American and Bermudan Options, Snell envelope, Minimax Duality

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January 2006, version 1.1

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1 American options, Snell envelope, Minimax duality

The main reference for this presentation is a July 2004 version of a paper by the author entitled:

Numeraire-invariant option pricing & american, bermudan, and trigger stream rollover.

I thank many participants at the Symposium on Optimal Stopping in Manchester University for helpful comments and constructive criticism.

Notation and a relevant result

We fix throughout a *finite horizon stochastic basis*

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), \ \ \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le m}, \ \ 0 < m < \infty.$$

The following won't be used, but its relevance will be apparent.

Theorem 1.1 *A right-continuous adapted process Z is of class-D if and only if there exists a right-continuous martingale M such that $|Z| \le M$.*

(The infinite horizon case also holds with M explicitly specified as uniformly integrable.)

This follows readily from the supermatingale property of the Snell envelope (as in Theorem 1.9 below) and its Doob-Meyer decomposition. If $|Z| \le M$, then Z is trivially of class D. The converse (for which we may assume $Z \ge 0$) follows for right-continuous class-D supermartingales by Doob-Meyer decomposition, and follows in general from this case combined with Theorem 1.9 below. Theorem 1.9 is evidently contained in the book of Dellacheri and Meyer (1977), but our proof approximates the Snell envelope by finite-period ones, and does not assume Z has left limits.

Financial terminology without financial context

In Part I, we use the following financially motivated terminology:

A ***Claim*** C is a right continuous martingale. We write $C \in \mathcal{C}$.

A ***Numeraire*** B is a positive claim. We write $B \in \mathcal{C}^+$.

A ***Payoff Process*** Z is a right-continuous, adapted, class-D process.

To add financial context, introduce an arbitrary positive process ξ (called state price density and embodying interest-rates and risk premia), and simply replace C, B, Z, V by their deflated values $\xi C, \xi B, \xi Z, \xi V$ in all formulae below. (But please simplify $\frac{\xi C}{\xi B}$ to $\frac{C}{B}$, etc.) Part II and III contain a proper financial context. Part I corresponds to the special case of Part II with $\xi = 1$.

Change of numeraire formula

A simple but useful formula related to choice of numeraire is

Proposition 1.2 *Let* Z *be a payoff process,* B *be a numeraire, and* $T \leq m$ *be a stopping time. Then*

$$\mathbb{E}\, Z_T = \mathbb{E}(B_m \frac{Z_T}{B_T}). \tag{1}$$

Proof. Set $\mathcal{F}_T := \{\Lambda \in \mathcal{F} : \Lambda \cap \{T \leq t\} \in \mathcal{F}_t \ \forall t\}$. Then

$$\mathbb{E}(B_m \frac{Z_T}{B_T}) = \mathbb{E}(\mathbb{E}(B_m \frac{Z_T}{B_T} \,|\, \mathcal{F}_T)) \qquad \text{(iterating expectation)}$$

$$= \mathbb{E}(\frac{Z_T}{B_T} \mathbb{E}(B_m \,|\, \mathcal{F}_T)) \qquad \text{(since } \frac{Z_T}{B_T} \text{ is } \mathcal{F}_T\text{-measurable)}$$

$$= \mathbb{E}(\frac{Z_T}{B_T} B_T) = \mathbb{E}\, Z_T. \qquad \text{(by optional sampling theorem)}$$

Domineering claims and numeraires

We say a claim C ***Domineers*** a payoff process Z if

$$\sup_{0\leq t\leq m}(Z_t - C_t) = 0.$$

Obviously a numeraire B domineers Z if and only if

$$\sup_{0\leq t\leq m}\frac{Z_t}{B_t} = 1.$$

Proposition 1.3 *A claim C domineers a payoff process Z if and only if*

$$C \geq Z \text{ and } C_0 = \sup_{T\leq m}\mathbb{E}(Z_T).$$

The supremum is taken over all stopping times $0 \leq T \leq m$.

Proof of Prop. 1.3 and Linear Programming

Proof. Assume $C \geq Z$ and $0 = \sup_{T \leq m} \mathbb{E}(Z_T) - C_0$. Then, using the optional sampling theorem,

$$0 = \sup_{T \leq m} \mathbb{E}(Z_T - C_T) \leq \mathbb{E} \sup_{0 \leq t \leq m} (Z_t - C_t).$$

Since $Z \leq C$, this implies $\sup_{0 \leq t \leq m}(Z_t - C_t) = 0$ a.s.

Conversely, assume $\sup_{0 \leq t \leq m}(Z_t - C_t) = 0$. Then $Z \leq C$, so $\mathbb{E}\, Z_T \leq \mathbb{E}\, C_T = C_0$. Hence, $\sup_{T \geq 0} \mathbb{E}\, Z_T \leq C_0$. To prove the reverse inequality, it suffices to show that for any $\varepsilon > 0$ there is a stopping time $T \leq m$ such that $\mathbb{E}(Z_T) \geq C_0 - \varepsilon$. Let $\varepsilon > 0$. The assumption implies that (pathwise) the set $\{t \in [0, m] : Z_t - C_t \geq -\varepsilon\}$ is nonempty a.s. Define the stopping time $T := \{t \in [0, m] : Z_t - C_t \geq -\varepsilon\}$. Since $Z - C$ is right-continuous, we must have $Z_T - C_T \geq -\varepsilon$. Therefore, $\mathbb{E}\, Z_T \geq \mathbb{E}\, C_T - \varepsilon = C_0 - \varepsilon$, as desired. $\square$

Domineering claims are the solutions of a *linear programming* problem in the space of all claims:

Linear Programming. *Let Z be a payoff process. Then a claim C domineers Z if and only if C is a solution to the following L.P. problem: Minimize C_0 subject to $C \geq Z$.*

Proof. The “only if” part follows because if $C \geq Z$, then, as above, $C_0 \geq \sup_T \mathbb{E}(Z_T)$. The “if” part is a direct consequence of the existence of domineering claims next. $\square$

Existence of domineering claims

One of our main results, derived below, is

Theorem 1.4 *Let Z be a payoff process. Then there exists a claim C that domineers Z.*

There are usually an infinite number of domineering claims. (An exception is when Z is a subclaim (submartingale) - then there is only one.) Theorem 1.10 below gives an explicit construction of practically all of them.

Basically, any numeraire β gives rise to a domineering claim C by the Doob-Meyer decomposition of the Snell envelope V associated to Z, where the decomposition is taken under the numeraire measure $\mathbb{P}^{\beta}$ defined by $\frac{d\mathbb{P}^{\beta}}{d\mathbb{P}} = \frac{\beta_m}{\beta_0}$. They admit an explicit representation in terms of β and Doob-Meyer decomposition of V under $\mathbb{P}$.

