Libor Market Model With Semimartingales

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Abstract

This paper extends the Libor market model to general semimartingales. Appealing simplifications occur for special semimartingales. The case where forward Libor rates are driven by a Brownian motion and a general integer-valued random measure is especially highlighted. Basically, from the existence of a state-price density, the drift of the forward Libor system is determined in terms of the other two characteristics, namely the covariation matrix of the continuous part and Levy measure of the system. Necessary and sufficient conditions are established, and formulae for forward Libor rates and deflated bond prices are derived in the actual, forward, and spot-Libor measures. Several other topics, such as continuous compounding as the limiting case and model construction, are discussed.

Key words: Semimartingale characteristics, random measure, stochastic exponential, forward Libor rates, state price density, forward and spot martingale measures.

1. Introduction

This paper extends to general semimartingales certain aspects of the *Libor market model* developed for continuous processes in Brace et. al. (1997), Miltersen et. al. (1997), Musiela and Rutkowski (1997), and Jamshidian (1997). Significant simplifications occur for special semimartingales. This is then applied to models driven by a Brownian motion and a general integer-valued random measure. We are primarily concerned with the dynamics of forward Libor rates as implied by the existence of a state price density. Specifically, we show the "drift" of the forward Libor rate system is determined by the other two characteristics, i.e., by the covariation matrix of continuous part, and the Levy system (measure) of the purely discontinuous part. The result is established through a statement about a general (or a special) semimartingale system, which may be of independent interest.

An extension of the Libor market model to diffusion-marked point processes was arrived at by Glasserman and Kou (1999). For continuously compounded interest rates – a more intricate theory is developed in Björk et. al. (1997a) for diffusion-multivariate point process, and in Björk et. al. (1997b) for general semimartingales. Dealing with trading strategies and stochastic integration against a continuum of bond prices, theirs is essentially an infinite dimensional theory, well suited to continuous compounding. But, for application to Libor rates, which are simple compounded and depend only on a finite number of bonds, such machinery is not necessary, nor necessarily the best approach. As other examples of literature dealing with jumps in continuously compounded term structure of interest rates, Shirakawa (1991) studies Gaussian diffusion and Poisson jumps, Duffie and Singelton (1996) analyse jumps arising from bond default, Babbs and Webber (1997) relate jumps to actions of central banks, Maghsoodi (1997) applies diffusion-marked point processes to exchange rates and two-country term structure, and Jarrow and Madan (1999) consider the question of hedging jumps.

While continuous compounding may ultimately lead to more unified theories, simple compounding has the of advantage of being grounded on standard finite-dimensional semimartingale theory, which is better understood and more developed. Additionally, it represents the interest-rate market more realistically. As such, it is arguably better suited for practical modelling of certain Libor and swap derivatives, such as caps, European and Bermudan swaptions. The Libor market model (and to somewhat lesser extent, the "swap market model") is receiving increasing theoretical attention, among which one may cite recent papers of Carr and Yang (1997), Andersen and Andreasen (1998), Glasserman, P., Zhao (1998), and Schoenmakers and Coffey (1999).

The paper closest to ours is Glasserman and Kou (1999), as it deals with both the Libor market model and diffusion-jump. In addition to determining the drift restriction for diffusion-marked point processes, they develop a concrete example of forward Libor rates following a mixed lognormal diffusion and lognormal jump arriving at Poisson intervals, and derive a formula for a cap, as an infinite sum of Black-Scholes terms, similar to that for stock options in Merton (1976). (For more on stock options with jumps, see, e.g., Gerber and Shiu (1996) and Bellamy and Jeanblanc (1997).) Otherwise (and aside from theirs being the first!), our paper differs in two ways. First, ours is more general, and also more comprehensive in its range of results.

Secondly, we believe our derivation is more lucid and efficient. While our treatment here is self-contained, Glasserman and Kou base their derivation on results of Björk, et. al. (1997a). But, as already mentioned, the latter's framework was adapted to continuous compounding and a continuum of bonds. As such, Glasserman and Kou go through rather lengthy calculations translating back and forth between simple and continuous compounding. It seems to us that this obscures the idea behind their formula. However, in our approach, the basic idea becomes apparent early on. A related difficulty is that, following Björk, et. al., they concentrate on the "risk-neutral measure" and its continuously compounded "money-market" numeraire. But, as amply argued in Jamshidian (1997), these are not suitable for studying forward Libor rates. Instead, we use an appropriate counterpart introduced there and termed "spot-Libor measure." (We also work in the "actual" and "forward" measures.)

In the next section we fix notation and review briefly some basic definitions and well-known results about semimartingales, including their Levy system and integer-valued random measures, which will be used throughout the paper.

Section 3 presents our main mathematical results. We have separated them from their application to Libor rates later in Sec. 4. We think this makes the exposition clearer. It is also a better organization for possible future reference. Let $X = (X_1,\ldots, X_m)$ be a positive semimartingale system with characteristics (B,C, ν). If $\prod_{j>i} X_j$ are local martingales for all i, then we show B can be explicitly calculated from C and ν. The formulae are particularly appealing in the case of special semimartingales, and are easily translated when X is driven by a Brownian motion and a general integer-valued random measure.

In Sec. 4, we provide necessary and sufficient conditions for the existence of a state-price density for a general price system, in terms of dynamics of the associated forward Libor system. Basically, the forward Libor drift is determined in terms of the other two characteristics. For special semimartingales, no truncation is necessary, and the formula simplifies nicely. Formulae for deflated bond prices in terms of forward Libor characteristics are also given. The case of diffusion-random measure is then highlighted. We provide formulae for three measures, the "actual", the "forward" (or "terminal"), and the "spot-Libor" measures. We try to minimize repetition by omitting substantially similar formulae. Yet cases with unique features are fully presented, at the risk of apparent repetition.

In Sec 5, we discuss several issues, including positivity of Libor rates, a heuristic derivation of an intriguing formula for continuous compounding as the limiting case, market prices of risk, and model construction.

In conclusion, we think this paper helps to demonstrate that simple compounding of interest entails basic and interesting mathematics that is overlooked in the continuously compounded framework. Further, as the former converges to the latter, it may be fruitful to investigate further the interplay between the two theories.

The body and statements in Sec. 4 are for the most part self-contained, only proofs referring to previous sections. So it should be possible to proceed directly to Sec. 4 from here.

2. Notation and Mathematical Background

This section introduces standard notation and briefly reviews some basic definitions and properties which will be used in the sequel. As reference, we refer the reader to Sections I.3, I.4, II.1, and II.2 of Jacod and Shiryaev (1987), and to various sections from Chapters 7, 10, 11, 12, 13, and 15 of Elliott (1982) and Chapters I and II of Protter (1990).

Let $T > 0$. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $0 \le t \le T$, $\mathcal{F} = \mathcal{F}_T$, be a complete, right-continuously filtered probability space, fixed throughout. The stochastic integral, as a process, will be denoted $\int H\,dX$. We use the convention that the values at $t = 0$ of $\int H\,dX$, $[X,Y]$, and ΔX are 0. Ito's product rule for semimartingales then states $XY = X_0Y_0 + \int X_-dY + \int Y_-dX + [X,Y]$. In what follows, an expression such as $\Sigma_{s\le t}(\Delta X)^2$ stands for the process $\Sigma_{s\le t}(\Delta X_s)^2$.

If Y is a semimartingale, its stochastic exponential $X = \mathcal{E}(Y)$ is defined as the unique solution to the equation $X = \exp(Y_0) + \int X_-dY$. Then $\Delta X = X_-\Delta Y$. Also, $X > 0$ iff $\Delta Y > -1$. Ito's product rule implies $\mathcal{E}(Y)\mathcal{E}(W) = \mathcal{E}(Y+W+[Y,W])$. The stochastic logarithm of a semimartingale $X > 0$ is defined by $\mathcal{L}(X) = \log(X_0) + \int dX / X_-$. The two operations are inverses of each other. By the product rule, $\mathcal{L}(XZ) = \mathcal{L}(X) + \mathcal{L}(Z) + [\mathcal{L}(X), \mathcal{L}(Z)]$. Also, $\Delta\mathcal{L}(X) = \Delta X / X_-$.

The set of all finite variation processes whose total variation is locally integrable is denoted by $\mathcal{A}_{\text{loc}}$. (Here, finite variation means adapted, corlol, and with paths of finite variation on $[0,T]$.) A finite variation local martingale belongs to $\mathcal{A}_{\text{loc}}$; so does a finite variation predictable process. A semimartingale X is called a special semimartingale (or simply special) if $X = N + A$ for some $A \in \mathcal{A}_{\text{loc}}$ and local martingale N. In this case, there is a unique such decomposition, called the canonical decomposition, such that A is predictable and $A_0 = 0$. This unique A is called the compensator of X and denoted X^p. Note, a finite variation process A is special iff $A \in \mathcal{A}_{\text{loc}}$.

A semimartingale X is called purely discontinuous if $[X,X] = \Sigma_{s\le t}(\Delta X)^2$. A finite variation process is purely discontinuous. For any semimartingale X, there exists a unique continuous local martingale X^c such that $X_0^c = X_0$ and $X - X^c$ is purely discontinuous. One has, $[X,Y] = [X^c,Y^c] + \Sigma_{s\le t}\Delta X\Delta Y$. In particular $\Delta[X,Y] = \Delta X\Delta Y$. It also follows that $[X,Y] = 0$ if (a) X is continuous and has finite variation or (b) X is continuous and Y is purely discontinuous. Note also $[X^c,Y^c] = [X^c,Y]$. If X is special, a purely discontinuous local martingale X^d is defined by $X^d = X - X^c - X^p$. If X and Y are special semimartingales, then $[X^p,Y^d]^p = 0$, and further, $[X,Y] \in \mathcal{A}_{\text{loc}}$ iff $[X^d,Y^d] \in \mathcal{A}_{\text{loc}}$, and in this case $[X,Y]^p = [X^c,Y^c] + [X^d,Y^d]^p + \sum_{s\le t}\Delta X^p\Delta Y^p$.

