}} \int_a^b \alpha(s)x ds = \infty$. But, this is equivalent to $\int_a^b \alpha(s)ds \neq 0$ for some $0 < a < b < t$, which holds because α does not vanish identically on $[0,t]$. □

Remark: In our application of Corollaries 2.7 and 2.9, the function $f(t,l)$ will be of the separable form $\beta(t)h(l)$ for some continuous bounded function h. In this case, simpler proofs are possible even with the continuity of β (and α) weakened to local integrability.

3. Application to a certain SDE system

3.1. **The SDE setup.** We assume in this section that L and Y are processes following

$$dL = L\sigma\, dW, \qquad (L_0 > 0)$$

$$dY = Y(f(t,L)dt + \lambda\, dW + \gamma dZ), \qquad (Y_0 > 0)$$

where W and Z are independent Brownian motions, f is a continuous function on $\mathbb{R}_+ \times (0,\infty)$, and $\sigma > 0$, λ, γ are optional processes with $\int_0^T (\sigma_t^2 + \lambda_t^2 + \gamma_t^2)dt < \infty$ a.s. all $T > 0$.

With $\mathcal{E}(X) := \exp(X - \frac{1}{2}[X])$ denoting the stochastic exponential of a continuous semi-martingale, the solution to this system is given by

$$L = L_0 \mathcal{E}(\int \sigma dW),$$

$$Y = Y_0 \exp(\int_0^\cdot f(t, L_t)dt)\mathcal{E}(\int \lambda dW)\mathcal{E}(\int \gamma dZ).$$

3.2. **The one-factor case.** The case $\gamma = 0$ is the interesting case. We show in particular that when σ and λ are deterministic, if $a := \frac{\lambda}{\sigma}$ is not a constant on $[0, t]$, then $\mathcal{S}(L_t, Y_t) = \mathbb{R}^2_+$, but otherwise, $\mathcal{S}(L_t, Y_t) = \{c_t l^a \le y \le C_t l^a\}$ is the region between two simple graphs for some explicit constants $0 < c_t \le C_t$ (depending on t, L_0, Y_0, a, σ, f).

Lemma 3.1. *Assume $\gamma = 0$ and $a := \frac{\lambda}{\sigma}$ is deterministic and C^1. Then, for all $t \ge 0$,*

$$Y_t = \frac{Y_0}{L_0^{a_0}} L_t^{a_t} e^{\int_0^t (f(s,L_s) - a_s' \log(L_s) + \frac{1}{2}a_s(1-a_s)\sigma_s^2)ds}, \tag{3.1}$$

where $a' := \frac{da}{dt}$. In particular, if $a := \frac{\lambda}{\sigma}$ is a constant, then

$$Y_t = \frac{Y_0}{L_0^a} L_t^a e^{\int_0^t (f(s,L_s) + \frac{1}{2}a(1-a)\sigma_s^2)ds}. \qquad (a' = 0) \tag{3.2}$$

Proof. Using basic Itô calculus on stochastic exponential and logarithm, we have

$$\mathcal{E}(\int \lambda dW) = \exp(\int a\frac{dL}{L} - \frac{1}{2}\int \lambda^2 dt)$$

$$= \exp(\int a\, d\log(L) + \frac{1}{2}\int (a\sigma^2 - \lambda^2)dt)$$

$$= \frac{L^a}{L_0^{a_0}} \exp(-\int \log(L)da + \frac{1}{2}\int (a\sigma^2 - \lambda^2)dt),$$

the last equality following by integration by parts (with boundary term $\frac{L^a}{L_0^{a_0}}$). As $a\sigma^2 - \lambda^2 = a(1-a)\sigma^2$, the desired result follows from this and the formula for Y in section 3.1. □

Both formulae for Y above are of the form of those Section 2. The corresponding results of Section 2 lead to our main result for the joint support of the solution to such SDE.

Theorem 3.2. *Assume $\gamma = 0$, $a := \frac{\lambda}{\sigma}$ is deterministic and C^1, and $\sigma_t = \sigma(t, L_t)$ for some continuous function $\sigma(t, l)$. Let $t > 0$. Assume further that L is log-prolific and the function $\hat{f}(s, l) := f(s, l) + \frac{1}{2}a_s(1 - a_s)\sigma^2(s, l)$ is bounded on $[0, t] \times (0, \infty)$. Then, $\mathcal{S}(L_t, Y_t) = \mathbb{R}^2_+$ if a is not a constant on $[0, t]$, and otherwise,*

$$\mathcal{S}(L_t, Y_t) = \text{closure}\ \{(l, y) \in (0, \infty)^2 : e^{\int_0^t \inf_{l>0} \hat{f}(s,l)ds} \le \frac{L_0^a y}{Y_0 l^a} \le e^{\int_0^t \sup_{l>0} \hat{f}(s,l)ds}\}.$$