Additive minimax duality

The following formula is due to Rogers (2001) and in the Bermudan case independently to Haugh and Kogan (2001).

Additive minimax duality. *Let Z be a payoff process. Then*

$$\sup_{T\le m}\mathbb{E}(Z_T)=\min_{C\in\mathcal{C}}(C_0+\mathbb{E}\sup_{0\le t\le m}(Z_t-C_t)).$$

The minimum is attained at any claim C that domineers Z.

Proof. For any claim C and stopping time $T\le m$,

$$\mathbb{E}\,Z_T=C_0+\mathbb{E}(Z_T-C_T)\le C_0+\mathbb{E}\sup_{0\le t\le m}(Z_t-C_t).$$

Taking sup over T, we see “$\le$" holds. Let C be a claim that domineers Z, which exists by Theorem 1.4. By Prop. 1.3, $\sup_{T\le m}\mathbb{E}\,Z_T=C_0=C_0+\mathbb{E}\sup_{0\le t\le m}(Z_t-C_t)$, using the definition of a domineering claim. Equality and the second statement thus follow. ☐

Multiplicative minimax duality.

Multiplicative minimax duality. *Let* Z *be a payoff process such that* $Z_m > 0$ *a.s. Then*

$$\sup_{T\le m} \mathbb{E}(Z_T) = \min_{B\in\mathcal{C}^+} \mathbb{E}(B_m \sup_{0\le t\le m} \frac{Z_t}{B_t}).$$

The minimum is attained at any numeraire B *that domineers* Z.

Proof. For any numeraire B, and stopping time $T \le m$, we have using Proposition 1,

$$\mathbb{E}(Z_T) = \mathbb{E}(B_m \frac{Z_T}{B_T}) \le \mathbb{E}(B_m \sup_{0\le t\le m} \frac{Z_t}{B_t}).$$

Taking sup over T, we see “$\le$" holds. Let B be a claim that domineers Z, which exists by Theorem 1.4. Then B is a numeraire as $B_m \ge Z_m > 0$. Prop. 1.3 and the definition imply $\sup_{T\le m} \mathbb{E}\, Z_T = B_0 = \mathbb{E}(B_m \sup_{0\le t\le m} \frac{Z_t}{B_t})$. Equality and the second statement follow. $\square$

So, when $Z_m > 0$, *both* minimax duality formulae hold.

Nonnegative multiplicative minimax duality

Let Z be a payoff process such that $Z_m \geq 0$ a.s. Then, applying the above to the payoff process $Z + \varepsilon\beta$, where β is any numeraire, and letting $\varepsilon \to 0$, we easily deduce

$$\sup_{T \leq m} \mathbb{E}(Z_T) = \inf_{B \in \mathcal{C}^+} \mathbb{E}(B_m \sup_{0 \leq t \leq m} \frac{Z_t}{B_t}).$$

The difference is that we have “inf” instead of “min”. Unless $Z_m > 0$, the infimum is generally not attained at any numeraire (e.g., $Z = (\beta - 1)^+$ for some numeraire β). However, the infimum *is* attained over a larger set of “*semipositive claims*” in a suitable sense, as in Part II.

Optimal stopping times

A stopping time $T^* \leq m$ is ***Optimal*** for a payoff process Z if

$$\mathbb{E}\, Z_{T^*} = \sup_{T \leq m} \mathbb{E}\, Z_T .$$

Proposition 1.5 *A stopping time T^* is optimal for Z if and only if $Z_{T^*} = C_{T^*}$ for some, hence all, claims C that domineer Z.*

Proof. Assume T^* is optimal for Z. By Theorem 1.4, there exist a claim C that domineers Z. By the optional sampling theorem, followed by Prop. 1.3, next optimality of T^*,

$$\mathbb{E}\, C_{T^*} = C_0 = \sup_{T \leq m} \mathbb{E}(Z_T) = \mathbb{E}\, Z_{T^*} .$$

Since $Z \leq C$, this implies $Z_{T^*} = C_{T^*}$. Conversely, if C is any claim that domineers Z and satisfies $Z_{T^*} = C_{T^*}$, then $\mathbb{E}\, Z_{T^*} = \mathbb{E}\, C_{T^*} = C_0 = \sup_{T \leq m} \mathbb{E}(Z_T)$. ☐

This combined with the optional sampling theorem implies

Corollary 1.6 *Let T^* be optimal for Z and $T \leq T^*$. Then $C_T = C'_T$ for any two domineering claims C and C'.*

Existence of optimal stopping times

Prop. 1.5 and the right-continuity of Z easily imply

Proposition 1.7 *A payoff process Z has an optimal stopping time if and only if for some, hence all, domineering claims C, the set $\{0 \leq t \leq m : Z_t(\omega) = C_t(\omega)\}$ is nonempty for almost all ω. In this case,*

$$T_* := \inf\{0 \leq t \leq m : Z_t = C_t\}$$

is an optimal stopping time too. Moreover, T_* does not depend on C and $T_* \leq T^*$ for all optimal stopping times T^*.

It is also the case that if an optimal stopping time exists, then $T_* = \inf\{0 \leq t \leq m : Z_t = V_t\}$, where V is the Snell envelope of Z, defined below. We show that

Theorem 1.8 *A continuous payoff process has an optimal stopping time.* (*So, T_* is optimal.*)

Conjecture and uniqueness of optimal stopping

We conjecture that the stopping time T_* is optimal if and only if Z has no negative jumps on the accessible part of T_*.

The classical example with no stopping time is an American call option on a stock that pays a single dividend before option expiration. In this case, Z has a negative jump at a fixed time. This example clearly generalizes to when the dividend is paid at a predictable time.

Usually, if an optimal stopping time exists, it is unique. An exception is when Z is a numeraire - then any stopping time is optimal by the optional sampling theorem.

If Z is a subclaim (submartingale), then $T = m$ is an optimal stopping time, and it is the only one if Z is not a numeraire (e.g., an American call on a zero-dividend stock).

The Snell envelope

The ***Snell Envelope*** process of a payoff process Z is defined by

$$V_t := \text{ess.} \sup_{t \le T \le m} \mathbb{E}(Z_T \,|\, \mathcal{F}_t).$$

A ***Superclaim*** is a right-continuous class-D supermartingale.

Theorem 1.9 *Let Z be a payoff process. Then its Snell envelope V is a superclaim. Moreover, for all $s \le t$,*

$$\mathbb{E}(V_s \,|\, \mathcal{F}_t) = \text{ess.} \sup_{s \le T \le m} \mathbb{E}(Z_T \,|\, \mathcal{F}_t). \tag{2}$$

Furthermore, V is the smallest superclaim satisfying $V \ge Z$.