Let E be Lusin space (e.g., $E = \mathbb{R}^d$ or $E = \mathbb{R}\times\{1,\ldots,d\}$). For any non-negative random measure μ on $\mathbb{R}\times E$, and any optional function $H=H(\omega,t,x)$ on $\Omega\times\mathbb{R}\times E$, one sets

$$H*\mu = \int_{[0,t]\times E} H(\omega,s,x)\mu(\omega,ds,dx) \quad \text{if} \quad \int_{[0,t]\times E} \left|H(\omega,s,x)\right|\mu(\omega,ds,dx) < \infty .$$

Let μ be an integer-valued random measure on $\mathbb{R}\times E$. Then there exists an optional process β and a sequence of stopping times T_n with disjoint graphs (meaning if $n \neq m$ then $T_n(\omega) \neq T_m(\omega)$ for all ω) such that $\mu = \sum_n 1_{T_n(\omega)=t}\delta_{(t,\beta_t(\omega))}$, where $\delta_{(t,x)}$ is the Dirac measure at point (t,x). Also, the compensator ν of μ is characterized as the unique predictable non-negative random measure on $\mathbb{R}\times E$ with the property that for any predictable function $H(\omega,t,x)$ satisfying $\left|H\right|*\nu<\infty$, the process $\left|H\right|*\mu$ (hence also $H*\mu$) belongs to $\mathcal{A}_{\text{loc}}$, and $(H*\mu)^p = H*\nu$. When $\nu(\omega,\{t\}\times E) = 0$ identically (or equivalently all T_n are totally inaccessible), let $G_{\text{loc}}(\mu)$ denote the set of all predictable functions H such that $\min(H^2,\left|H\right|)*\nu<\infty$ a.s. Then, $H*(\mu-\nu)$ is defined to be the unique purely discontinuous local martingale X such that ΔX is indistinguishable from the process $\sum_n 1_{T_n(\omega)=t}H(\omega,t,\beta_t(\omega))$. If $H, K\in G_{\text{loc}}(\mu)$, then $A \equiv$ $[H*(\mu-\nu),K*(\mu-\nu)] = (HK)*\mu$. If $\left|HK\right|*\nu<\infty$, then $A\in\mathcal{A}_{\text{loc}}$, and $A^p = (HK)*\nu$. When $\nu(\omega,dt,dx)$ is not dt-absolutely continuous, $G_{\text{loc}}(\mu)$ and $H*(\mu-\nu)$ can still be defined, but the definitions and the formula for A^p are more complicated. In general, if $K\in G_{\text{loc}}(\mu)$ and H is a locally bounded predictable process, then $\int H\,d(K*(\mu-\nu)) = (HK)*(\mu-\nu)$.

Let $X = (X_1,\ldots,X_n)$ be an n-dimensional semimartingale. Associated to X is an integer-valued random measure μ_X on $\mathbb{R}\times\mathbb{R}^n$ defined by $\mu_X = \sum_{t>0} 1_{\Delta X(t)\neq 0}\delta_{(t,\Delta X(t,\omega))}$. One has, $\sum_{s\le t}\Delta X_i \Delta X_j =$ $(x_i x_j)*\mu_X$. Let ν_X denote the compensator of μ_X. Then $\min(\left|x\right|^2,1)*\nu_X<\infty$. Also, $[X_i,X_j]\in\mathcal{A}_{\text{loc}}$ iff $\left|x_i x_j\right|*\nu_X<\infty$, in which case $[X_i,X_j]^p = [X_i^c,X_j^c]+(x_i x_j)*\nu_X$. If X is (component-wise) predictable, then $\nu_X=\mu_X$. The semimartingale vector X is special iff $(1_{|x|>1}\left|x\right|)*\nu_X<\infty$. In this case $X_i^d = x_i*(\mu_X-\nu_X)$, and if further, $\left|x_i x_j\right|*\nu_X<\infty$, then $(x_i x_j)*\nu_X = (x_i x_j)*(\mu_{X^p}+\nu_{X^d})$. If X is a local martingale of finite variation, then $x_i*\nu_X$ is continuous, and $\sum_{s\le t}\Delta X_i = x_i*\mu_X = X_i + x_i*\nu_X - X_i(0)$.

Let μ be an integer-valued random measure on $\mathbb{R}\times E$ such that $\nu(\omega,\{t\}\times E) = 0$ identically. Let $H_i \in G_{\text{loc}}(\mu)$, $i = 1,\ldots,n$, and set $X_i = H_i*(\mu-\nu)$. Then, for any optional function K on $\Omega\times\mathbb{R}\times\mathbb{R}^n$, $K*\mu_X = (K\circ H)*\mu$ (provided either side is defined), and $K*\nu_X = (K\circ H)*\nu$. Further, if $K\in G_{\text{loc}}(\mu_X)$, then $K*(\mu_X-\nu_X) = (K\circ H)*(\mu-\nu)$.

3. Preliminary Results

Lemma 1. Let $X_i > 0$ be semimartingales, $i = 1,\ldots,m$. The following equations hold for all i.

(1)
$$\Delta \mathcal{L}(\prod_{j=i}^{m} X_j) = \prod_{j=i}^{m}(1+\frac{\Delta X_j}{X_{j-}}) - 1;$$

(2)
$$[X_i, \mathcal{L}(\prod_{j=i+1}^{m} X_j)] = \sum_{j=i+1}^{m}\int \frac{d[X_i^c, X_j^c]}{X_{j-}} + \sum_{s\le t}\Delta X_i\,(\prod_{j=i+1}^{m}(1+\frac{\Delta X_j}{X_{j-}}) - 1))\,;$$

(3)
$$\mathcal{L}(\prod_{j=i}^{m} X_j) = \sum_{j=i}^{m}(\mathcal{L}X_j + \sum_{k=j+1}^{m}\int \frac{d[X_j^c, X_k^c]}{X_{j-}X_{k-}} + \sum_{s\le t}\frac{\Delta X_j}{X_{j-}}(\prod_{k=j+1}^{m}(1+\frac{\Delta X_k}{X_{k-}}) - 1))\,.$$

Proof.

$$\Delta \mathcal{L}(\prod_{j=i}^{m} X_j) = \frac{\Delta \prod X_j}{\prod X_{j-}} = \frac{\prod (1+\Delta X_j) - 1}{\prod X_{j-}} = \prod_{j=i}^{m}(1+\frac{\Delta X_j}{X_{j-}}) - 1.$$

This gives (1). Next, for any semimartingale Y, we have,

$$[Y, \mathcal{L}(\prod_{j=i+1}^{m} X_j)] = [Y^c, \mathcal{L}(\prod_{j=i+1}^{m} X_j)] + \sum_{s\le t}\Delta Y\,\Delta\mathcal{L}(\prod_{j=i+1}^{m} X_j)] = \sum_{j=i+1}^{m}\int \frac{d[Y^c, X_j^c]}{X_{j-}} + \sum_{s\le t}\Delta Y(\prod_{j=i+1}^{m}(1+\frac{\Delta X_j}{X_{j-}}) - 1)\,,$$

where for the last equality we used (1) and also the fact that $[Y^c, \mathcal{L}\prod X_j] = \sum [Y^c, \mathcal{L}X_j]$. (Indeed, $[Y^c, \mathcal{L}(ZW)] = [Y^c, \mathcal{L}Z + \mathcal{L}W + [\mathcal{L}Z, \mathcal{L}W]]$. But, $[Y^c,[\mathcal{L}Z, \mathcal{L}W]] = 0,$ because Y^c is continuous and $[\mathcal{L}Z, \mathcal{L}W]$ has finite variation.) This gives Eq. (2). Eq. (2), substituted in the following equation, which can easily be established by induction, yields Eq. (3).

$$\mathcal{L}(\prod_{j=i}^{m} X_j) = \sum_{j=i}^{m}(\mathcal{L}(X_j) + [\mathcal{L}(X_j), \mathcal{L}(\prod_{k=j+1}^{m} X_k)]).$$ □

In our applications, X_j will represent (essentially) forward Libor rates, and their products represent deflated bond prices. The above result is then relevant to the relationship between bond prices and Libor rates. The peculiar order of the sums and products has to do with applicability to the “terminal measure”. For the “spot measure”, we need the following result.

Lemma 2. Let $X_i > 0$ be semimartingales, $i = 1,\ldots,m$. The following equations hold for all i.

(4)
$$\Delta \mathcal{L}(\prod_{j=1}^{i} \frac{1}{X_j}) = \prod_{j=1}^{i}(1+\frac{\Delta X_j}{X_{j-}})^{-1} - 1;$$

(5)
$$[X_i, \mathcal{L}(\prod_{j=1}^{i} \frac{1}{X_j})] = -\sum_{j=1}^{i}\int \frac{d[X_i^c, X_j^c]}{X_{j-}} + \sum_{s\le t}\Delta X_i\,(\prod_{j=1}^{i}(1+\frac{\Delta X_j}{X_{j-}})^{-1} - 1))\,;$$

(6)
$$\mathcal{L}(\prod_{j=1}^{i} \frac{1}{X_j}) = -\sum_{j=1}^{i}(\mathcal{L}X_j - \sum_{k=1}^{j}\int \frac{d[X_j^c, X_k^c]}{X_{j-}X_{k-}} + \sum_{s\le t}\frac{\Delta X_j}{X_{j-}}(\prod_{k=1}^{j}(1+\frac{\Delta X_k}{X_{k-}})^{-1} - 1))\,.$$

Proof. For $X > 0$, we have $\Delta(1/X) = -\Delta X/(X_- X)$, or written differently,

$$1+\frac{\Delta X^{-1}}{X_-^{-1}} = (1+\frac{\Delta X}{X_-})^{-1}.$$

Hence,

$$\Delta \mathcal{L}(\prod_{j=1}^{i} X_j^{-1}) = \frac{\Delta(\prod X_j^{-1})}{\prod X_j^{-1}} = \prod (1+\frac{\Delta X_j^{-1}}{X_{j-}^{-1}}) - 1 = \prod (1+\frac{\Delta X_j}{X_{j-}})^{-1} - 1.$$

Eq (5) now follows too, using $[Y^c, \mathcal{L}\prod X_j] = \sum [Y^c, \mathcal{L}X_j]$ and $[Y^c, \mathcal{L}(1/Z)] = -[Y^c, \mathcal{L}Z]$.