Proof. If a is not a constant, then in view of Eq. (3.1), Corollary 2.9 applied to $\hat{f}$, with $g(t,l) := \frac{Y_0}{L_0^{a_0}} l^{a_t}$ and $\alpha := -a'$, yields $\mathcal{S}(L_t, Y_t) = \mathbb{R}^2_+$.[2] The statement for the case of constant a follows immediately by combining Eq. (3.2) with Cor. 2.7 applied to $\hat{f}$. □

Remark: A similar argument shows that if more generally $a := \frac{\lambda}{\sigma}$ is a C^1 function of L (i.e., $\lambda_t = \sigma(t, L_t)a(L_t)$) and $\tilde{f}(t,l) := f(t,l) + \frac{1}{2}a(l)(1-a(l))\sigma^2(t,l) - \frac{1}{2}la'(l)$ is bounded, then $\mathcal{S}(L_t, Y_t) = \{(l,y) \in (0,\infty)^2 : e^{\int_0^t \inf_l \tilde{f}(s,l)ds} \leq \frac{y}{Y_0} e^{-\int_{L_0}^l a(z)\frac{dz}{z}} \leq e^{\int_0^t \sup_l \tilde{f}(s,l)ds}\}$.

3.3. **The two-factor case.** In this not so interesting case, γ is not identically zero, implying $\mathcal{S}(L_t, Y_t) = \mathbb{R}^2_+$, as shown below, or seen alternatively by changing measure to that under which $Z + \int \frac{f(t,L)}{\gamma} dt$ is a Brownian motion, where (L_t, Y_t) are bivariately lognormal.

Lemma 3.3. *Let K, X, L be three positive random variables. Assume that K is independent of both X and L, and $\mathcal{S}(L) = \mathcal{S}(K) = \mathbb{R}_+$. Then $\mathcal{S}(L, XK) = \mathbb{R}^2_+$.*

Proof. Set $Y = XK$. Let $y > 0$, $l > 0$, and $\varepsilon > 0$. It is sufficient to show that $\mathbb{P}\Lambda > 0$, where $\Lambda := \{|\log \frac{Y}{y}| < \varepsilon\} \cap \{|L - l| < \varepsilon\}$. Since $l \in \mathbb{R}_+ = \mathcal{S}(L)$, there clearly exists $x > 0$ such that $(l,x) \in \mathcal{S}(L,X)$. It follows $\mathbb{P}\Gamma > 0$, where $\Gamma := \{|\log \frac{X}{x}| < \frac{\varepsilon}{2}\} \cap \{|L - l| < \varepsilon\}$. Set $k := \frac{y}{x}$. Since $\mathcal{S}(K) = \mathbb{R}_+$, we have $\mathbb{P}\Pi > 0$, where $\Pi := \{|\log \frac{K}{k}| < \frac{\varepsilon}{2}\}$. Hence, by the independence assumption, $\mathbb{P}(\Pi \cap \Gamma) = \mathbb{P}(\Pi)\mathbb{P}(\Gamma) > 0$. But $\Pi \cap \Gamma \subset \Lambda$, since $|\log \frac{Y}{y}| = |\log \frac{X}{x} + \log \frac{K}{k}| \leq |\log \frac{X}{x}| + |\log \frac{K}{k}|$. Hence, $\mathbb{P}\Lambda > 0$, as desired. □

Proposition 3.4. *Assume σ, λ, and γ deterministic. Let $t > 0$, and assume γ is not identically zero a.e. on $[0,t]$. Then, $\mathcal{S}(L_t, Y_t) = \mathbb{R}^2_+$.*

Proof. Set $K_t := \mathcal{E}(\int \gamma dZ)_t$ and $X_t := Y_0 \exp(\int_0^t f(s, L_s)ds)\mathcal{E}(\int \lambda dW)_t$. Clearly, K_t is independent of both X_t and L_t. Also, $\mathcal{S}(L_t) = \mathcal{S}(K_t) = \mathbb{R}_+$, as both L_t and K_t are lognormal with positive variance. Since $Y_t = X_t K_t$, Lemma 3.3 yields $\mathcal{S}(L_t, Y_t) = \mathbb{R}^2_+$. □

4. Application to the three-period Libor and swap market models

4.1. **Arbitrage-free market model.** All martingales here are presumed right continuous.

Proposition 4.1. *Let I be a nonempty set and $(B_i)_{i\in I}$ be a family of positive functions on $\mathbb{R}_+ \times \Omega$. Then, for any $T > 0$, the following three conditions are equivalent.*

(a) For some $j \in I$, there exists an equivalent probability measure $\mathbb{P}^j$ such that $\frac{B^i}{B^j}$ is a $\mathbb{P}^j$-martingale on $[0,T]$ for all $i \in I$.

(b) For all $j \in I$, there exists an equivalent probability measure $\mathbb{P}^j$ such that $\frac{B^i}{B^j}$ is a $\mathbb{P}^j$-martingale on $[0,T]$ for all $i \in I$.