Proof outline. Eq. (2) easily implies $\mathbb{E}(V_s \,|\, \mathcal{F}_t) \le V_t$, as well as class-D property V using that of Z. We show (2) by reducing to the Bermudan case and convergence arguments. Specifically, when the supremum in V is taken over stopping times with finite number of values, we prove (2) using induction. We then show the corresponding (bermudan) snell envelopes converge to V, using uniform integrability properties. Right continuity is also shown by convergence arguments. The last statement now is an easy consequence of optional sampling theorem. □

Invariant Doob-Meyer decomposition

Theorem 1.10 *Let V be a superclaim and β be a numeraire. Then there exist a unique claim C with $C_0 = V_0$ and an increasing predictable process A with $A_0 = 0$ such that $V = C - \beta A$. Moreover,*

$$A = -\int \frac{dV^p}{\beta_-},$$

$$C = V^m + \int A_- d\beta + [A, \beta],$$

where $V = V^p + V^m$, $V_0^p = 0$ is the Doob-Meyer decomposition of V, i.e., V^p is the (*decreasing*) *compensator of V.*

Theorem 1.4 follows immediately from this and Theorem 1.9 .

Indeed, the C above domineers Z since $C \geq V \geq Z$ and $C_0 = V_0 := \sup_{T \leq m} \mathbb{E}(Z_T)$.

Proof of Theorem 1.10 and remarks

Proof. Define the numeraire measure $\mathbb{P}^\beta$ by $\frac{d\mathbb{P}^\beta}{d\mathbb{P}} = \frac{\beta_m}{\beta_0}$. One can easily show $\frac{V}{\beta}$ is a $\mathbb{P}^\beta$ class-D supermartingale. The desired decomposition $V = C - \beta A$ follows by applying the classical Doob-Meyer decomposition to $\frac{V}{\beta}$ in the measure $\mathbb{P}^\beta$. As for the formula, set $A' = -\int \frac{dV^p}{\beta_-}$ and $C' = V^m + \int A'_- d\beta + [A', \beta]$. Clearly, A' predictable and increasing and C' is a local martingale. Simple Itö calculus yields $\frac{V}{\beta} = \frac{C'}{\beta} - A'$. By the uniqueness canonical decomposition under $\mathbb{P}^\beta$, we conclude $C' = C$ and $A' = A$. $\square$

Remark. The Snell envelope V equals the essential infimum of all claims that domineer its payoff process Z.

Remark. Even when Z is continuous, the Snell envelope V may have jumps. One can construct an American option on a stock with continuous price but a volatility that jumps at a totally inaccessible time T, causing V to jump at T.

Multiplicative Doob-Meyer decomposition

For a positive payoff process Z, there is a special domineering (local) numeraire B. It is obtained by decomposing its (positive) Snell envelope V uniquely as $V = \frac{B}{A}$, where A is increasing, predictable, $A_0 = 1$, and B is a local numeraire. This B has an intuitive financial interpretation, especially in the Bermudan case, as a "*rollover trading strategy*" (see Part III).

The following formula appears new.

Lemma 1.11 *Let X and Y be semimartingales with $1 + \Delta X \neq 0$ everywhere. Then*

$$\mathcal{E}(X+Y) = e^{-[X^c, Y^c]} \mathcal{E}(X) \mathcal{E}(Y - \sum_{s \leq \cdot} \frac{\Delta X_s \Delta Y_s}{1 + \Delta X_s}),$$

where X^c is the continuous local martingale part of X.

Proof. Set $Z := Y - \sum_{s\leq \cdot} \frac{\Delta X_s \Delta Y_s}{1+\Delta X_s}$ and use $\mathcal{E}(X)\mathcal{E}(Z) = \mathcal{E}(X + Z + [X, Z])$ and simplify the bracket.

Multiplicative Doob-Meyer decomposition $\cdots$

Corollary 1.12 *Let X be a special semimartingale with canonical decomposition $X = X^p + X^m$, $X_0^p = 0$. (So, X^p is the compensator of X.) Assume $1 + \Delta X \neq 0$ everywhere. Then $\mathcal{E}(X)$ has the multiplicative decomposition*

$$\mathcal{E}(X) = \mathcal{E}(X^p)\mathcal{E}(X^m - \sum_{s\leq\cdot} \frac{\Delta X_s^p \Delta X_s^m}{1 + \Delta X_s^p}).$$

Proof. Apply the lemma with X equal X^p and $Y = X^m$, and simplify. $\square$

Note that $\sum_{s\leq\cdot} \frac{\Delta X_s^p \Delta X_s^m}{1+\Delta X_s^p}$ is a local martingale, because it equals the bracket of local martingale X^m with predictable finite variation process $\sum_{s\leq\cdot} \frac{\Delta X_s^p}{1+\Delta X_s^p}$. (Of course, the sum is absolutely convergent). $\square$

Multiplicative Doob-Meyer decomposition $\cdots$

The next formula (to our knowledge new) furnishes the local multiplicative Doob-Meyer decomposition of a special semimartingale.

Theorem 1.13 *Let Y be a special semimartingale such that $Y_0 \neq 0$ and $Y_- \neq 0$ everywhere. Then,*

$$Y = Y_0 \mathcal{E}(\int \frac{dY^p}{Y_-}) \mathcal{E}(\int \frac{dY^m}{Y_-} - \sum_{s \leq \cdot} \frac{\Delta Y^p_s \Delta Y^m_s}{Y^p_s + Y^m_{s-}}),$$

where $Y = Y_0 + Y^p + Y^m$ is the canonical decomposition of Y.

Proof. Set $X = \int \frac{dY}{Y_-}$, so that $Y = Y_0 \mathcal{E}(X)$. Since, $Y \neq 0$ everywhere, $1 + \Delta X \neq 0$ everywhere. Therefore Corollary 1.12 is applicable. Finally use $X^p = \int \frac{dY^p}{Y_-}$, $X^m = \int \frac{dY^m}{Y_-}$, $\Delta X^p = \frac{\Delta Y^p}{Y_-}$, and simplify the sum. $\square$

The second stochastic exponential is always a local martingale. It is of interest to know when it is an actual martingale. We show this is so when Y is continuous and $\mathbb{E} \exp(\int_0^m \frac{d[Y_t]}{2Y_t^2}) < \infty$.

Hedging interpretation and option rollover

In order to hedge the sale of an American option with payoff process Z, the seller can buy any domineering claim C, assuming such a claim is available in the market place or can be replicated by available instruments (e.g., assuming complete markets). Then, if the buyer exercises optimally at an optimal exercise time T^*, then the option payout Z_{T^*} that the seller pays equals C_{T^*} by Prop. 1.5. So, the seller sells B and with the proceeds pays the buyer. If the buyer exercises suboptimally before an optimal stopping time T^*, then the payout Z_{T^*} will be less than C_{T^*} by Prop 1.7; so the seller liquidates and pockets the difference. If the buyer does not exercise at an optimal stopping time T^*, then, the seller sells C at T^* and buys a new domineering claim hedge at an infinitesimally lower price, and continually repeats until the buyer exercises, at which time he has pocketed a continuous stream of cash as profit. (This last case is more evident in the Bermudan case.)