Substituted in the following equation, which can easily be established by induction, yields (6).

$$\mathcal{L}(\prod_{j=1}^{i} X_j^{-1}) = -\sum_{j=1}^{i}(\mathcal{L}(X_j) + [\mathcal{L}(X_j), \mathcal{L}(\prod_{k=1}^{j} X_k^{-1})]).$$

Alternatively, first establish (6) for i = 1. Then use (3), with a reversed order and substitute. □

Remark 1. Setting $Y_i = \mathcal{L}(X_i)$ in Eq. (3) and (6) above, we obtain

$$\prod_{j=i}^{m} \mathcal{E}(Y_j) = \mathcal{E}\,(\sum_{j=i}^{m}(Y_j + \sum_{k=j+1}^{m}[Y_j^c, Y_k^c] + \sum_{s\le t}\Delta Y_j(\prod_{k=j+1}^{m}(1+\Delta Y_k) - 1)));$$

$$\prod_{j=1}^{i} \mathcal{E}(Y_j)\,\mathcal{E}\,(-\sum_{j=1}^{i}(Y_j - \sum_{k=1}^{j}[Y_j^c, Y_k^c] + \sum_{s\le t}\Delta Y_j(\prod_{k=1}^{j}(1+\Delta Y_k)^{-1} - 1))) = 1.$$

Although simpler and more general – they do not require $\mathcal{E}(Y_i) > 0$ - the form given earlier is actually more convenient for our purposes. (Same comment applies to other formulae below.) We can also write these equations in the more symmetric form

$$\prod_{j=1}^{m} \mathcal{E}(Y_j) = \mathcal{E}\,(\sum_{j=1}^{m} Y_j + \sum_{1\le j<k\le m}[Y_j^c, Y_k^c] + \sum_{s\le t}(\prod_{j=1}^{m}(1+\Delta Y_k) - \sum_{j=1}^{m}\Delta Y_j - 1));$$

$$\prod_{j=1}^{m} \mathcal{E}(Y_j)\,\mathcal{E}\,(-\sum_{j=1}^{m} Y_j + \sum_{1\le j\le k\le m}[Y_j^c, Y_k^c] + \sum_{s\le t}(\prod_{j=1}^{m}(1+\Delta Y_j)^{-1} + \sum_{j=1}^{m}\Delta Y_j - 1)) = 1.$$ □

We wish to draw conclusion about the dynamics of the (essentially) forward Libor rates X_j from the postulate that deflated bond prices are local martingales. The following results yield a "drift" restriction on the forward Libor system $X = (X_1,\ldots, X_m)$ in terms of continuous volatility and the size and intensity of the jumps. First we need a simple Lemma.

Lemma 3. Let Y be a semimartingale and $Z > 0$ be a local martingale. Then YZ is a local martingale if and only if $Y + [Y, \mathcal{L}(Z)]$ is a local martingale.

Proof. Write $A \approx B$ if A-B is a local martingale. Set $N = Y + [Y, \mathcal{L}(Z)]$ and $M = \int Z_- dY + [Y, Z]$. Note, $N = \int Z_- dM + Y_0$ and $M = \int dN/Z_-$. So, $N \approx 0$ iff $M \approx 0$. But, since $Z \approx 0$, by Ito's product rule, $YZ \approx M$. So $YZ \approx 0$ iff $N \approx 0$. □

Theorem 1. Let $X_i > 0$ be semimartingales, $i = 1,\ldots,m$. The following three conditions are equivalent. (i) $\prod_{j=i}^{m} X_j$ are local martingales for all i. (ii) For all i, Eq. (7) and (8) below hold.

(7) $$\left| 1_{|x|\geq 1} x_i \prod_{j=i+1}^{m} (1+\frac{x_j}{X_{j-}}) \right| * \nu_X < \infty;$$

(8) $$X_i = -\sum_{j=i+1}^{m} \int \frac{d[X_i^c, X_j^c]}{X_{j-}} - (1_{|x|<1} x_i (\prod_{j=i+1}^{m} (1+\frac{x_j}{X_{j-}}) - 1)) * \nu_X - (1_{|x|\geq 1} x_i \prod_{j=i+1}^{m} (1+\frac{x_j}{X_{j-}})) * \nu_X + X_i^c + (1_{|x|<1} x_i) * (\mu_X - \nu_X) + \sum_{s\leq t} 1_{|\Delta X|\geq 1} \Delta X_i .$$

(iii) For all i, the following equation holds.

(9) $$\mathcal{L}(\prod_{j=i}^{m} \frac{X_j}{X_j(0)}) = \sum_{j=i}^{m} \int \frac{dX_j^c}{X_{j-}} + (\prod_{j=i}^{m} (1+\frac{x_j}{X_{j-}}) - 1) * (\mu_X - \nu_X) .$$

Proof. Set $F_i = \prod_{j=i}^{m} X_j$. Write $A \approx B$ if A-B is a local martingale.
(iii) ⇒(i). (iii) implies all $\mathcal{L}(F_i) \approx 0$, hence all $F_i \approx 0$.
(i) ⇒(ii). Assume all $F_i \approx 0$. Set $M_i = X_i + [X_i, \mathcal{L}F_{i+1}]$. By lemma 3, $M_i \approx 0$. Now, by Eq. (2), $[X_i, \mathcal{L}F_{i+1}] = C_i + g_i * \mu_X$, where

$$g_i = x_i (\prod_{j=i+1}^{m} (1+\frac{x_j}{X_{j-}(t,\omega)}) - 1), \qquad C_i = \sum_{j=i+1}^{m} \int \frac{d[X_i^c, X_j^c]}{X_{j-}} .$$

So, $X_i + C_i + g_i * \mu_X \approx 0$. The canonical representation of X states (cf, Jacod and Shiryaev (1987), Theorem II.2.34),

(10) $$X_i = B_i + X_i^c + (1_{|x|<1} x_i) * (\mu_X - \nu_X) + \sum_{s\leq t} 1_{|\Delta X|\geq 1} \Delta X_i , \qquad B_i \equiv (X_i - \sum_{s\leq t} 1_{|\Delta X|\geq 1} \Delta X_i)^p .$$

Hence, $B_i + (1_{|x|\geq 1} x_i) * \mu_X + C_i + g_i * \mu_X \approx 0$. But $g_i * \mu_X \approx (1_{|x|\geq 1} g_i) * \mu_X + (1_{|x|<1} g_i) * \nu_X$, since $g_i = O(|x|^2)$ near $x = 0$. Hence, $B_i + C_i + (1_{|x|\geq 1}(x_i + g_i)) * \mu_X + (1_{|x|<1} g_i) * \nu_X \approx 0$. This equation furnishes a canonical decomposition for $(1_{|x|\geq 1}(x_i + g_i)) * \mu_X$, showing it is a special semimartingale, and since it already is of finite variation, that it has locally integrable variation. Thus (7) holds. It now further follows, $B_i + C_i + (1_{|x|\geq 1}(x_i + g_i)) * \nu_X + (1_{|x|<1} g_i) * \nu_X = 0$. Substituting from this for B_i into (10) yields at once Eq. (8).

(ii) ⇒(iii). Calculate $\mathcal{L}(X_j)$, $j \geq i$ from (8), and substitute it into (3). The integral appearing in (3) cancels. The discontinuous sum in (3) combines with that in (8) and the two terms in (8) involving "$* \nu_X$" turning these into two terms involving "$*(\mu_X - \nu_X)$", which combine with the term in (8) involving "$*(\mu_X - \nu_X)$" to produce a single such term. With this substitution, we get

$$\mathcal{L}\Big(\prod_{j=i}^{m}\frac{X_j}{X_j(0)}\Big)=\sum_{j=i}^{m}\Big(\int\frac{dX_j^c}{X_{j-}}+\Big(\frac{x_j}{X_{j-}}\prod_{k=j+1}^{m}\Big(1+\frac{x_k}{X_{k-}}\Big)-1\Big)\Big)*(\mu_X-\nu_X)\Big).$$

Now the algebraic identity (11) below gives (9).

$$\sum_{j=i}^{m}\frac{x_j}{X_{j-}}\prod_{k=j+1}^{m}\Big(1+\frac{x_k}{X_{k-}}\Big)=\prod_{j=i}^{m}\Big(1+\frac{x_j}{X_{j-}}\Big)-1. \qquad (11) \qquad \square$$

Eq. (8), while general, looks rather convoluted. Moreover, it does not quite furnish a “drift”, as a sum of jumps appears in it. To obtain a proper drift, in the sense of a predictable, finite variation process, X_i must be special semimartingales, i.e., $(1_{|x|\geq 1}x_i)*\nu_X<\infty$. With this, the discontinuous sum in (8) which is $(1_{|x|\geq 1}x_i)*\mu_X$ can be written as $(1_{|x|\geq 1}x_i)*\nu_X$ + $(1_{|x|\geq 1}x_i)*(\mu_X-\nu_X)$. Each term then combines with other terms in (8) to yield the simpler and more appealing Eq (13) below (for which an alternative, direct derivation will be given).

Theorem 2. Let $X_i > 0$ be semimartingales, $i = 1,\ldots,m$. The following five conditions are equivalent.

(i) $\prod_{j=i}^{m}X_j$ are local martingales and X_i are special semimartingales for all $i = 1,\ldots,m$.

(ii) The two equations below hold for all $i = 1,\ldots,m$.

$$\left|x_i\Big(\prod_{j=i+1}^{m}\Big(1+\frac{x_j}{X_{j-}}\Big)-1\Big)\right|*\nu_X<\infty \quad \text{a.s.} \qquad (12)$$

$$X_i=-\sum_{j=i+1}^{m}\int\frac{d[X_i^c,X_j^c]}{X_{j-}}-\Big(x_i\Big(\prod_{j=i+1}^{m}\Big(1+\frac{x_j}{X_{j-}}\Big)-1\Big)\Big)*\nu_X+X_i^c+x_i*(\mu_X-\nu_X). \qquad (13)$$

(iii) For all $i = 1,\ldots,m$, X_i are special semimartingales and

$$\mathcal{L}\Big(\prod_{j=i}^{m}\frac{X_j}{X_j(0)}\Big)=\sum_{j=i}^{m}\int\frac{dX_j^c}{X_{j-}}+\Big(\prod_{j=i}^{m}\Big(1+\frac{x_j}{X_{j-}}\Big)-1\Big)*(\mu_X-\nu_X). \qquad (14)$$

(ii)^d The two equations below hold for all $i = 1,\ldots,m$.

$$\left|x_i\Big(\prod_{j=i+1}^{m}\Big(1+\frac{x_j}{X_{j-}}\Big)-1\Big)\right|*\nu_{X^d}<\infty \quad \text{a.s.} \qquad (12)^d$$

$$X_i=-\sum_{j=i+1}^{m}\int\frac{d[X_i^c,X_j^c]}{X_{j-}}-\Big(x_i\Big(\prod_{j=i+1}^{m}\Big(1+\frac{x_j}{X_{j-}}\Big)-1\Big)\Big)*\nu_{X^d}+X_i^c+x_i*(\mu_{X^d}-\nu_{X^d}). \qquad (13)^d$$

(iii)^d For all $i = 1,\ldots,m$, X_i are special semimartingales and

$$(14)^d \qquad \mathcal{L}\Big(\prod_{j=i}^{m}\frac{X_j}{X_j(0)}\Big) = \sum_{j=i}^{m}\int\frac{dX_j^c}{X_{j-}} + \Big(\prod_{j=i}^{m}\Big(1+\frac{x_j}{X_{j-}}\Big)-1\Big)*(\mu_{X^d}-\nu_{X^d}).$$

Moreover, if these conditions are satisfied, then $X_i^p = -[X_i, \mathcal{L}(\prod_{j=i+1}^{m} X_j)]^p$.