(c) There exists a positive function ξ on $\mathbb{R}_+ \times \Omega$ such that ξB^i is a $\mathbb{P}$-martingale on $[0,T]$ for all $i \in I$.

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[2] When σ and λ are deterministic but not necessarily continuous, a more direct proof of this case that does not utilize Eq. (3.1) and Corollary 2.9 seems possible from following observation. The condition $a := \frac{\lambda}{\sigma}$ is not a constant on $[0,t]$ is equivalent to $\int_0^t |\sigma\lambda| ds < \sqrt{\int_0^t \sigma^2 ds}\sqrt{\int_0^t \lambda^2 ds}$, which is in turn equivalent to L_t and $\mathcal{E}_t(\int \frac{\lambda}{\sigma}\frac{dL}{L})$, which are then bivariately lognormally distributed, to have joint support equal to $\mathbb{R}^2_+$.

Proof. (b) ⇒ (a) is obvious. (a) ⇒ (c): Given $j \in I$ with the said property, define $\xi = \frac{M}{B^j}$, where M is the unique (right-continuous) $\mathbb{P}$-martingale such that $M_t = \mathbb{E}(\frac{d\mathbb{P}^j}{d\mathbb{P}} \mid \mathcal{F}_t)$ a.s. for all t. As is well known, N is a $\mathbb{P}^j$-martingale on on $[0,T]$ if and only if NM is a $\mathbb{P}$-martingale on $[0,T]$. Applied to $N = \frac{B^i}{B^j}$, it follows $\xi B^i = NM$ is a $\mathbb{P}$-martingale on $[0,T]$ for all i.

(c) ⇒ (b). Suppose such ξ exists and let $j \in I$. On $[0,T]$, set $M := \frac{\xi B^j}{\mathbb{E}(\xi_0 B_0^j)}$. Let $\mathbb{P}^j$ be the probability measure with $\frac{d\mathbb{P}^j}{d\mathbb{P}} = M_T$. Then, $\frac{B^i}{B^j}$ is a $\mathbb{P}^j$-martingale on $[0,T]$ for any $i \in I$, because $M\frac{B^i}{B^j} = \xi B^i$ is a $\mathbb{P}$-martingale on $[0,T]$. □

We call a family $(B^i)_{i\in I}$ of positive functions on $\mathbb{R}_+ \times \Omega$ an ***arbitrage-free price system*** if for every $T > 0$ it satisfies any, hence all, of the above three equivalent conditions.

While the "prices" B^i need not be measurable, their ratios are always semimartingales:

Proposition 4.2. *Let $(B^i)_{i\in I}$ be an arbitrage-free price system. Then $\frac{B^i}{B^j}$ are positive semimartingales, each with positive left limits, for all $i, j \in I$.*

Proof. Let $T > 0$. On $[0,T]$, since $\frac{B^i}{B^j}$ is a $\mathbb{P}^j$-martingale, it has positive left limits, and it is a $\mathbb{P}$-semimartingale on $[0,T]$. This proves the desired result since T was arbitrary. □

4.2. **Forward Libor and swap rate processes.** *Henceforth, throughout the paper, we fix an arbitrage-free price system* (B^1, B^2, B^3). We define

$$L := \frac{B^2}{B^3} - 1;$$

$$S := \frac{B^1 - B^3}{B^2 + B^3};$$

$$L^* := \frac{B^1}{B^2} - 1.$$

The processes L, S, and L^* are semimartingales by Proposition 4.2. Given an integer $n \geq 1$, if we think of B^i as the zero-coupon maturing in year $i+n-1$ (although we do not require $B^i_{i+n-1} = 1$), then L, S, and L^* respectively represent the annual $(n+1)\times 1$ forward Libor rate , the $n \times 2$ forward swap rate, and the $n \times 1$ forward-Libor rate processes.

One easily verifies that $S = \frac{L^*+L+L^*L}{2+L}$, and by inversion that $L^* = \frac{(2+L)S-L}{1+L}$.[3]

The first equation shows that S is positive if L and L^* are so. But, the second shows L^* becomes negative whenever $S < \frac{L}{2+L}$, even though L and S may be positive.

In what follows, we fix a time interval $[0,T]$ and an equivalent measures $\mathbb{P}^3$ under which $\frac{B^1}{B^3}$ and $\frac{B^2}{B^3}$ are martingales on $[0,T]$. In general, the choice of $\mathbb{P}^3$ depends on T, and even then there may be many such measures $\mathbb{P}^3$. But, in so far as the support is independent of

[3]Diving top and bottom of the defining formula of S by B^2,

$$S = \frac{B^1/B^2 - B^3/B^2}{1 + B^3/B^2} = \frac{1 + L^* - 1/(1+L)}{1 + 1/(1+L)} = \frac{L^* + L + L^*L}{2 + L}.$$

Similarly, dividing the top and bottom of the defining formula of S by B^3 yields $\frac{B^1}{B^3} = 1 + (2+L)S$.

the choice of equivalent measure and T is arbitrary, the results to be derived for $\mathcal{S}(L_t, S_t)$ and $\mathcal{S}(L_t^*)$ will hold under $\mathbb{P}$ for all times $t > 0$, regardless of the choice of measure $\mathbb{P}^3$ and the interval $[0, T]$. So, for convenience, we will suppress the notational dependence on T.