When Z is positive, the financial interpretation of the domineering numeraire B arising from the *multiplicative* Doob-Meyer decomposition is as follows. At time 0 buy $A_0 = 1$ unit of the American option with payoff process Z. Later, when exercise becomes optimal at some time t, do exercise, and with the proceeds $Z_t = V_t$ buy A_t units of the American option with payout Z exercise dates between $t + dt$ and m. As the unit price of this American option is infinitesimally less than V_t, the number of shares A_t increases infinitesimally. This self-financing trading strategy describes B. Continuing in the way, we see $B = AV$ is the multiplicative decomposition, as A is increasing. This is an instance of *option stream rollover* of Part III.

2 An invariant option pricing framework

A conceptual definition of an option

An *Option* is a pair $\mathcal{O} = (T, O)$ consisting of an *Expiry* T and a *Payoff* O (in base currency) paid at T. The expiry T is a stopping time (bounded by $m > 0$), and the payoff O is a random variable known at time T, i.e., measurable w.r.t. to σ-algebra $\mathcal{F}_T$ of events at or before T:

$$\mathcal{F}_T := \{\Lambda \in \mathcal{F} : \Lambda \cap \{T \leq t\} \in \mathcal{F}_t\}.$$

- For a European option, the expiry T is deterministic.
- American and Bermudan options: T is the optimal exercise time.
- Barrier options: T is the first passage time to the barrier.
- Credit derivatives with recovery : T is the default time.

A conceptual definition of an option $\cdots$

- The definition so far depends only on the filtration $(\mathcal{F}_t)_{t=0}^m$. To talk about the option price, a probability measure $\mathbb{P}$ and an integrability condition are required.

- Think of a *Numeraire* β as a claim which pays no dividend and has a positive price $\beta_t > 0$ at all times $t \leq m$, e.g., a zero-dividend stock, or the m-maturity zero-coupon bond.

- To each numeraire β, there is associated a *Numeraire Measure* $\mathbb{P}^\beta$, characterized by the property that if B is any other numeraire, then the relative price process (B_t/β_t) is a (right-continuous) $\mathbb{P}^\beta$-martingale.

- The required integrability condition on option $\mathcal{O} = (T, O)$ is this: O/β_T is $\mathbb{P}^\beta$-integrable for some numeraire β.

Numeraire invariance of the option definition

The important aspect of this integrability condition is that if it holds for some numeraire, then it holds for *all* numeraires:

$$\frac{B_0}{\beta_0}\mathbb{E}^B[\frac{O}{B_T}] = \mathbb{E}^\beta[\frac{O}{B_T}\frac{B_m}{\beta_m}] \qquad \text{(change of numeraire)}$$

$$= \mathbb{E}^\beta[\mathbb{E}^\beta[\frac{O}{B_T}\frac{B_m}{\beta_m}\,|\,\mathcal{F}_T]] \qquad \text{(iterating expectation)}$$

$$= \mathbb{E}^\beta[\frac{O}{B_T}\mathbb{E}^\beta[\frac{B_m}{\beta_m}\,|\,\mathcal{F}_T]] \qquad (\text{by } \mathcal{F}_T \text{ measurability of } \frac{O}{B_T})$$

$$= \mathbb{E}^\beta[\frac{O}{B_T}\frac{B_T}{\beta_T}] \qquad \text{(optional sampling theorem)}$$

$$= \mathbb{E}^\beta[O/\beta_T] < \infty.$$

Price process $\mathcal{O}_t$ of an option

Consider investing the option payoff O at expiry T in a numeraire β and holding this position until the terminal date m. We end up with a claim that pays $\beta_m O/\beta_T$ at time m.
At a time and state before expiry (i.e., $t < T(\omega)$), the option has not yet been invested in the numeraire, so this claim is identical with the option itself. We are thus forced to define the option price $\mathcal{O}_t$ to be the price of this claim, i.e., $\beta_t \mathbb{E}^\beta[O/\beta_T \,|\, \mathcal{F}_t]$.

After expiry, the option has ceased to exist and has no price. In this case it is convenient to define the option price to be zero. We thus arrive at the following definition of an option price:

$$\mathcal{O}_t := 1_{t \leq T} \beta_t \mathbb{E}^\beta[\frac{O}{\beta_T} \,|\, \mathcal{F}_t].$$

This definition is independent of choice of numeraire β.

Price Transitivity Law

Using the optional sampling theorem, the option price at any stopping time τ (bounded by m) is given by

$$\mathcal{O}_\tau = 1_{\tau \le T} \beta_\tau \mathbb{E}^\beta [\frac{O}{\beta_T} \mid \mathcal{F}_\tau] \ \text{ a.s.}$$

The pair $(\tau, \mathcal{O}_\tau)$ is a τ-expiry option with payoff $\mathcal{O}_\tau$. Let S be another stopping time. What can we say about the time S price $(\tau, \mathcal{O}_\tau)_S$ of this option? When $S \le \tau \le T$, we simply have

$$(\tau, \mathcal{O}_\tau)_S = \mathcal{O}_S.$$

That is, pricing to time τ and then pricing to time S is the same as pricing directly to time S.

Indistinguishable options

Two options $\mathcal{O}$ and $\mathcal{O}'$ *Indistinguishable* if the price processes $(\mathcal{O}_t)$ and $(\mathcal{O}'_t)$ are indistinguishable, i.e., a.s. $\mathcal{O}_t = \mathcal{O}'_t$ all t.

Theorem: Options $\mathcal{O} = (T, O)$ and $\mathcal{O}' = (T', O)$ are indistinguishable if and only if $O = O'$ a.s. and $\{T \neq T'\} \subset \{O = 0\}$ a.s.

So, T and T' need not be the same, but they can differ only at zero payments.

Notation: We write $\mathcal{O} \leq \mathcal{O}'$ if a.s. $\mathcal{O}_t \leq \mathcal{O}'_t$ all t.

Nonnegative options and nonnegative arbitrage

An option $\mathcal{O} = (T, O)$ is *Nonnegative* if $O \geq 0$ a.s. It is *Positive* if $O > 0$ a.s.

(In practice, most options are nonnegative, but are not positive either, for they can have zero payoff in some states (i.e., $\mathbb{P}[O = 0] > 0$) , e.g., call and put options and swaptions.)

Nonnegative Arbitrage. *Once the price of a nonnegative option becomes zero, it stays zero thereafter*, i.e., for almost all paths ω,

- if $\mathcal{O}_t(\omega) = 0$, then $\mathcal{O}_s(\omega) = 0$ for all $s \geq t$.
- if $O(\omega) > 0$, then $\mathcal{O}_t(\omega) > 0$ for all $t \leq T(\omega)$.