Proof. We show (iii) ⇒(i) ⇒(ii) ⇒(iii) and (ii)^d ⇒ (iii)^d ⇒(i) ⇒(ii)^d.
Set $F_i = \prod_{j=i}^{m} X_j$. Write $A \approx B$ if A-B is a local martingale.

(iii) ⇒(i). (iii) implies all $\mathcal{L}(F_i) \approx 0$, hence all $F_i \approx 0$.
(i) ⇒(ii). Assume all $F_i \approx 0$. Set $M_i = X_i + [X_i, \mathcal{L}F_{i+1}]$. By Lemma 3, $M_i \approx 0$. Since X_i is a special semimartingale, it follows $[X_i, \mathcal{L}F_{i+1}]$ is special and $X_i^p = -[X_i, \mathcal{L}F_{i+1}]^p$. But by (2), $[X_i, \mathcal{L}F_{i+1}] = C_i + g_i * \mu_X$, where

$$g_i = x_i\Big(\prod_{j=i+1}^{m}\Big(1+\frac{x_j}{X_{j-}(t,\omega)}\Big)-1\Big), \qquad C_i = \sum_{j=i+1}^{m}\int\frac{d[X_i^c, X_j^c]}{X_{j-}}.$$

It follows that $|g_i| * \nu_X < \infty$ and $X_i^p = -[X_i, \mathcal{L}F_{i+1}]^p = -C_i - g_i * \nu_X$, i.e. (12) and (13) hold.
(ii)⇒(iii). (ii) clearly exhibits X_i as special. Now, calculating $\mathcal{L}(X_j)$ from (13), and substituting it into (3) and using the identity (11) simplifies to (14).
(ii)^d⇒(iii)^d. This follows by an argument identical to that in (ii) ⇒(iii).
(iii)^d ⇒(i) . (14)^d shows all $\mathcal{L}(F_i) \approx 0$, hence all $F_i \approx 0$.
(i) ⇒(ii)^d. By induction, (ii)^d holds for $j > i$; so the argument (ii)^d⇒(iii)^d above implies (14)^d holds with i replaced by i +1. This implies $[X_i^d, \mathcal{L}F_{i+1}] = g_i * \mu_{X^d}$, since $X_i^d = x_i * (\mu_{X^d} - \nu_{X^d})$.
Proceeding as in (i) ⇒(ii), we have $X_i^p = -[X_i, \mathcal{L}F_{i+1}]^p$. But, $[X_i, \mathcal{L}F_{i+1}]^p = C_i + [X_i^d, \mathcal{L}F_{i+1}]^p$ because $\mathcal{L}(F_{i+1}) \approx 0$. Thus, $X_i^p = -C_i - g_i * \nu_{X^d}$, as desired. □

Remark 2. The formulae in the above Theorems can also be expressed in terms $Y_i \approx \mathcal{L}(X_i)$.
For example, condition (ii) clearly becomes

$$\Big|y_i\Big(\prod_{j=i+1}^{m}(1+y_j)-1\Big)\Big| * \nu_Y < \infty \text{ a.s.}$$

$$Y_i = -\sum_{j=i+1}^{m}[Y_i^c, Y_j^c] - \Big(y_i\Big(\prod_{j=i+1}^{m}(1+y_j)-1\Big)\Big)*\nu_Y + Y_i^c + y_i*(\mu_Y - \nu_Y). \qquad \square$$

Note, if X_i^d has finite variation, or equivalently $|x_i| * \nu_X < \infty$, then we may rewrite (13) as

$$X_i = -\sum_{j=i+1}^{m}\int\frac{d[X_i^c, X_j^c]}{X_{j-}} - \Big(x_i\prod_{j=i+1}^{m}\Big(1+\frac{x_j}{X_{j-}}\Big)\Big)*\nu_X + X_i^c + x_i*\mu_X, \qquad |x_i| * \nu_X < \infty.$$

We will apply Theorems 1 and 2 in the "terminal measure" and in the actual measure, but with view to the "terminal numeraire." For the "spot measure" and "spot numeraire", we need to apply Lemma 2. For the sake of brevity, we forgo the analogue of Theorem 1, and provide only the special semimartingale result. Once this is done, the omitted case will be amply clear.

Theorem 3. Let $X_i > 0$ be semimartingales, $i = 1,\ldots,m$. The following five conditions are equivalent.

(i) $\prod_{j=1}^{i} 1/X_j$ are local martingales and X_i are special semimartingales for all i.

(ii) The two equations below hold for all $i = 1,\ldots,m$.

(15) $$\left| x_i\left(\prod_{j=1}^{i}\left(1+\frac{x_j}{X_{j-}}\right)^{-1}-1\right)\right| * \nu_X < \infty \quad \text{a.s.}$$

(16) $$X_i = \sum_{j=1}^{i}\int \frac{d[X_i^c, X_j^c]}{X_{j-}} - \left(x_i\left(\prod_{j=1}^{i}\left(1+\frac{x_j}{X_{j-}}\right)^{-1}-1\right)\right) * \nu_X + X_i^c + x_i * (\mu_X - \nu_X).$$

(iii) For all $i = 1,\ldots,m$, X_i are special semimartingale and

(17) $$\mathcal{L}\left(\prod_{j=1}^{i}\frac{X_j(0)}{X_j}\right) = -\sum_{j=1}^{i}\int \frac{dX_j^c}{X_{j-}} + \left(\prod_{j=1}^{i}\left(1+\frac{x_j}{X_{j-}}\right)^{-1}-1\right) * (\mu_X - \nu_X).$$

$(\text{ii})^d$ The two equations below hold for all $i = 1,\ldots,m$.

$(15)^d$ $$\left| x_i\left(\prod_{j=1}^{i}\left(1+\frac{x_j}{X_{j-}}\right)^{-1}-1\right)\right| * \nu_{X^d} < \infty \quad \text{a.s.}$$

$(16)^d$ $$X_i = \sum_{j=1}^{i}\int \frac{d[X_i^c, X_j^c]}{X_{j-}} - \left(x_i\left(\prod_{j=1}^{i}\left(1+\frac{x_j}{X_{j-}}\right)^{-1}-1\right)\right) * \nu_{X^d} + X_i^c + x_i * (\mu_{X^d} - \nu_{X^d}).$$

$(\text{iii})^d$ For all $i = 1,\ldots,m$, X_i are special semimartingale and

$(17)^d$ $$.\ \mathcal{L}\left(\prod_{j=1}^{i}\frac{X_j(0)}{X_j}\right) = -\sum_{j=1}^{i}\int \frac{dX_j^c}{X_{j-}} + \left(\prod_{j=1}^{i}\left(1+\frac{x_j}{X_{j-}}\right)^{-1}-1\right) * (\mu_{X^d} - \nu_{X^d}).$$

Moreover, if these conditions are satisfied, then $X_i^p = -[X_i, \mathcal{L}(\prod_{j=1}^{i} 1/X_j)]^p$.

Proof. We show (iii) ⇒(i) ⇒(ii) ⇒(iii), $(\text{ii})^d$⇒$(\text{iii})^d$⇒(i), (ii)&$(\text{iii})^d$ ⇒$(\text{ii})^d$, and (iii)⇒$(\text{iii})^d$. Set $F_i = \prod_{j=1}^{i} 1/X_j$. Write $A \approx B$ if A-B is a local martingale.

(iii) ⇒(i). (iii) implies all $\mathcal{L}(F_i) \approx 0$, hence all $F_i \approx 0$.
(i) ⇒(ii). Assume all $F_i \approx 0$. Set $M_i = X_i + [X_i, \mathcal{L}F_i]$. By Lemma 3, $M_i \approx 0$. Since X_i is a special semimartingale, it follows $[X_i, \mathcal{L}F_i]$ is special and $X_i^p = -[X_i, \mathcal{L}F_i]^p$. But by (5), $[X_i, \mathcal{L}F_i] = -C_i + g_i * \mu_X$, where

$$g_i = x_i\Big(\prod_{j=1}^{i}(1+\frac{x_j}{X_{j-}(t,\omega)})^{-1} - 1\Big), \qquad C_i = \sum_{j=1}^{i}\int \frac{d[X_i^c, X_j^c]}{X_{j-}}.$$

It follows that $|g_i| * \nu_X < \infty$, and $X_i^p = -[X_i, \mathcal{L}F_i]^p = C_i - g_i * \nu_X$, i.e. (15) and (16) hold.
(ii)⇒(iii). (ii) clearly exhibits X_i as special. Now, calculating $\mathcal{L}(X_j)$ from (15), and substituting it into (6) and using the identity below (which follows from 11) simplifies to (17).

$$\sum_{j=1}^{i}\frac{x_j}{X_{j-}}\prod_{k=1}^{j}(1+\frac{x_k}{X_{k-}})^{-1} = 1 - \prod_{j=1}^{i}(1+\frac{x_j}{X_{j-}})^{-1}.$$

(ii)^d⇒(iii)^d. This follows by an argument identical to that in (ii) ⇒(iii).
(iii)^d ⇒(i) . (16)^d shows all $\mathcal{L}(F_i) \approx 0$, hence all $F_i \approx 0$.
(ii)&(iii)^d ⇒(ii)^d. (ii) shows $g_i * \nu_X = [X_i, \mathcal{L}F_i]^p + C_i$. Observe, $[X_i, \mathcal{L}F_i]^p + C_i = [X_i^d, \mathcal{L}F_i]^p$ because $\mathcal{L}(F_i) \approx 0$. But, using (iii)^d we have $[X_i^d, \mathcal{L}F_i]^p = g_i * \nu_{X^d}$. Hence, $g_i * \nu_X = g_i * \nu_{X^d}$. Further, $x_i * (\mu_X - \nu_X) = x_i * (\mu_{X^d} - \nu_{X^d})$. Making these two substitutions in (16) gives (16)^d.
(iii)⇒(iii)^d In general if $H \in G_{\text{loc}}(\mu_X)$ then $H * (\mu_X - \nu_X) = H * (\mu_{X^d} - \nu_{X^d})$. Hence (17)⇒(17)^d.□

Note, the integrability conditions (12) and (15) are restrictions only on intensity of *large* jumps (as is the condition of being a special semimartingale). As for the counterpart of Theorem 1, it is now clear that all $\prod_{j=1}^{i} 1/X_j$ are local martingales iff

$$X_i = \sum_{j=1}^{i}\int \frac{d[X_i^c, X_j^c]}{X_{j-}} - \Big(1_{|x|<1}x_i\Big(\prod_{j=1}^{i}(1+\frac{x_j}{X_{j-}})^{-1} - 1\Big)\Big) * \nu_X -$$
$$\Big(1_{|x|\geq 1}x_i\prod_{j=1}^{i}(1+\frac{x_j}{X_{j-}})^{-1}\Big) * \nu_X + X_i^c + (1_{|x|<1}x_i) * (\mu_X - \nu_X) + \sum_{s\leq t} 1_{|\Delta X|\geq 1}\Delta X_i .$$

A satisfactory feature of these results is that we may replace "ν_X" by "ν_{X^d}" , namely, the version (ii)^d of Theorems 2 and 3 hold. This means that the local martingale part of the system X completely determines its drift X^p . It is therefore possible to specify the local martingale part exogenously, and then simply "read" the drift, as now done below.