Proposition 4.3. *The processes* L, $S + \int \frac{d[L,S]}{2+L_-}$, *and* $L^* + \int \frac{d[L,L^*]}{1+L_-}$ *are* $\mathbb{P}^3$*-local martingales.*

Proof. As $L = \frac{B^2}{B^3} - 1$ and $(2+L)S = \frac{B^1}{B^3} - 1$ are $\mathbb{P}^3$-martingales, and by Itô's product rule,

$$d\left((2+L)S\right) = (2+L_-)dS + S_-dL + d[L,S],$$

it follows that $S + \int \frac{d[L,S]}{2+L_-}$ is a $\mathbb{P}^3$-local martingale. As $\frac{B^1}{B^3} = (1+L^*)(1+L)$ is a $\mathbb{P}^3$-local martingale, so is $L^*(1+L) - \int L^*_- dL = \int (1+L)_- dL^* + [L^*, L]$; hence so is $L^* + \int \frac{d[L,L^*]}{1+L_-}$. $\square$

4.3. **Libor and Swap market model SDE.** Henceforth we assume $dL = L\sigma dW$ and $dS = \mu dt + S(\lambda dW + \gamma dZ)$ for some $\mathbb{P}^3$-independent Brownian motions W and Z and optional processes σ, λ and γ with $\int_0^T (\sigma_t^2 + \lambda_t^2 + \gamma_t^2)dt < \infty$ a.s. for all T. We must have $\mu = -\frac{LS\sigma\lambda}{2+L}$ because $S + \int \frac{d[L,S]}{2+L}$ is a $\mathbb{P}^3$-local martingale by Proposition 4.3, while $d[L,S] = LS\sigma\lambda dt$. It follows that (L,S) satisfies the SDE system of Section 3 with $f(t,l) = -\frac{\sigma_t\lambda_t l}{2+l}$:

$$dL = L\sigma dW,$$
$$dS = S(-\frac{\sigma\lambda L}{2+L}dt + \lambda dW + \gamma dZ).$$

Assuming L^* follows $dL^* = \mu^* dt + L^*(\lambda^* dW + \gamma^* dZ)$, we similarly get $\mu^* = -\frac{LL^*\sigma\lambda^*}{1+L^*}$ since $L^* + \int \frac{d[L,L^*]}{1+L}$ is a $\mathbb{P}^3$-local martingale. This yields a similar SDE for L^* with $f(t,l) = -\frac{\sigma_t\lambda_t^* l}{1+l}$,

4.4. **Joint support of forward Libor and swap rates.** We call the system (B^1, B^2, B^3) above a ***lognormal SMM*** (resp. ***lognormal LMM***) if σ is positive, deterministic and continuous, λ and γ (resp. λ^* and γ^*) are deterministic, and $a := \frac{\lambda}{\sigma}$ (resp. $a^* := \frac{\lambda^*}{\sigma}$) is C^1.[4] Theorem 3.2 is applicable to both models, and yields the same joint support formula in both cases due to the fact that $\inf_{l>0} \frac{l}{2+l} = \inf_{l>0} \frac{l}{1+l} = 0$ and $\sup_{l>0} \frac{l}{2+l} = \sup_{l>0} \frac{l}{1+l} = 1$.

Theorem 4.4. *In the lognormal SMM,* $\mathcal{S}(L_t, S_t)$ *is given for any* $t > 0$ *as follows.*

(*a*) $\mathcal{S}(L_t, S_t) = \mathbb{R}_+^2$ *if* γ *is not identically zero a.e. on* $[0,t]$.

(*b*) $\mathcal{S}(L_t, S_t) = \mathbb{R}_+^2$ *if* $a := \frac{\lambda}{\sigma}$ *is not a constant on* $[0,t]$.

(*c*) *If* $\gamma = 0$ *on* $[0,t]$ *and* $a := \frac{\lambda}{\sigma}$ *is a constant on* $[0,t]$, *then, for* $a > 0$, *we have*

$$\mathcal{S}(L_t, S_t) = \{(l,y) \in \mathbb{R}_+^2 : e^{-\frac{1}{2}a(1+a)\int_0^t \sigma_s^2 ds} \le \frac{L_0^a y}{S_0 l^a} \le e^{\frac{1}{2}a(1-a)\int_0^t \sigma_s^2 ds}\}, \qquad (a > 0) \tag{4.1}$$

and for $a = 0$, *we have* $\mathcal{S}(L_t, S_t) = \mathbb{R}_+ \times \{S_0\}$, *and for* $a < 0$, *we have*

$$\mathcal{S}(L_t, S_t) = \{(l,y) \in (0,\infty)^2 : e^{\frac{1}{2}a(1-a)\int_0^t \sigma_s^2 ds} \le \frac{L_0^a y}{S_0 l^a} \le e^{-\frac{1}{2}a(1+a)\int_0^t \sigma_s^2 ds}\}. \qquad (a < 0) \tag{4.2}$$

[4]The assumptions that σ is continuous and a is C^1 are likely superfluous: $\sigma\lambda$ is already locally integrable by Schwartz inequality, which should be enough. See also the remark after Cor. 2.9 and footnote 2.