Semipositive options

A nonnegative option $\mathcal{O}$ is *Semipositive* if its price is positive before expiry T, i.e., at each time t, $\mathcal{O}_t > 0$ a.s. on $\{t < T\}$.

This then implies a.s. $\mathcal{O}_t > 0$ on $\{t < T\}$ all t, i.e., for almost all paths ω, $\mathcal{O}_t(\omega) > 0$ at all times $t < T(\omega)$.

A nonnegative option is indistinguishable from a semipositive option : Let $\mathcal{O} = (T, O)$ be a nonnegative option. Then, there exists a unique semipositive option that is indistinguishable from $\mathcal{O}$, namely the option (T^0, O), where

$$T^0 := \inf\{t > 0 : \mathcal{O}_t = 0\}.$$

(It turns out that O is automatically $\mathcal{F}_{T^0}$-measurable - in fact, $O = \mathcal{O}_{T^0}$ a.s.)

Payoff processes and trigger options

A progressively measurable process $Z = (Z_t)$ is a *Payoff Process* if a.s. $|Z_t| \leq \beta_t$ all t for some numeraire β.
Example: $Z_t = (\beta_t - K)^+, \ K > 0.$

Proposition: Let Z be a payoff process. Then for any numeraire β, the β-deflated process (Z_t/β_t) is of $\mathbb{P}^\beta$-class D. In particular, for any stopping time T, the pair (T, Z_T) is an option.

We call the T-expiry option (T, Z_T) as a Z-*Trigger Option*.

Examples of trigger options: American and Bermudan options, barrier options, credit derivatives with recovery process.

Dominated and trigger option convergence

Dominated option convergence: Let $(\mathcal{O}^n)_{n=1}^{\infty}$, $\mathcal{O}^n = (T_n, O^n)$ by a sequence of options such that $T^n \searrow T$ a.s., $O^n \to O$ a.s. to some random variales T and O. Assume there exists a numeraire β such that $|\mathcal{O}_t^n| \leq \beta_t$ a.s. for all t and n. Then, $\mathcal{O} = (T, O)$ is an option, and a.s. $\mathcal{O}_t^n \to \mathcal{O}_t$, all t.

When the dominating numeraire β can be chosen continuous, then the condition $T^n \searrow T$ can be weakened to $T^n \to T$.

Trigger option convergence: Let (Z_t) be a right continuous payoff process and $T_n \searrow T$. The theorem implies that a.s. $(T_n, Z_{T_n})_t \to (T, Z_T)_t$, all t.

Doob-Meyer decomposition of superclaims

A *Superclaim* is a right-continuous payoff process (V_t) such that $V_t \geq (s, V_s)_t$ a.s. for all $t \leq s$. A *Supernumeraire* is a positive superclaim. As we saw, the Snell envelope is a superclaim.

Proposition: A right-continuous payoff process (V_t) is a superclaim if and only if the process (V_t/β_t) is a right-continuous $\mathbb{P}^{\beta}$-supermartingale for some (*hence all*) numeraire β.

Doob-Meyer decomposition: Let (V_t) be a supernumeraire and β be a numeraire. Then there exist a unique numeraire B with $B_0 = V_0$ and a decreasing predictable process (A_t) such that a.s. $V_t = \beta_t A_t + B_t$ all t.

The Snell envelope

The *Snell Envelope* process of a right continuous payoff process $Z = (Z_t)$ is defined by

$$V_t := \sup_{T \geq t} (T, Z_T)_t,$$

where supremum is taken over the set all stopping times T satisfying $t \leq T \leq m$.

Theorem. (V_t) is a right continuous payoff process, and

$$(s, V_s)_t = \sup_{T \geq s} (T, Z_T)_t. \quad (t \leq s)$$

Corollary. (V_t) is a superclaim. Indeed, for $t \leq s$,

$$V_t := \sup_{T \geq t} (T, Z_T)_t \geq \sup_{T \geq s} (T, Z_T)_t = (s, V_s)_t.$$

The american option in the continuous case

Simple examples show that the supremum $\sup_{T\geq 0}(T, Z_T)_0$ in definition of Snell envelope is not necessarily attained at any stopping time T. In continuous case we show the supremum *is* attained (and suspect this to be the case when jumps of Z are totally inaccessible). Set

$$T_t^* := \inf\{s \in [t,m] : Z_s = V_s\}.$$

Theorem. Let Z be a continuous payoff process that is dominated by a continuous numeraire. Then $Z_{T_t^*} = V_{T_t^*}$ and

$$V_t = (T_t^*, Z_{T_t^*})_t.$$

Further for all times t and s, we have

$$(T_s^*, Z_{T_s^*})_t = 1_{t\leq T_s} \sup_{T\geq \max(t,s)} (T, Z_T)_t.$$

In particular, $\mathcal{A}_t = 1_{t\leq T_0^*} V_t$, all t, where $\mathcal{A} := (T_0^*, Z_{T_0^*})$.

Multiplicative minimax duality

Let (Z_t) be a *positive* right-continuous payoff process. The Snell envelope (V_t) is then a supernumeraire, and Doob-Meyer decomposition implies there are many *Domineering Numeraire*, i.e., numeraires B such that $B_0 = V_0$ and $B_t \geq V_t$ for all t.

Proposition: Let B be a domineering numeraire. Then

$$\sup_{t \geq 0} \left(\frac{Z_t}{B_t}\right) = 1.$$

Multiplicative minimax duality formula: a.s., all t,

$$V_t = \inf_{\beta \in \mathcal{C}^+} \beta_t \mathbb{E}^{\beta}[\sup_{s \geq t} (\frac{Z_s}{\beta_s}) \,|\, \mathcal{F}_t].$$

Multiplicative minimax duality $\cdots$

- In particular, $V_0 = \inf_\beta \beta_0 \mathbb{E}^\beta[\sup_{t\geq 0}(Z_t/\beta_t)]$.
- The infimum is attained at any domineering numeraire.
- So, $\beta_0 \mathbb{E}^\beta[\sup_{t\geq 0}(Z_t/\beta_t)]$ is an *upper bound* for the American option price V_0, for any numeraire β.
- Such an upper bound can be computed by Monte-Carlo simulation. An suitable numeraire must first be chosen.
- These results extend to nonnegative payoff processes (Z_t).
- An “additive version” of minimax duality was previously derived by Rogers (2001) and Haugh & Kogan (2001), and further studied in Andersen & Broadie (2001), Joshi & Theis (2002), and Kolodko & Schoenmakers (2003).

Multiplicative Doob-Meyer decomposition

A *Local Numeraire* is an adapted, right continuous, positive process (B_t) such that (B_t/β_t) is a $\mathbb{P}^\beta$-local martingale for some (*hence all*) numeraire β.