Assume the positive semimartingale system $X = (X_1,\ldots,X_m) > 0$ is driven by a multidimensional (column-vector) Brownian motion W, and an integer-valued random measure μ on $\mathbb{R}\times E$, E a Lusin space. So,

$$X_i = X_i(0) + A_i + \int \beta_i dW + \phi_i * (\mu - \nu), \qquad i = 1,\ldots,m,$$

where ν is the compensator of μ, A_i is a predictable, finite variation process, β_i is a predictable (row) vector process with $\int|\beta_i|^2 dt < \infty$ a.s., and $\phi_i \in G_{\text{loc}}(\mu)$. Assume $\nu(\omega,\{t\}\times E) = 0$ identically, or equivalently $\nu(\omega,dt,dx)$ is dt-absolutely continuous. Then, by the equivalence between (i) and (iii)^{d} in Theorem 2, $\prod_{j=i}^{m} X_j$ are local martingales for all i, iff

$$\left|\phi_i\left(\prod_{j=i+1}^{m}\left(1+\frac{\phi_j}{X_{j-}}\right)-1\right)\right| * \nu < \infty \qquad \text{a.s.};$$

$$X_i = X_i(0) - \sum_{j=i+1}^{m}\int \frac{\beta_i\,\beta_j^{\mathrm{t}}}{X_{j-}}\,dt - \left(\phi_i\left(\prod_{j=i+1}^{m}\left(1+\frac{\phi_j}{X_{j-}}\right)-1\right)\right) * \nu + \int \beta_i\,dW + \phi_i * (\mu-\nu).$$

Note, our assumption on ν now implies that this implies $A_i = X_i^p$ is in fact continuous, and X_i is quasi-left continuous. Eq. (14)^{d} translates to

$$\mathcal{L}\left(\prod_{j=i}^{m}\frac{X_j}{X_j(0)}\right) = \sum_{j=i}^{m}\int \frac{\beta_j\,dW}{X_{j-}} + \left(\prod_{j=i}^{m}\left(1+\frac{\phi_j}{X_{j-}}\right)-1\right) * (\mu-\nu)\,.$$

Similar equations can be written for the "spot" case, i.e., Eq. (16)^{d} and (17)^{d} of Theorem 3.

The property of being a special semimartingale is not necessarily preserved under change of measure. We will need a condition under which it is preserved.

Proposition 1. Let Q be a measure equivalent to P and $Z_t = dQ\big|_{\mathcal{F}_t} / dP\big|_{\mathcal{F}_t} = E_t^P[dQ/dP]$ be its martingale density. Let X be any semimartingale. (i) Suppose $[X, \mathcal{L}(Z)]$ is P-special. Then X is P-special if and only if X is Q-special.

(ii) If X is both P and Q special, then $[X, \mathcal{L}(Z)]$ is P-special. (Also, of course, $[X, \mathcal{L}(Z)]$ is P-special iff $[X, Z]$ is P-special, because Z_- is locally bounded.)

Proof. Write $A \approx^P B$ ($A \approx^Q B$) if A-B is a P-local (Q-local) martingale. Note $Z \approx^P 0$. As is well-known, $Y \approx^Q 0$ iff $YZ \approx^P 0$. Hence by Lemma 3, $Y \approx^Q 0$ iff $Y + [Y,\mathcal{L} Z] \approx^P 0$. For any Q-special semimartingale Y, let Y^q denote the unique predictable process of finite variation such that $Y^q(0)=0$ and $Y^q \approx^Q Y$. Now, assume X is a Q-special semimartingale, and set $Y = X - X^q$. Then $Y \approx^Q 0$. Hence $0 \approx^P Y + [Y,\mathcal{L} Z] \approx^P Y + [X,\mathcal{L} Z]$, since $[X^q,\mathcal{L} Z] \approx^P 0$. Hence $0 \approx^P X - X^q + [X,\mathcal{L} Z]$. This implies X is P-special iff $[X, \mathcal{L}Z]$ is P-special. Thus follow (ii) and the only if part of (i). As for the if part, set $Y = X - X^p - [X,\mathcal{L} Z]^p$. Note, $[Y,\mathcal{L}(Z)] \approx^P [X,\mathcal{L} Z]$. Hence $Y \approx^P -[Y,\mathcal{L} Z]$. It follows $Y \approx^Q 0$, implying X is Q-special, and

$$X^q = X^p + [X,\mathcal{L}(Z)]^p\,. \qquad \square$$

4. Arbitrage-Free Forward Libor Dynamics

Consider a semimartingale system $B = (B_1,\ldots,B_{n+1}) > 0$, i.e., a vector of positive semimartingales $B_i > 0$. We interpret the B_i as zero-coupon bond prices of increasing maturities. (But, they can also be considered as zero-dividend stock prices.). A *day-count fraction* δ is a sequence $\delta = (\delta_1,\ldots,\delta_n)$ of positive numbers $\delta_i > 0$. In practice, δ_i are usually 0.25 (quarterly) or 0.50 (semi-annually). Associated to such data is the system of *forward Libor rates* $L = (L_1,\ldots,L_n)$ defined by

(18) $$L_i = (\frac{B_i}{B_{i+1}} - 1)/\delta_i, \qquad i = 1,\ldots,n.$$

We do not insist $L_i > 0$, as not to exclude Gaussian models. But in practice and concrete examples, L_i are normally positive. At any case, clearly $L_i > -1/\delta_i$. Note

$$\frac{B_i}{B_{n+1}} = \prod_{j=i}^{n} (1+\delta_j L_j), \qquad \frac{B_{i+1}}{B_1} = \prod_{j=1}^{i} (1+\delta_j L_j)^{-1}.$$

Therefore, Lemmas 1 and 2 provide

$$\mathcal{L}(\frac{B_i B_{n+1}(0)}{B_{n+1} B_i(0)}) = \sum_{j=i}^{n} (\int \frac{\delta_j dL_j}{1+\delta_j L_{j-}} + \sum_{k=j+1}^{m} \int \frac{\delta_j \delta_k d[L_j^c, L_k^c]}{(1+\delta_j L_{j-})(1+\delta_k L_{k-})} + \sum_{s\le t} \frac{\delta_j \Delta L_j}{1+\delta_j L_{j-}} (\prod_{k=j+1}^{m} (1+\frac{\delta_k \Delta L_k}{1+\delta_k L_{k-}}) - 1))$$

$$\mathcal{L}(\frac{B_{i+1} B_1(0)}{B_1 B_{i+1}(0)}) = -\sum_{j=1}^{i} (\int \frac{\delta_j dL_j}{1+\delta_j L_{j-}} - \sum_{k=1}^{j} \int \frac{\delta_j \delta_k d[L_j^c, L_k^c]}{(1+\delta_j L_{j-})(1+\delta_k L_{k-})} + \sum_{s\le t} \frac{\delta_j \Delta L_j}{1+\delta_j L_{j-}} (\prod_{k=1}^{j} (1+\frac{\delta_k \Delta L_k}{1+\delta_k L_{k-}})^{-1} - 1))$$

Definition.. Let $B = (B_1,\ldots,B_{n+1}) > 0$ be a semimartingale system. A *state price density* for B is a semimartingale $\xi > 0$ such that $\xi(0) = 1$ and $\xi B_i / B_i(0)$ is a martingale for each i.

Note, if B has a state price density, then so does $\alpha B = (\alpha B_1,\ldots, \alpha B_{n+1})$, for any semimartingale $\alpha > 0$. Hence, existence of a state price density ξ is a property only of the ratios B_i / B_j, i.e., of the forward Libor system L. It is well known that existence of a state price density ξ is related to absence of arbitrage. For example, the price ξC of a "self-financing portfolio" will then be a martingale. So, if $C_T \ge 0$ and $P\{C_T > 0\} > 0$, then $C_t = E_t[\xi_T C_T]/\xi_t > 0$ for $t < T$. Similarly, the uniqueness of ξ is related to "completeness", i.e., that all payoffs can be replicated by self-financing portfolios. Here though, we are primarily concerned with the implication of the existence of a state price density on the dynamics of forward Libor system L.

In what follows, if Q is a measure equivalent to P, we denote the Q-compensator of μ_L by ν_L^Q, and the continuous Q-local martingale part of L_i by $L_i^{c,Q}$. (Note, $[L_i^{c,Q}, L_j^{c,Q}] = [L_i^c, L_j^c]$.)

Theorem 4. Let $B_i > 0$ be semimartingales, $i = 1,\ldots,n+1$. Define the system $L = (L_1,\ldots,L_n)$ as in Eq. (18). Then, the following three conditions are equivalent.

(i) There exists a state price density ξ for B.

(ii) There exists a measure Q equivalent to P (called the *terminal measure*) such that for all $i = 1,\ldots,n$, $E^Q[B_i(T)B_{n+1}(0)/(B_{n+1}(T)B_i(0))] \geq 1$ and the following two equations hold.