Moreover, in the lognormal LMM, $\mathcal{S}(L_t, L_t^*)$ is given by replacing all occurrences the symbols S_t, S_0, a, λ, γ in the statements above by L_t^*, L_0^*, a^*, λ^*, γ^* respectively.

Proof. Since L is prolific by Prop. 2.3, and in view of the SDE derived for (L, S) in Section 4.3, the results of Section 3 are applicable with the function $f(t, l) := -\frac{\sigma_t \lambda_t l}{2+l}$. Parts (a) and (b) thus follow immediately from Prop. 3.4 and Theorem 3.2 respectively. As for part (c), if $a > 0$ then $\inf_{l>0} f(t, l) = -a\sigma_t^2$ and $\sup_{l<0} f(t, l) = 0$, while if $a < 0$ then $\inf_{l>0} f(t, l) = 0$ and $\sup_{l<0} f(t, l) = -a\sigma_t^2$. As such, (c) is immediate from Theorem 3.2. The case of LMM follows by applying the same argument to the function $f(t, l) = -\frac{\sigma_t \lambda_t^* l}{1+l}$. □

5. The support of forward Libor in the lognormal swap market model

5.1. **The threshold.** We now apply Theorem 4.4 to the lognormal swap market model to analyze the support of forward Libor L^* and find out the first time t^* after which L^* may become negative. In most cases we find $\mathbb{P}\{L_t^* < 0\} > 0$ for all $t > 0$. The most interesting case is when $0 < a \leq 1$ is a constant, with the case $a = 1$ particularly tractable.

Proposition 5.1. *Given $t > 0$, in the lognormal SMM, $L_t^* > 0$ a.s. if and only if $\gamma = 0$ on $[0, t]$, $a := \frac{\lambda}{\sigma}$ is constant on $[0, t]$, $0 \leq a \leq 1$, and $e^{\frac{1}{2}a(1+a)\int_0^t \sigma_s^2 ds} \leq S_0 L_0^{-a} a^*$, where $a^* := (\frac{2}{a})^a (1-a)^{a-1}$ if $0 < a < 1$, $a^* := 2$ if $a = 1$, and $a^* := 1$ if $a = 0$.*

Proof. Since $L^* = \frac{(2+L)S-L}{1+L}$, we have, for any $t \geq 0$,

$$\{L_t^* < 0\} = \{S_t < \frac{L_t}{2 + L_t}\}.$$

It follows $\mathbb{P}\{L_t^* < 0\} > 0$ if and only if $\mathcal{S}(L_t, S_t) \cap \{(l, y) \in \mathbb{R}_+^2 : y < \frac{l}{2+l}\}$ is nonempty.

This holds in particular if $\mathcal{S}(L_t, S_t) = \mathbb{R}_+^2$, which is the case by Theorem 4.4 if either γ is not identically zero, or a is not a constant on $[0, t]$.

We are thus reduced to the case of $\gamma = 0$ and constant a, which we henceforth assume.

Case $a = 0$. In this case, $\mathcal{S}(L_t, S_t) = \mathbb{R}_+ \times \{S_0\}$. Since the function $0 < l \mapsto \frac{l}{2+l}$ is increasing with asymptote 1, it follows that $\mathbb{P}\{L_t^* < 0\} > 0$ if and only if $S_0 < 1$, which is equivalent to the statement in theorem for this case (as $a^* := 1$).

Case $a < 0$. In this case, Eq. (4.2) of Theorem 4.4 is applicable. We have $\mathcal{S}(L_t, S_t) = \{(l, y) \in (0, \infty)^2 : c_t l^a \leq y \leq C_t l^a\}$ for some positive constants c_t and C_t. Since $a < 0$, this set is the region between two decreasing graphs that approach 0 (the l-axis) for large l. Therefore, it clearly intersects $\{(l, y) \in \mathbb{R}_+^2 : y < \frac{l}{2+l}\}$. Hence, in this case, $\mathbb{P}\{L_t^* < 0\} > 0$.

Case $a > 1$. In this case, Eq. (4.1) of Theorem 4.4 is applicable. We have $\mathcal{S}(L_t, S_t) = \{(l, y) \in \mathbb{R}_+^2 : c_t l^a \leq y \leq C_t l^a\}$ for some positive constants c_t and C_t. Since $a > 1$, the curve $l \mapsto Cl^a$ has slope 0 at $l = 0$ for any $C > 0$. But, the curve $l \mapsto \frac{l}{2+l}$ has slope $\frac{1}{2}$ at $l = 0$. This implies that the graph the former lies strictly below the graph of the latter for sufficiently small $l > 0$. Hence, the intersection of the joint support with $\{y < \frac{l}{2+l}\}$ is nonempty, implying that in this case $\mathbb{P}\{L_t^* < 0\} > 0$.