Theorem: Let (V_t) be a supernumeraire. Then there exists a unique decomposition $V_t = D_t B_t$, where (D_t) is a decreasing predictable process and (B_t) is a local numeraire with $B_0 = V_0$.

A supernumeraire (V_t) *Multiplicative*, if the local numeraire (B_t) in the multiplicative decomposition $V_t = D_t B_t$ is actually the price process of a numeraire B. We then refer to this numeraire B as the *Rollover Numeraire* associated to (V_t). It is clearly a domineering numeraire, and as such relevant to minimax duality.

An application of multiplicative decomposition

Corollary: Assume that discount factors satisfy $(s,1)_t \leq 1$ for all $t \leq s$. Then there exists a unique increasing, predictable, local numeraire (B_t) with $B_0 = 1$.

The assumption is equivalent to the identically one process being a supernumeraire. Applying the multiplicative Doob-Meyer decomposition, we obtain a decomposition $1 = D_t B_t$. The resulting local numeraire (B_t) is increasing, as it equals $(1/D_t)$.

This increasing local numeraire (B_t) can be interpreted as the "*continuous money market account*". If it is further assumed to be absolutely continuous, then, taking its logarithmic derivative, we obtain the "*instantaneous interest-rate process*" r_t, which is nonnegative and satisfies $B_t = \exp(\int_0^t r_s ds)$.

3 Bermudan options, stream rollover, minimax duality

Option streams

A *(Finite) Option Stream* (or simply a *Stream*) is a finite sequence $\mathcal{O}^{\cdot} = (\mathcal{O}^n)_{n=1}^k$ of options $\mathcal{O}^n = (T_n, O^n)$ with increasing expiries $T_1 \le \cdots \le T_k$. Stream $\mathcal{O}^{\cdot}$ is called:

- *Positive* if $O^n > 0$ a.s., all n. Ditto, *Nonnegative*.
- *Decreasing* if $\mathcal{O}^{n+1}_{T_n} \le O^n$ a.s., all n, or equivalently, if a.s., $1_{t \le T_n} O^{n+1}_t \le O^n_t$ all t, n. It also equivalent to the condition that for some (hence all) numeraire β, the discrete deflated envelope process $(O^n/\beta_{T_n})_{n=1}^k$ is a supermartingale under the finite filtration $(\mathcal{F}_{T_n})_{n=1}^k$.
- a *Trigger Stream* if $O^n(\omega) = O^{n+1}(\omega)$ whenever $T_n(\omega) = T_{n+1}(\omega)$. (Equivalently, $1_{T_i = T_j} O^i = 1_{T_i = T_j} O^j$, all i, j).

Bermudan option stream

Henceforth, we take as given a payoff process Z.

e.g., $Z_t = (K - S_t)^+$.

Let $\tau_\cdot = (\tau_1, \cdots, \tau_k)$ be a *Tenor*, i.e., sequence of increasing stopping times $\tau_1 \leq \cdots \leq \tau_k$. Define the trigger stream

$$\mathcal{E}^\cdot := (\mathcal{E}^n)_{n=1}^k, \quad \mathcal{E}^n := (\tau_n, Z_{\tau_n}).$$

We view $\mathcal{E}^\cdot$ as "*the underlying european option stream*."

(In practice, the expiries τ_n are deterministic, such as annually or daily.)

Define the *Associated Bermudan Stream* $\mathcal{B}^\cdot := (\mathcal{B}^n)_{n=1}^k$ by $\mathcal{B}^n := (T_n^*, Z_{T_n^*})$, where the *Optimal Exercise Tenor* $T_\cdot^* := (T_1^*, \cdots, T_k^*)$ is defined by either of the following **two equivalent definitions**.

The optimal exercise tenor

Backward inductive definition: Set $T_k^* := \tau_k$. For $n < k$, define inductively (setting $\mathcal{B}^{n+1} := (T_{n+1}, Z_{T_{n+1}})$),

$$T_n^* := 1_{Z_{\tau_n} \geq \mathcal{B}^{n+1}_{\tau_n}} \; \tau_n + 1_{Z_{\tau_n} < \mathcal{B}^{n+1}_{\tau_n}} \; T_{n+1}^*.$$

Snell envelope definition: Define

$$T_n^* := \min\{t \in \{\tau_n, \cdots, \tau_k\} : Z_{\tau_n} = \sup_{T \in \{\tau_n, \cdots, \tau_k\}} (T, Z_T)_{\tau_n}\}.$$

The well-known equivalence is easily proved by induction.

Recall, the bermudan stream is defined as

$$\mathcal{B}^{\cdot} := (\mathcal{B}^n)_{n=1}^k, \;\; \mathcal{B}^n := (T_n^*, Z_{T_n^*}).$$

Bermudan pricing formula and consquences

Bermudan option pricing formula:

$$\mathcal{B}_t^n = 1_{t \leq T_n^*} \sup_{t \leq T \in \{\tau_n, \cdots, \tau_k\}} (T, Z_T)_t.$$

Corollaries:

Time-0 Pricing: $\mathcal{B}_0^n = \sup_{T \in \{\tau_n, \cdots, \tau_k\}} (T, Z_T)_0.$

Snell envelope formula: $\mathcal{B}_{\tau_n}^n = \sup_{T \in \{\tau_n, \cdots, \tau_k\}} (T, Z_T)_{\tau_n}.$

Principle of Dynamic Programming: $\mathcal{B}_{\tau_n}^n = \max(Z_{\tau_n}, \mathcal{B}_{\tau_n}^{n+1}).$

Bermudan stream $\mathcal{B}^{\cdot}$ is decreasing: $\mathcal{B}_{T_n^*}^n \geq \mathcal{B}_{T_n^*}^{n+1}.$

Rolling an option over a positive option

Let $\mathcal{O} = (T, O)$ be an option and $\mathcal{O}' = (T', O')$ be a *positive* option with $T \leq T'$. Define *Rollover Option* $\mathcal{O} \Rsh \mathcal{O}'$ by

$$\mathcal{O} \Rsh \mathcal{O}' := (T', \frac{OO'}{\mathcal{O}'_T}).$$

We interpret $\mathcal{O} \Rsh \mathcal{O}'$ as the option obtained by investing at time T the payoff O in option $\mathcal{O}'$.

The rollover operator $\Rsh$ is **associative** on *positive* options:

$$(\mathcal{O}^1 \Rsh \mathcal{O}^2) \Rsh \mathcal{O}^3 = \mathcal{O}^1 \Rsh (\mathcal{O}^2 \Rsh \mathcal{O}^3).$$

We denote this by $\mathcal{O}^1 \Rsh \mathcal{O}^2 \Rsh \mathcal{O}^3$, etc.