(19) $$\left| 1_{|x|\geq 1} x_i \prod_{j=i+1}^{n} (1 + \frac{\delta_j x_j}{1+\delta_j L_{j-}}) \right| * \nu_L^Q < \infty;$$

(20) $$L_i = -\sum_{j=i+1}^{n} \int \frac{\delta_j d[L_i^c, L_j^c]}{1+\delta_j L_{j-}} - (1_{|x|<1} x_i (\prod_{j=i+1}^{n} (1 + \frac{\delta_j x_j}{1+\delta_j L_{j-}}) - 1)) * \nu_L^Q - (1_{|x|\geq 1} x_i \prod_{j=i+1}^{m} (1 + \frac{\delta_j x_j}{1+\delta_j L_{j-}})) * \nu_L^Q + L_i^{c,Q} + (1_{|x|<1} x_i) * (\mu_L - \nu_L^Q) + \sum_{s\leq t} 1_{|\Delta L|\geq 1} \Delta L_i .$$

(iii) There exists a martingale $\zeta > 0$ such that $\zeta(0) = 1$, and for all $i = 1,\ldots,n$, $E[B_i(T)\zeta(T)B_{n+1}(0)/(B_{n+1}(T)B_i(0))] \geq 1$ and, denoting $\tilde{L} = (L, \mathcal{L}\zeta)$, the following two equations hold.

(21) $$\left| 1_{|x|\geq 1} x_i (1 + x_{n+1}) \prod_{j=i+1}^{n} (1 + \frac{\delta_j x_j}{1+\delta_j L_{j-}}) \right| * \nu_{\tilde{L}} < \infty \qquad \text{a.s.}$$

(22) $$L_i = -\int \frac{d[L_i^c, \zeta^c]}{\zeta_-} - \sum_{j=i+1}^{n} \int \frac{\delta_j d[L_i^c, L_j^c]}{1+\delta_j L_{j-}} - (1_{|x|<1} x_i (\prod_{j=i+1}^{n} (1 + \frac{\delta_j x_j}{1+\delta_j L_{j-}})(1 + x_{n+1}) - 1)) * \nu_{\tilde{L}} - (1_{|x|\geq 1} x_i (1 + x_{n+1}) \prod_{j=i+1}^{n} (1 + \frac{\delta_j x_j}{1+\delta_j L_{j-}})) * \nu_{\tilde{L}} + L_i^c + (1_{|x|<1} x_i) * (\mu_{\tilde{L}} - \nu_{\tilde{L}}) + \sum_{s\leq t} 1_{|\Delta\tilde{L}|\geq 1} \Delta L_i .$$

.

Moreover, if these conditions hold, then all $B_i B_{n+1}(0)/(B_{n+1}B_i(0))$ are Q-martingales, and

(23) $$\mathcal{L}(\frac{B_i B_{n+1}(0)}{B_i(0) B_{n+1}}) = \sum_{j=i}^{n} \int \frac{\delta_j dL_j^{c,Q}}{1+\delta_j L_{j-}} + (\prod_{j=i}^{n} (1 + \frac{\delta_j x_j}{1+\delta_j L_{j-}}) - 1) * (\mu_L - \nu_L^Q);$$

(24) $$\mathcal{L}(\frac{\xi B_i}{B_i(0)}) = \int \frac{d\zeta^c}{\zeta_-} + \sum_{j=i}^{n} \int \frac{\delta_j dL_j^c}{1+\delta_j L_{j-}} + (\prod_{j=i}^{n} (1 + \frac{\delta_j x_j}{1+\delta_j L_{j-}})(1 + x_{n+1}) - 1) * (\mu_{(L,\mathcal{L}\zeta)} - \nu_{(L,\mathcal{L}\zeta)});$$

and the following relationship exists between various quantities

(25) $$\zeta_t = \frac{\xi_t B_{n+1}(t)}{B_{n+1}(0)} = E_t[\frac{dQ}{dP}]; \qquad \frac{dQ}{dP} = \zeta_T; \qquad h * \nu_L^Q = (h(1 + x_{n+1})) * \nu_{(L,\mathcal{L}\zeta)}$$

for any predictable function $h(\omega,t,x_1,\ldots,x_n)$ such that $|h|*\nu_L<\infty$ and $|h\,x_{n+1}|*\nu_{(L,\mathcal{L}\zeta)}<\infty$.

Proof. Set $X_i=(1+\delta_i L_i),\ i=1,...,n$. Note, $\prod_{j=1}^{n} X_j = B_i / B_{n+1}$.

(i) $\Rightarrow$ (ii). Set $\zeta=\xi B_{n+1}/B_{n+1}(0)$. Define Q by $dQ/dP=\zeta(T)$. By (i), $\zeta B_iB_{n+1}(0)/(B_{n+1}B_i(0))$ is a P-martingale, hence $B_iB_{n+1}(0)/(B_{n+1}B_i(0))$ is a Q-martingale. So, the desired expectation inequality holds with equality. Now, with $m=n$, apply Theorem 1 in measure Q. Condition (i) there holds, so condition (ii) and (iii) there are valid, yielding (19) , (20) and (23).

(ii) $\Rightarrow$ (i). Set $\zeta_t=E_t[dQ/dP]$. With $m=n$, apply Theorem 1 in measure Q. Eq. (19) and (20) show condition (ii) in Theorem 1 holds; so condition (i) there is valid, stating all $B_iB_{n+1}(0)/(B_{n+1}\,B_i(0))$ are Q-local martingales. As they are positive, they are Q-super martingales. The assumed expectation inequality now implies they are Q-martingales. Hence $\xi\equiv\zeta B_{n+1}(0)/B_{n+1}$ is a state price density.

(i) $\Leftrightarrow$ (iii). Condition (iii) here is identical to condition (ii) in Theorem 1, with $m=n+1$ and $X_m=\zeta$. But (ii) in Theorem 1 is equivalent to condition (i) there, stating $\zeta B_iB_{n+1}(0)/(B_{n+1}B_i(0))$ are all local martingales. Further, as argued above, they will martingales iff the expectation inequality in (iii) here holds. This completes the equivalence. In addition, Eq. (24) follows, as this is the condition (iii) of Theorem 1.

There remains to show the very last statement in Eq. (25). By assumption both $h*\mu$ and $[h*\mu,\mathcal{L}(\zeta)]=(h\,x_{n+1})*\mu_{(L,\mathcal{L}\zeta)}$ are P-special. Hence, by Proposition 1, $h*\mu$ is Q-special, and by the last equation in its proof, we have $h*\nu^Q=h*\nu+(h\psi)*\nu$. □

As in the previous section, a substantial simplification results for the case of special semimartingales, at the expense only of some mild local integrability assumptions.

Theorem 5. Let $B_i>0$ be semimartingales, $i=1,\ldots,n+1$. Assume L_i, as defined in (18), is a special semimartingale for all $i=1,\ldots,n$. Then, the following five conditions are equivalent.

(i) There exists a state price density ξ for B, and $[L_i,\mathcal{L}(\xi B_{n+1})]$ is special for all i.

(ii) There exists a measure Q equivalent to P (called the *terminal measure*) such that for all $i=1,\ldots,n$, $E^Q[B_i(T)B_{n+1}(0)/(B_{n+1}(T)B_i(0))]\geq 1$ and the following two equations hold.

(26) $$\left| x_i\Big(\prod_{j=i+1}^{n}\Big(1+\frac{\delta_j x_j}{1+\delta_j L_{j-}}\Big)-1\Big)\right| * \nu_L^Q<\infty \qquad \text{a.s.}$$

(27) $$L_i=-\sum_{j=i+1}^{n}\int\frac{\delta_j\,d[L_i^c,L_j^c]}{1+\delta_j L_{j-}}-\Big(x_i\Big(\prod_{j=i+1}^{n}\Big(1+\frac{\delta_j x_j}{1+\delta_j L_{j-}}\Big)-1\Big)\Big)*\nu_L^Q+L_i^{c,Q}+x_i*(\mu_L-\nu_L^Q).$$

(iii) There exists a martingale $\zeta > 0$ such that $\zeta(0) = 1$, and, for all $i = 1,\dots,n$, $[L_i, \mathcal{L}(\zeta)]$ is special, $E[B_i(T)\zeta(T)B_{n+1}(0)/(B_{n+1}(T)B_i(0))] \geq 1$, and the following two equations hold.

$$\left| x_i\left(\prod_{j=i+1}^{n}\left(1+\frac{\delta_j x_j}{1+\delta_j L_{j-}}\right)-1\right)(1+x_{n+1})\right| * \nu_{(L,\mathcal{L}\zeta)} < \infty \quad \text{a.s.} \tag{28}$$

$$L_i = -[L_i, \mathcal{L}\zeta]^p - \sum_{j=i+1}^{n}\int \frac{\delta_j\, d[L_i^c, L_j^c]}{1+\delta_j L_{j-}} - \left(x_i\left(\prod_{j=i+1}^{n}\left(1+\frac{\delta_j x_j}{1+\delta_j L_{j-}}\right)-1\right)(1+x_{n+1})\right) * \nu_{(L,\mathcal{L}\zeta)} + L_i^c + x_i * (\mu_L - \nu_L)\,. \tag{29}$$

(ii)[d] The same condition as (ii) above, but with μ_L and ν_L^Q in (26) and (27) replaced by μ_{L^d} and $\nu_{L^d}^Q$ respectively.

(iii)[d] The same condition as (iii) above, but with μ_L, ν_L and $\nu_{(L,\mathcal{L}\zeta)}$ in (28) and (29) replaced by μ_{L^d}, ν_{L^d} and $\nu_{(L^d,(\mathcal{L}\zeta)^d)}$ respectively.

Moreover, if these conditions hold, then $B_iB_{n+1}(0)/(B_{n+1}B_i(0))$ are Q-martingales, and Eq. (23), (24), and (25) hold, as does the following modification of Eq. (24).

$$\mathcal{L}\left(\frac{\xi B_i}{B_i(0)}\right) = \mathcal{L}\zeta + \sum_{j=i}^{n}\int \frac{\delta_j dL_j^c}{1+\delta_j L_{j-}} + \left(\left(\prod_{j=i}^{n}\left(1+\frac{\delta_j x_j}{1+\delta_j L_{j-}}\right)-1\right)(1+x_{n+1})\right) * (\mu_{(L,\mathcal{L}\zeta)} - \nu_{(L,\mathcal{L}\zeta)})\,; \tag{24$'$}$$

Proof. Set $X_i = (1+\delta_i L_i),\ i = 1,\dots,n$. Note, $\prod_{j=1}^{n} X_j = B_i / B_{n+1}$.