Case $a = 1$. In this case Eq (4.1) simplifies to

$$\mathcal{S}(L_t, S_t) = \{(l, y) \in \mathbb{R}^2_+ : e^{-\int_0^t \sigma_s^2 ds} l \leq \frac{L_0}{S_0} y \leq l\}.$$

We note that the intersection of this set with $\{y < \frac{l}{2+l}\}$ is nonempty if and only if the slope of the linear map $l \mapsto \frac{S_0}{L_0} e^{-\int_0^t \sigma_s^2 ds} l$ is less than $\frac{1}{2}$ (which is the slope of $l \mapsto \frac{l}{2+l}$ at $l = 0$). Therefore, $\mathbb{P}\{L_t^* < 0\} > 0$ if and only if $\frac{S_0}{L_0} e^{-\int_0^t \sigma_s^2 ds} < \frac{1}{2}$, which is equivalent to the statement in the theorem in this special case (as $a^* := 2$).

Case $0 < a < 1$. In this case, let us rewrite Eq. (4.1) as

$$\mathcal{S}(L_t, S_t) = \{(l, y) \in \mathbb{R}^2_+ : c_t l^a \leq y \leq C_t l^a\},$$

where

$$c_t := \frac{S_0}{L_0^a} e^{-\frac{1}{2}a(1+a)\int_0^t \sigma_s^2 ds}; \qquad C_t := \frac{S_0}{L_0^a} e^{\frac{1}{2}a(1-a)\int_0^t \sigma_s^2 ds}.$$

Clearly, this set intersects $\{y < \frac{l}{2+l}\}$ if and only if the graph of the (lower) curve $l \mapsto c_t l^a$ (which has slope ∞ at $l = 0$) intersects $\{y < \frac{l}{2+l}\}$. Moreover, since $0 < a < 1$, this will happen if and only if graph of the curve $l \mapsto c_t l^a$ intersects the graph of the curve $l \mapsto \frac{l}{2+l}$ in precisely two points. We conclude that $\mathbb{P}\{L_t^* < 0\} > 0$ if and only if the equation $(2 + l)l^{a-1} = 1/c_t$ has precisely two solutions $l > 0$. But, by taking derivative and setting it to zero, it is easy to see that the convex function $0 < l \mapsto (2 + l)l^{a-1}$ (which is near ∞ at near $l = 0$ and $l = \infty$) has a unique global minimum at $l = \frac{2}{a} - 2$, with the value of $a^* := (\frac{2}{a})^a(1-a)^{a-1}$. Therefore, $\mathbb{P}\{L_t^* < 0\} > 0$ if and only if $1/c_t > a^*$, which is equivalent to the statement of the theorem for this case.

All the cases have been covered, and the proof is complete. ☐

Thus $\mathbb{P}\{L_t^* < 0\} > 0$ if any of the stated conditions fails, e.g., if $a^* S_0 \leq L_0^a$ or if $\gamma \neq 0$.

Consider the case where $\gamma = 0$, σ and λ are constants, $0 < a \leq 1$, and $S_0 L_0^{-a} a^* > 1$. Then, the result implies that $\mathbb{P}\{L_t^* < 0\} > 0$ if and only if $t > \frac{2}{a(a+1)\sigma^2} \log(S_0 L_0^{-a} a^*)$.

The special case $a = 1$, $\gamma = 0$ gives $\mathbb{P}\{L_t^* < 0\} > 0$ if and only if $\int_0^t \sigma_s^2 ds > \log \frac{2S_0}{L_0}$. (In this case, it is also easy to see that $\{L_t^* < 0\} \subset \{L_t < \frac{L_0}{S_0} e^{\int_0^t \sigma_s^2 ds} - 2\}$ for all $t \geq 0$).

The special case $a = \gamma = 0$ gives $\mathbb{P}\{L_t^* < 0\} > 0$ if and only if $S_0 < 1$.

The significance of the number $a^* := (\frac{2}{a})^a(1-a)^{a-1}$, $0 < a < 1$, is that it is the minimum of the function $0 < l \mapsto (2 + l)l^{a-1}$. We have $1 < a^* \leq 3$. This a^* is a concave function of a, its limits at $a = 0$ is 1 and at $a = 1$ is 2, and it reaches the maximum of 3 at $a = \frac{2}{3}$.