Rolling an option over a nonnegative option

Let $\mathcal{O} = (T, O)$ be an option and $\mathcal{O}' = (T', O')$ be a *nonnegative* option with $T \leq T'$. Define *Rollover Option* $\mathcal{O} \Rsh \mathcal{O}'$ by

$$\mathcal{O} \Rsh \mathcal{O}' := (1_{\mathcal{O}'_T=0} T + 1_{\mathcal{O}'_T>0} T', 1_{\mathcal{O}'_T=0} O + \frac{O}{\mathcal{O}'_T} O').$$

Pricing is given by

$$(\mathcal{O} \Rsh \mathcal{O}')_t = \mathcal{O}_t + 1_{t>T} \frac{O}{\mathcal{O}'_T} \mathcal{O}'_t.$$

The rollover operator is *not* associative in general. A few properties of $\Rsh$ are :

$$(\mathcal{O} \Rsh \mathcal{O}')_S = \mathcal{O}_S, \quad S \leq T.$$

$$\mathcal{O} \Rsh \mathcal{O} = \mathcal{O}; \quad \mathcal{O} \Rsh (\mathcal{O} \Rsh \mathcal{O}') = \mathcal{O} \Rsh \mathcal{O}', \text{ etc.}$$

$$\mathcal{O} \Rsh (\mathcal{O}' + \mathcal{O}'') \leq \mathcal{O} \Rsh \mathcal{O}' + (\mathcal{O} \Rsh \mathcal{O}'').$$

$$\lim_{\varepsilon \to 0} (\mathcal{O} \Rsh (\mathcal{O}' + \varepsilon \mathcal{O}''))_t = (\mathcal{E} \Rsh \mathcal{O}')_t. \quad \text{(by dominated option convergence)}$$

Minimax duality of nonnegative payoff processes

Henceforth, we assume $Z \geq 0$ (i.e., a.s. $Z_t \geq 0$, all t.)
Let $\mathbf{O}^+$ (resp. $\mathbf{O}^{++}$) denote the set of all nonnegative (resp. positive) options (T, O) with $T \geq \tau_k$.

***Multiplicative minimax duality formula*:**

$$\mathcal{B}_0^1 = \inf_{\mathcal{O}\in\mathbf{O}^{++}} \Big(\max_{n=1,\cdots,k} (\mathcal{E}^n \Rsh \mathcal{O})\Big)_0$$

$$= \min_{\mathcal{O}\in\mathbf{O}^{+}} \Big(\max_{n=1,\cdots,k} (\mathcal{E}^n \Rsh \mathcal{O})\Big)_0.$$

$$\mathcal{B}_0^1 = \min_{\mathcal{O}\in\mathbf{O}^{++}} \Big(\max_{n=1,\cdots,k} (\mathcal{E}^n \Rsh \mathcal{O})\Big)_0 \ \text{ if } Z > 0.$$

(The “$\leq$” part is easy.) More generally, for any $1 \leq i \leq k$ and stopping time $\tau \leq \tau_i$,

$$\mathcal{B}_\tau^i = \inf_{\mathcal{O}\in\mathbf{O}^{++}} \Big(\max_{n=i,\cdots,k} (\mathcal{E}^n \Rsh \mathcal{O})\Big)_\tau = \min_{\mathcal{O}\in\mathbf{O}^{+}} \Big(\max_{n=i,\cdots,k} (\mathcal{E}^n \Rsh \mathcal{O})\Big)_\tau.$$

Domineering options and minimax duality

A *Domineering* option is a nonnegative option $\mathcal{O}$ such that $\mathcal{O}_{\tau_n} \geq Z_{\tau_n}$ and $\mathcal{O}_0 = \mathcal{B}_0^1$.

A nonnegative option $\mathcal{O}$ is domineering if and only if

$$\mathcal{O} = \max_{n=1,\cdots,k} (\mathcal{E}^n \Rsh \mathcal{O}).$$

Corollary. A domineering option $\mathcal{O}$ satisfies

$$\mathcal{B}_0^1 = (\max_{n=1,\cdots,k} (\mathcal{E}^n \Rsh \mathcal{O}))_0.$$

So, in the minimax formula $\mathcal{B}_0^1 = \min_{\mathcal{O} \in \mathbf{O}^+} (\max_{n=1,\cdots,k} (\mathcal{E}^n \Rsh \mathcal{O}))_0$, the minimum is attained at *every* domineering option $\mathcal{O}$.

Our next goal is to construct a domineering option denoted $\mathcal{B}^{\cdot \Rsh}$.

Rollover of nonnegative option stream

Rollover option $\mathcal{O}^{\cdot\Rsh}$ of a *positive* stream $\mathcal{O}^{\cdot} = (\mathcal{O}^n)_{n=1}^k$ is

$$\mathcal{O}^{\cdot\Rsh} := \mathcal{O}^1 \Rsh \cdots \Rsh \mathcal{O}^k = (T_k, O^k \prod_{n=1}^{k-1} \frac{O^n}{\mathcal{O}^{n+1}_{T_n}}),$$

For a nonnegative option streams we define the *Left and Right Associative Rollover Options* respectively by

$$\mathcal{O}^{\Rsh\cdot} = (\cdots((\mathcal{O}^1 \Rsh \mathcal{O}^2) \Rsh \mathcal{O}^3) \cdots \Rsh \mathcal{O}^{k-1}) \Rsh \mathcal{O}^k.$$

$$\mathcal{O}^{\cdot\Rsh} = \mathcal{O}^1 \Rsh (\mathcal{O}^2 \Rsh (\mathcal{O}^3 \cdots \Rsh (\mathcal{O}^{k-1} \Rsh \mathcal{O}^k) \cdots)).$$

Of course, for positive streams, $\mathcal{O}^{\Rsh\cdot} = \mathcal{O}^{\cdot\Rsh}$.

Stream rollover associativity and pricing formula

We call a nonnegative stream $\mathcal{O}^{\cdot}$ is *Associative* if $\mathcal{O}^{\Rsh\cdot} = \mathcal{O}^{\cdot\Rsh}$ and this holds for all “substreams” of $\mathcal{O}^{\cdot}$ too.

Proposition. Decreasing and increasing streams are associative. Semipositive trigger streams are associative.

***Rollover pricing formula*:** For a nonnegative stream, a.s. all t,

$$\mathcal{O}_t^{\cdot\Rsh} = \mathcal{O}_t^1 + \sum_{n=2}^{k} 1_{T_{n-1}<t} \prod_{i=1}^{n-1} \frac{\mathcal{O}_{T_i}^i}{\mathcal{O}_{T_i}^{i+1}} \mathcal{O}_t^n .$$

(A similar but more complex formula can be written for $\mathcal{O}_t^{\Rsh\,\cdot}$.)