(i) $\Rightarrow$ (ii). Set $\zeta = \xi B_{n+1}/B_{n+1}(0)$. Define Q by $dQ/dP = \zeta(T)$. By (i), $\zeta B_iB_{n+1}(0)/(B_{n+1}B_i(0))$ is a P-martingale, hence $B_iB_{n+1}(0)/(B_{n+1}B_i(0))$ is a Q-martingale. So, the desired expectation inequality holds with equality. By Proposition 1, (i) also implies L_i is Q-special. So, $L_i^{d,Q} = x_i * (\mu_L - \nu_L^Q)$. Now, with $m = n$, apply Theorem 2 in measure Q. Condition (i) there holds, so condition (ii) and (iii) there are valid, yielding (26), (27), and (23).

(ii) $\Rightarrow$ (i). Set $\zeta_t = E_t[dQ/dP]$. With $m = n$, apply Theorem 2 in measure Q. Eq. (26) and (27) show condition (ii) in Theorem 2 holds; so (i) there is valid, stating all $B_iB_{n+1}(0)/(B_{n+1}B_i(0))$ are Q-local martingales. As they are positive, they are Q-super martingales. The assumed expectation inequality now implies they are Q-martingales. Hence $\xi \equiv \zeta B_{n+1}(0)/B_{n+1}$ is a state price density. Proposition 1, part (ii) implies $[L_i, \mathcal{L}(\xi B_{n+1})]$ is special.

(i) $\Leftrightarrow$ (iii). Eq. (28) and the assumption that $[L_i, \mathcal{L}(\zeta)]$ is special allow us to write

$$[L_i, \mathcal{L}\zeta]^p + \left(x_i\left(\prod_{j=i+1}^{n}\left(1+\frac{\delta_j x_j}{1+\delta_j L_{j-}}\right)-1\right)(1+x_{n+1})\right) * \nu_{L,\mathcal{L}\zeta} = \left(x_i\left(\prod_{j=i+1}^{n}\left(1+\frac{\delta_j x_j}{1+x_j L_{j-}}\right)(1+x_{n+1})-1\right)\right) * \nu_{L,\mathcal{L}\zeta} + [L_i, (\mathcal{L}\zeta)^c].$$

Make this substitution in Eq. (29). The resulting formula is identical to condition (ii) in Theorem 2, with $m = n+1$ and $X_m = \zeta$. But (ii) in Theorem 2 is equivalent to condition (i) there, stating $\zeta B_i B_{n+1}(0)/(B_{n+1}B_i(0))$ all are local martingales. Further, as argued above, they will martingales iff the expectation inequality in (iii) here holds. This completes the equivalence. In addition, Eq. (24)′ follows, as this is condition (iii) of Theorem 2.

(i) $\Leftrightarrow$ (ii)^d $\Leftrightarrow$ (iii)^d. These follow exactly as above, except that condition (ii)^d of Theorem 2 is applied instead of condition (ii) there.

Finally, the various relationships in Eq. (25) are already shown, save for the last statement, which follows similarly as in (or just from) Theorem 4, using Proposition 1. □

We now consider the case L is driven by a multidimensional (column-vector) Brownian motion W and an integer-valued random measure μ on $\mathbb{R}\times E$, E a Lusin space, (e.g., $E = \mathbb{R}^d$ or $E = \mathbb{R}\times\{1,\ldots,d\}$), with compensator ν. As in Sec. 2, we assume $\nu(\omega,\{t\}\times E) = 0$ identically (or equivalently, $\nu(\omega,dt,dx)$ is dt-absolutely continuous). Let $\mathcal{H}(\mu)$ denote the set of all pairs (α,ψ) where α is a predictable row-vector process satisfying $\int|\alpha|^2\,dt < \infty$ a.s., and $\psi(\omega,t,x)$ is a predictable function $\psi\in G_{\text{loc}}(\mu)$. Then, $\int \alpha\, dW + \psi * (\mu-\nu)$ will be a local martingale. Again, the key conditions to be applied from Theorem (5) are (ii)^d and (iii)^d, rather than (ii) and (iii).

Theorem 6. Let $B_i > 0$ be semimartingales, $i = 1,\ldots,n+1$. Let W, μ, ν, and $\mathcal{H}(\mu)$ be as above. Assume L_i, as defined in (18) satisfies, for all $i = 1,\ldots,n$,

(30) $$L_i = L_i(0) + A_i + \int \beta_i\, dW + \phi_i * (\mu-\nu), \qquad i = 1,\ldots,n,$$

where A_i is some predictable process of finite variation and $(\beta_i, \phi_i) \in \mathcal{H}(\mu)$. Then the following three conditions are equivalent.

(i) There exists a state price density ξ for B such that if we set $\zeta = \xi B_{n+1}/B_{n+1}(0)$ then $\mathcal{L}(\zeta) = \int \alpha\, dW + \psi * (\mu-\nu)$ for some $(\alpha,\psi)\in\mathcal{H}(\mu)$ satisfying $\psi > -1$ and $|\phi_i\psi| * \nu < \infty$.

(ii) There exists a measure Q equivalent to P and a Q-Brownian motion W^Q such that $E^Q[B_i(T)B_{n+1}(0)/(B_{n+1}(T)B_i(0))] \geq 1$, $\mathcal{L}(\zeta) = \int \alpha\, dW + \psi * (\mu-\nu)$ for some $(\alpha, \psi)\in\mathcal{H}(\mu)$ satisfying $\psi > -1$, where $\zeta_t = E_t[dQ/dP]$, and the following two equations hold for $i = 1,\ldots,n$.

(31) $$\left|\phi_i\Big(\prod_{j=i+1}^{n}\Big(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}}\Big)-1\Big)\right| * \nu^Q < \infty \qquad \text{a.s.}$$

(32) $$L_i = L_i(0) - \int \sum_{j=i+1}^{n}\frac{\delta_j\beta_i\beta_j^{\mathrm{t}}}{1+\delta_j L_{j-}}\,dt - \Big(\phi_i\Big(\prod_{j=i+1}^{n}\Big(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}}\Big)-1\Big)\Big) * \nu^Q + \int \beta_i\, dW^Q + \phi_i * (\mu-\nu^Q).$$

(iii) There exists $(\alpha,\psi)\in \mathcal{H}(\mu)$, $\psi > -1$, such that $E[\zeta(T)] \geq 1$, and for all $i = 1,\ldots,n$, $E[B_i(T)\zeta(T)B_{n+1}(0)/(B_{n+1}(T)B_i(0))] \geq 1$, where $\zeta = \mathcal{E}(\int \alpha\, dW + \psi * (\mu - \nu))$, and the following two equations hold.

(33) $$|\phi_i\psi| * \nu < \infty \quad \text{a.s.}, \qquad \left|\phi_i(\prod_{j=i+1}^{n}(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}})-1)(1+\psi)\right| * \nu < \infty \quad \text{a.s.}$$

(34) $$L_i = L_i(0) - \int \beta_i(\alpha^{\mathrm{t}} + \sum_{j=i+1}^{n}\frac{\delta_j\beta_j^{\mathrm{t}}}{1+\delta_j L_{j-}})dt - (\phi_i\psi) * \nu - (\phi_i(\prod_{j=i+1}^{n}(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}})-1)(1+\psi)) * \nu + \int \beta_i\, dW + \phi_i * (\mu - \nu).$$

Moreover, if these conditions hold, then $B_iB_{n+1}(0)/(B_{n+1}B_i(0))$ are Q-martingales, and

(35) $$\mathcal{L}(\frac{B_iB_{n+1}(0)}{B_i(0)B_{n+1}}) = \int\sum_{j=i}^{n}\frac{\delta_j\beta_j dW^Q}{1+\delta_j L_{j-}} + (\prod_{j=i}^{n}(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}})-1) * (\mu - \nu^Q);$$

(36) $$\mathcal{L}(\frac{\xi B_i}{B_i(0)}) = \mathcal{L}(\zeta) + \int\sum_{j=i}^{n}\frac{\delta_j\beta_j}{1+\delta_j L_{j-}}dW + ((\prod_{j=i}^{n}(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}})-1)(1+\psi)) * (\mu - \nu);$$

and the following relationships hold: the ζ, α, ψ in (i), (ii) and (iii) are the same, and

(37) $$\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = \zeta_t = \xi_t\frac{B_{n+1}(t)}{B_{n+1}(0)}, \qquad W^Q = W - \int \alpha^{\mathrm{t}}\, dt, \qquad h * \nu^Q = (h(1+\psi)) * \nu,$$

for any predictable function $h(\omega,t,x_1,\ldots,x_n)$ such that $(|h| + |h\psi|) * \nu < \infty$; and if further $\left|\psi(\prod_{j=i}^{n}(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}})-1)\right| * \nu < \infty$, then

(38) $$\mathcal{L}(\frac{B_iB_{n+1}(0)}{B_i(0)B_{n+1}}) = -\int\sum_{j=i}^{n}\frac{\delta_j\beta_j\alpha^{\mathrm{t}}}{1+\delta_j L_{j-}}dt - (\psi(\prod_{j=i}^{n}(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}})-1)) * \nu + \int(\sum_{j=i}^{n}\frac{\delta_j\beta_j}{1+\delta_j L_{j-}})\, dW + (\prod_{j=i}^{n}(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}})-1) * (\mu - \nu).$$

Proof. Applying the equivalence between $(\mathrm{i})^{\mathrm{d}}$, $(\mathrm{ii})^{\mathrm{d}}$, $(\mathrm{iii})^{\mathrm{d}}$ in Theorem 5, all that remains to be shown is (a) that the various brackets in the formulae of Theorem 5 reduce to the formulae of this theorem, and (b) that in the implication (i) $\Rightarrow$ (ii), $L_i^{c,Q} = \int \beta_i dW^Q$ for some Q-Brownian motion W^Q, and (c) Eq. (38). But, (a) is clear (e.g., in (iii), $[L_i, \mathcal{L}(\zeta)]^p = \int \beta_i\alpha^{\mathrm{t}}\, dt + (\phi_i\psi) * \nu$.) As for (b), set $W^Q = W - \int \alpha^{\mathrm{t}}dt$. Set $M = \int \zeta\, dW$. Note $M = \int \zeta\, dW^Q + [W^Q,\zeta]$. Hence by the product rule, $\zeta W^Q = M + \int W^Q d\zeta$. So, ζW^Q is P-local martingale. It follows W^Q is a Q-local martingale. So, $L_i^{c,Q} = \int \beta_i dW^Q$. Also, since $[W^Q, W^Q] = t$, by Levy's characterization W^Q is a Q-Brownian motion. Finally, Eq. (38) follows, for instance, from (35) and (37). □

It is interesting that from basically mere assumption that ξB_i are (local) martingales, we get a such rather nice representation of these same (local) martingales as in Eq. (36).