Corollary 5.2. *In the lognormal SMM,* $L_t^* > 0$ *a.s. for all* $t > 0$ *if and only if* $\gamma = 0$, a *is constant,* $0 \leq a \leq 1$, *and* $\exp(\frac{1}{2}a(1+a)\int_0^\infty \sigma_s^2 ds) \leq S_0 L_0^{-a} a^*$. ☐

5.2. **Support of L_t^*.** We conclude by showing $\mathcal{S}(L_t^*) = [l_t^*, \infty)$ in the lognormal SMM (unless S is deterministic) for some $-1 \le l_t^* \le 0$, and classify the cases $l_t^* = -1$ or $l_t^* = 0$.

Theorem 5.3. *In the lognormal SMM, for any $t > 0$, we have $\mathcal{S}(L_t^*) = [-1, \infty)$ if γ is not identically zero or $a := \frac{\lambda}{\sigma}$ is not constant on $[0, t]$, and otherwise we have:*

(*a*) *If $a = 0$, then $\mathcal{S}(L_t^*) = [S_0 - 1, 2S_0]$.*
(*b*) *If $a < 0$, then $\mathcal{S}(L_t^*) = [-1, \infty)$.*
(*c*) *If $a > 0$, then $\mathcal{S}(L_t^*) = [l_t^*, \infty)$, where*

$$l_t^* := \inf_{x>0} \frac{S_0 L_0^{-a} e^{-\frac{1}{2}a(1+a)\int_0^t \sigma_s^2 ds}(2+x)x^a - x}{1+x}. \qquad (a > 0)$$

(*d*) *If $a > 1$, then $-1 < l_t^* < 0$.*
(*e*) *Assume $0 < a \le 1$. Set $a^* := (\frac{2}{a})^a (1-a)^{a-1}$ if $a < 1$ and $a^* := 2$ if $a = 1$. If $L_0^a e^{\frac{1}{2}a(1+a)\int_0^t \sigma_s^2 ds} \le a^* S_0$, then $l_t^* = 0$, that is, $\mathcal{S}(L_t^*) = \mathbb{R}_+$; otherwise we have $-1 < l_t^* < 0$.*
(*f*) *Assume $a = 1$. Then $\mathcal{S}(L_t^*) = \mathbb{R}_+$ if $L_0 e^{\int_0^t \sigma_s^2 ds} \le 2S_0$, and otherwise*

$$\mathcal{S}(L_t^*) = \Big[\frac{2 - 2\frac{S_0}{L_0} e^{-\int_0^t \sigma_s^2 ds}}{\sqrt{\frac{L_0}{S_0} e^{\int_0^t \sigma_s^2 ds} - 1}} - 1,\ \infty\Big). \qquad (a = 1,\ L_0 e^{\int_0^t \sigma_s^2 ds} > 2S_0).$$

Proof. Define the continuous function $f(x, y) := \frac{(2+x)y - x}{1+x}$ on $\mathbb{R}_+^2$. Then, $L^* = f(L, S)$, which implies $\mathcal{S}(L^*)$ is the closure of $f(\mathcal{S}(L, S))$. Note also, $f(x, y) \ge f(x, 0) = \frac{-x}{1+x} > -1$, which easily implies $f(\mathbb{R}_+^2) = (-1, \infty)$. Hence, $\mathcal{S}(L_t^*) = [-1, \infty)$ when $\mathcal{S}(L, S) = \mathbb{R}_+^2$, which is the case by Theorem 4.4 when γ is not identically zero or a is not constant on $[0, t]$. Assuming $\gamma \equiv 0$ and a is constant on $[0, t]$, it remains to prove (a)-(f).

(a). Assume $a = 0$. Then $S_t = S_0$; hence $L_t^* = f(L_t, S_0)$. But, it is easy to see that the function $0 \le x \mapsto f(x, S_0)$ is decreasing, implying its image is $(f(\infty, S_0), f(0, S_0)] = (S_0 - 1, 2S_0]$. Hence, $\mathcal{S}(L_t^*) = [S_0 - 1, 2S_0]$.

(b)-(f). Assume $a \ne 0$. Then, by Theorem 4.4, $\mathcal{S}(L_t, S_t)$ is connected, contains the graph of the function $0 < x \mapsto f(x, c_t x^a)$ for some $c_t > 0$, and the rest of $\mathcal{S}(L_t, S_t)$ lies above this graph. Hence, $\mathcal{S}(L^*)$ is connected, and since f is increasing in y, we have $-1 \le \inf_{\mathcal{S}(L_t, S_t)} f = \inf_{x>0} f(x, c_t x^a)$. Setting $l_t^* := \inf_{x>0} f(x, c_t x^a)$, it follows that $\mathcal{S}(L_1^t) = [l_t^*, \infty)$ once we show $\sup_{x>0} f(x, c_t x^a) = \infty$.

(b). Assume $a < 0$. Then, $-1 \le l_t^* \le \lim_{x\to\infty} f(x, c_t x^a) = \lim_{x\to\infty} \frac{-x}{1+x} = -1$. So, $l_t^* = -1$. Also, $\sup_{x>0} f(x, c_t x^a) \ge \lim_{x\downarrow 0} f(x, c_t x^a) = \lim_{x\downarrow 0} 2c_t x^a = \infty$. So, $\mathcal{S}(L_t^*) = [-1, \infty)$.