Some consequences of rollover pricing formula

$$\mathcal{O}_0^{\cdot\Rsh} = \mathcal{O}_0^{\Rsh\cdot} = \mathcal{O}_0^1.$$

Denote $\mathcal{O}^{\cdot\Rsh}$ also by $\mathcal{O}^{\cdot\Rsh k}$. Ditto for $n \leq k$. Then

$$\mathcal{O}_S^{\cdot\Rsh} = \mathcal{O}_S^{\cdot\Rsh n}, \qquad S \leq T_n.$$

(Ditto $\mathcal{O}_S^{\Rsh\cdot}$). For positive streams we have,

$$\mathcal{O}_{T_n}^{\cdot\Rsh} = O^n \prod_{j=1}^{n-1} \frac{O^j}{\mathcal{O}_{T_j}^{j+1}};$$

$$\mathcal{O}^n \Rsh \mathcal{O}^{\cdot\Rsh} = \mathcal{O}^n \Rsh \cdots \Rsh \mathcal{O}^k.$$

Proposition. If $\mathcal{O}^{\cdot}$ is a decreasing nonnegative stream then $\mathcal{O}^k \leq \mathcal{O}^{\cdot\Rsh}$, and $\mathcal{O}^n \Rsh \mathcal{O}^{\cdot\Rsh} \leq \mathcal{O}^{\cdot\Rsh}$ for all n.

The bermudan rollover option $\mathcal{B}^{\cdot\Rsh}$

Since $\mathcal{B}^{\cdot}$ is decreasing, $\mathcal{B}^{\cdot\Rsh}_{\tau_n} \geq (\mathcal{B}^n \Rsh \mathcal{B}^{\cdot\Rsh})_{\tau_n} = \mathcal{B}^n_{\tau_n} \geq Z_{\tau_n}$.
As $\mathcal{B}^{\cdot\Rsh}_0 = \mathcal{B}^1_0$, **it follows $\mathcal{B}^{\cdot\Rsh}$ is a domineering option**. Hence,

$$\mathcal{B}^{\cdot\Rsh} = \max_{n=1,\cdots,k}(\mathcal{E}^n \Rsh \mathcal{B}^{\cdot\Rsh});$$

and

$$\mathcal{B}^1_0 = (\max_{n=1,\cdots,k}(\mathcal{E}^n \Rsh \mathcal{B}^{\cdot\Rsh}))_0.$$

The minimax duality formula $\mathcal{B}^1_0 = \min_{\mathcal{O}\in\mathbf{O}^+}(\max_{n=1,\cdots,k}(\mathcal{E}^n \Rsh \mathcal{O}))_0$ now easily follows.

The bermudan stream $\mathcal{B}^{\cdot}$ (is associative and) satisfies the “stream rollover strategy” formula:

$$\mathcal{B}^{\cdot\Rsh} = (\tau_1, \mathcal{B}^1_{\tau_1}) \Rsh \cdots \Rsh (\tau_k, \mathcal{B}^1_{\tau_k}).$$

Regenerative tenors and streams

A tenor $T^{\cdot} = (T_n)_{n=1}^{k}$ *is Regenerative at* at tenor $\tau^{\cdot} = (\tau_n)_{n=1}^{k}$ if $\tau_k = T_k$, and for $n < k$, $\tau_n \leq T_n$ and $1_{\tau_n < T_n} T_n = 1_{\tau_n < T_n} T_{n+1}$.

Last condition can also be written as $T_n = 1_{\tau_n = T_n} \tau_n + 1_{\tau_n < T_n} T_{n+1}$, or, if preferred,

$1_{\tau_n < T_n} = 1_{\tau_n < T_n = T_{n+1}}$.

Then $T_n \in \{\tau_n, \cdots, \tau_k\}$, all n. By induction one shows

$$1_{t > \tau_n} (T_n, Z_n)_t = \sum_{i=n+1}^{k-1} 1_{\tau_{i-1} < t \leq \tau_i \wedge T_n} (T_i, Z_i)_t .$$

Define the associated *Regenerative Stream* $Z^{\cdot}_{T_\cdot}$ by

$$Z^{\cdot}_{T_\cdot} := (Z^{n}_{T_\cdot})_{n=1}^{k}, \quad Z^{n}_{T_\cdot} := (T_n, Z_{T_n}).$$

Example: The optimal exercise tenor $T^{*}_{\cdot}$, and the bermudan stream $\mathcal{B}^{\cdot} = Z^{\cdot}_{T^{*}_{\cdot}}$ are regenerative.

Primal-dual bounds for regenerative streams

For any regenerative stream, we have ***primal-dual bounds:***

$$(T_1, Z_{T_1})_0 \le \mathcal{B}^1_0 \le (\max_{n=1,\cdots,k} (\mathcal{E}^n \Rsh Z^{\cdot\Rsh}_{T_\cdot}))_0.$$

Various examples of regenerative streams in the literature aim at a tight (additive) primal-dual bounds (or a tight “duality gap”).

A regenerative stream satisfies

$$Z^{\cdot\Rsh}_{T_\cdot} \le \max_{n=1,\cdots,k} (\mathcal{E}^n \Rsh Z^{\cdot\Rsh}_{T_\cdot}).$$

If $Z_{\tau_n} \le (Z^{\cdot\Rsh}_{T_\cdot})_{\tau_n}$ for all n, this implies $Z^{\cdot\Rsh}_{T_\cdot}$ is domineering, hence equality holds and $(T_1, Z_{T_1})_0 = \mathcal{B}^1_0 = (\max_{n=1,\cdots,k} (\mathcal{E}^n \Rsh Z^{\cdot\Rsh}_{T_\cdot}))_0$.

Stream rollover strategies

A *Stream Rollover Strategy* is a pair $(\tau_\cdot, \mathcal{O}^\cdot)$, where $\tau_\cdot$ is a tenor and $\mathcal{O}^\cdot$ is a stream such that its expiry tenor is regenerative at $\tau_\cdot$, and $1_{\tau_n < T_n} O^n = 1_{\tau_n < T_n} O^{n+1}$ for all $n < k$.

Examples are $(\tau_\cdot, Z_{T_\cdot})$, where $Z_{T_\cdot}$ is a regenerative trigger stream as above, e.g., $(\tau_\cdot, \mathcal{B}^\cdot)$

The “strategy” is to rollover $\mathcal{O}^\cdot$ at the tenor $\tau_\cdot$:

Theorem. Let $(\tau^\cdot, \mathcal{O}^\cdot)$ be a stream rollover strategy such that $\mathcal{O}^\cdot$ is nonnegative and associative. Then

$$\mathcal{O}^{\cdot\Rsh} = (\tau_1, \mathcal{O}^1_{\tau_1}) \Rsh \cdots \Rsh (\tau_k, \mathcal{O}^1_{\tau_k}).$$