(c). Assume $a > 0$. Then, $\sup_{x>0} f(x, c_t x^a) \ge \lim_{x\to\infty} f(x, c_t x^a) = \lim_{x\to\infty} \frac{c_t x^{a+1}}{1+x} = \infty$. Hence, $\mathcal{S}(L_t^*) = [l_*, \infty)$. Also in this case by Theorem 4.4, $c_t = S_0 L_0^{-a} e^{-\frac{1}{2}a(1+a)\int_0^t \sigma_s^2 ds}$; hence the formula for l_t^* in (c) follows from the definition $l_t^* := \inf_{x>0} f(x, c_t x^a)$.

(d). Assume $a > 1$. Then by Proposition 5.1, $l_t^* < 0$. This together with part (c) imply that $l_t^* = f(x_*, c_t x_*^a)$ for some $x_* > 0$ because $f(x, c_t x^a)$ is near 0 for small x and is large for x large (so, the inf, which is $l_t^* < 0$, in attained at an $x_* > 0$). Hence, $l_t^* > -1$ as $f > -1$.

(e). Assume $0 < a \leq 1$. If $L_0^a\, e^{\frac{1}{2}a(1+a)\int_0^t \sigma_s^2 ds} \leq a^* S_0$, then $l_t^* = 0$ by Proposition 5.1. Otherwise, by Proposition 5.1, $l_t^* < 0$, and this implies that $l_t^* = f(x_*, c_t x_*^a)$ for some $x_* > 0$ (as $f(x, c_t x^a)$ is near 0 for small x and is large for x large), implying $l_t^* > -1$ as $f > -1$.

(f). Assume $a = 1$. By (e), if $L_0\, e^{\int_0^t \sigma_s^2 ds} \leq 2S_0$, then $l_t^* = 0$, as desired, and if otherwise, then $l_t^* = f(x_*, c_t x_*)$ for some $x_* > 0$. But then, since x_* minimizes $f(x, c_t x)$, and

$$\frac{d}{dx} f(x, c_t x) = \frac{(2c_t + 2c_t x - 1)(1+x) - (2c_t x + c_t x^2 - x)}{(1+x)^2} = \frac{2c_t - 1 + 2c_t x + c_t x^2}{(1+x)^2},$$

we must have $2 - \frac{1}{c_t} + 2x_* + x_*^2 = 0$. The positive solution is $x_* = -1 + \sqrt{\frac{1}{c_t} - 1}$. Therefore,

$$l_t^* = \frac{c_t(2x_* + x_*^2) - x_*}{1 + x_*} = \frac{1 - 2c_t - x_*}{1 + x_*} = \frac{2 - 2c_t}{\sqrt{\frac{1}{c_t} - 1}} - 1 = \frac{2 - 2\frac{S_0}{L_0} e^{-\int_0^t \sigma_s^2 ds}}{\sqrt{\frac{L_0}{S_0} e^{\int_0^t \sigma_s^2 ds} - 1}} - 1.$$

(We used $c_t = \frac{S_0}{L_0} e^{-\int_0^t \sigma_s^2 ds}$ for $a = 1$. Note also $0 < c_t < \frac{1}{2}$ as $L_0 e^{\int_0^t \sigma_s^2 ds} > 2S_0$ here.) □

The theorem gives explicit expressions for $\mathcal{S}(L_t^*)$ in all cases except the case $a > 1$ and the case $0 < a < 1$ and $L_0^a\, e^{\frac{1}{2}a(1+a)\int_0^t \sigma_s^2 ds} > a^* S_0$. In these two cases, the theorem shows that $\mathcal{S}(L_t^*) = [l_t^*, \infty)$, for some $-1 < l_t^* < 0$. Part (c) yields that in both cases

$$l_t^* = \frac{S_0 L_0^{-a} e^{-\frac{1}{2}a(1+a)\int_0^t \sigma_s^2 ds}(2 + x_*)x_*^a - x_*}{1 + x_*},$$

for some $x_* > 0$ that (setting the derivative to zero) is easily seen to satisfy the equation

$$2a x_*^{a-1} + (3a - 1)x_*^a + a x_*^{a+1} = \frac{L_0^a}{S_0} e^{\frac{1}{2}a(1+a)\int_0^t \sigma_s^2 ds}.$$

For $a = 1$, we found the single positive solution x_* of this equation in part (f). Evidently, one also gets a closed-form solution for $a = 2$, $a = 3$, and $a = \frac{1}{2}$, but not for other $a > 0$. It is not difficult to show that for $a > 1$ the positive solution x_* is unique, while for $0 < a < 1$ and $L_0^a\, e^{\frac{1}{2}a(1+a)\int_0^t \sigma_s^2 ds} > a^* S_0$, there are precisely two positive solutions, one larger than $\frac{2(1-a)}{1+a}$ and the other smaller, with the valid solution (a minimum) the larger one.

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