We now proceed with the dynamics in the *spot measure*. Whereas the terminal measure above had B_{n+1} as numeraire, the numeraire of the spot measure is B_1. This may come as a surprise, but we will explain it below. First, however, we present the result.

For the sake of brevity, we forgo the analogues of Theorem 4 and Theorem 5, which provide a representation in terms of characteristics, and present only the analogue of Theorem 6, which provides a similar representation in term of a Brownian motion and random measure. Indeed, the ideas and formulae are similar, as we just use Theorem 3 instead of Theorems 1 and 2.

Theorem 7. Let $B_i > 0$ be semimartingales, $i = 1,\ldots,n+1$. Set $B^* = B_1/B_1(0)$. Let W, μ, ν, and $\mathcal{H}(\mu)$ be as above. Assume L_i, as defined in (18) satisfies, for all $i = 1,\ldots,n$,

(39) $$L_i = L_i(0) + A_i + \int \beta_i \, dW + \phi_i * (\mu - \nu), \qquad i = 1,\ldots,n,$$

where A_i is some predictable process of finite variation and $(\beta_i, \phi_i) \in \mathcal{H}(\mu)$. Then the following three conditions are equivalent.

(i) There exists a state price density ξ for B such that if we set $\zeta^* = \xi B^*$ then $\mathcal{L}(\zeta^*) = \int \alpha^* dW + \psi^* * (\mu - \nu)$ for some $(\alpha^*, \psi^*) \in \mathcal{H}(\mu)$ satisfying $\psi^* > -1$ and $|\phi_i \psi^*| * \nu < \infty$.

(ii) There exists a measure P^* equivalent to P and a P^*-Brownian motion W^* such that $E^{P^*}[B_i(T)/(B^*(T)B_i(0))] \geq 1$, $\mathcal{L}(\zeta^*) = \int \alpha^* dW + \psi^* * (\mu - \nu)$ for some $(\alpha^*, \psi^*) \in \mathcal{H}(\mu)$ satisfying $\psi > -1$, where $\zeta_t^* = E_t[dP^*/dP]$, and the following two equations hold for $i = 1,\ldots,n$.

(40) $$\left| \phi_i \Big(\prod_{j=1}^{i} \big(1 + \frac{\delta_j \phi_j}{1 + \delta_j L_{j-}}\big)^{-1} - 1 \Big) \right| * \nu^{P^*} < \infty \quad \text{a.s.}$$

(41) $$L_i = L_i(0) + \sum_{j=1}^{i} \int \frac{\delta_j \beta_i \beta_j^{\mathrm{t}}}{1 + \delta_j L_{j-}} dt - \Big(\phi_i \Big(\prod_{j=1}^{i} \big(1 + \frac{\delta_j \phi_j}{1 + \delta_j L_{j-}}\big)^{-1} - 1\Big)\Big) * \nu^{P^*} + \int \beta_i \, dW^* + \phi_i * (\mu - \nu^{P^*}).$$

(iii) There exists $(\alpha^*, \psi^*) \in \mathcal{H}(\mu)$, $\psi^* > -1$, such that $E[\zeta^*(T)] \geq 1$, and for all $i = 1,\ldots,n$, $E[B_i(T)\zeta^*(T)/(B^*(T)B_i(0))] \geq 1$, where $\zeta^* = \mathcal{E}(\int \alpha^* dW + \psi^* * (\mu - \nu))$, and the following hold.

(42) $$|\phi_i \psi^*| * \nu < \infty \quad \text{a.s.}, \qquad \left| \phi_i \Big(\prod_{j=1}^{i} \big(1 + \frac{\delta_j \phi_j}{1 + \delta_j L_{j-}}\big)^{-1} - 1 \Big)(1 + \psi^*) \right| * \nu < \infty \quad \text{a.s.}$$

(43) $$L_i = L_i(0) + \int \beta_i(-\alpha^{*t} + \sum_{j=1}^{i} \frac{\delta_j \beta_j^t}{1+\delta_j L_{j-}})dt - (\phi_i \psi^*) * \nu - (\phi_i (\prod_{j=1}^{i}(1+\frac{\delta_j \phi_j}{1+\delta_j L_{j-}})^{-1} - 1)(1+\psi^*)) * \nu + \int \beta_i dW + \phi_i * (\mu - \nu) .$$

Moreover, if these conditions hold, then $B_i/(B^*B_i(0))$ are P^*-martingales, and

(44) $$\mathcal{L}(\frac{B_{i+1}}{B_{i+1}(0)B^*}) = -\int \sum_{j=1}^{i} \frac{\delta_j \beta_j dW^*}{1+\delta_j L_{j-}} + (\prod_{j=1}^{i}(1+\frac{\delta_j \phi_j}{1+\delta_j L_{j-}})^{-1} - 1) * (\mu - \nu^{P^*}) ;$$

(45) $$\mathcal{L}(\frac{\xi B_{i+1}}{B_{i+1}(0)}) = \mathcal{L}(\zeta^*) - \int \sum_{j=1}^{i} \frac{\delta_j \beta_j}{1+\delta_j L_{j-}} dW + ((\prod_{j=1}^{i}(1+\frac{\delta_j \phi_j}{1+\delta_j L_{j-}})^{-1} - 1)(1+\psi^*)) * (\mu - \nu) ;$$

and the following relationships hold: the ζ^*, α^*, ψ^* in (i), (ii) and (iii) are the same, and

(46) $$\frac{dP^*}{dP}\Big|_{\mathcal{F}_t} = \zeta_t^* = \xi_t B_t^*, \qquad W^{P^*} = W - \int \alpha^{*t} dt, \qquad h * \nu^{P^*} = (h(1+\psi^*)) * \nu ,$$

for any predictable function $h(\omega,t,x_1,\ldots,x_n)$ such that $(|h| + |h\psi^*|) * \nu < \infty$; and if further $|\psi^*(\prod_{j=1}^{i}(1+\frac{\delta_j \phi_j}{1+\delta_j L_{j-}})^{-1} - 1)| * \nu < \infty$, then

(47) $$\mathcal{L}(\frac{B_{i+1}}{B_{i+1}(0)B^*}) = \int \sum_{j=1}^{i} \frac{\delta_j \beta_j \alpha^{*t}}{1+\delta_j L_{j-}} dt - (\psi^* (\prod_{j=1}^{i}(1+\frac{\delta_j \phi_j}{1+\delta_j L_{j-}})^{-1} - 1)) * \nu + - \int (\sum_{j=1}^{i} \frac{\delta_j \beta_j}{1+\delta_j L_{j-}}) dW + (\prod_{j=1}^{i}(1+\frac{\delta_j \phi_j}{1+\delta_j L_{j-}})^{-1} - 1) * (\mu - \nu) .$$

Proof. (i) $\Rightarrow$ (ii). Define P^* by $dP^*/dP = \zeta^*(T)$. By (i), $\zeta^* B_i/(B^*B_i(0))$ is a P-martingale, hence $B_i/(B^*B_i(0))$ is a P^*-martingale. So, the desired expectation inequality holds in fact with equality. Set $W^* = W - \int \alpha^{*t}dt$. Set $M = \int \zeta^* dW$. Note, $M = \int \zeta^* dW^* + [W^*, \zeta^*]$. Thus, $\zeta^* W^* = M + \int W^* d\zeta^*$. So, $\zeta^* W^*$ is P-local martingale. It follows W^* is a P^*-local martingale. Hence, $L_i^{c,P^*} = \int \beta_i dW^*$. Also, since $[W^*, W^*] = t$, by Levy's characterization W^* is a P^*-Brownian motion. Since $|\phi_i \psi^*| * \nu < \infty$, by Proposition 1, L_i is P^*-special. So, $L_i^{d,P^*} = x_i * (\mu_L - \nu_L^{P^*}) = \phi_i * (\mu - \nu^{P^*})$. Now, set $X_i = (1+\delta_i L_i)$, $i = 1,\ldots,n$. Note, $\prod_{j=1}^{i} X_j^{-1} = B_{i+1}/B_1$, which we showed is a P^*-local martingale. Hence condition (i) in Theorem 3, applied to measure P^* and with $m = n$, holds. So, conditions $(\text{ii})^d$ and $(\text{iii})^d$ there hold, yielding (40), (41), and (44).

(ii) $\Rightarrow$ (i). Since L_i are special under both P and P^*, by Proposition 1, $|\phi_i \psi^*| * \nu < \infty$. With $m = n$ and X_i as above, apply Theorem 3 in measure P^*. Condition $(\text{ii})^d$ in Theorem 3 holds, so (i) there is valid, stating all $B_i/(B^*B_i(0))$ are P^*-local martingales. As they are positive, they are P^*-super martingales. The assumed expectation inequality now implies they are P^*-martingales. Hence $\xi \equiv \zeta^*/B^*$ is a state price density.

(i) $\Leftrightarrow$ (iii). Set $\eta = -\psi^*/(1+\psi^*)$. Since $1+\psi^* = (1+\eta)^{-1}$, by (42) we can write

$$(\phi_i\psi^*)*\nu + (\phi_i(\prod_{j=1}^{i}(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}})^{-1} - 1)(1+\psi^*))*\nu = (\phi_i((1+\eta)^{-1}\prod_{j=1}^{i}(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}})^{-1} - 1))*\nu .$$

Make this substitution in Eq. (43). Now apply Theorem 3 with $m = n + 1$, $X_1 = 1/\zeta^*$, $X_{i+1} = (1+\delta_i L_i),\ i = 1,\dots,n$. Noting that $\prod_{j=1}^{i} X_j^{-1} = \zeta^* B_i / B_1$, $(\mathcal{L}X_1)^c = -\int \alpha^* dW$, and $(\mathcal{L}X_1)^d = \eta*(\mu-\nu)$, the equivalence between (i) and (ii)^d in Theorem 3 shows that $\zeta^* B_i / B_1$ are local martingales iff (42) and (43) hold. Further, as argued above, they will martingales iff the expectation inequality in (iii) here holds. This completes the equivalence. In addition, Eq. (45) follows, as this is condition (iii)^d in Theorem 3.

As for the last statement in Eq. (46), by assumption both $h*\mu$ and $[h*\mu, \mathcal{L}(\zeta)] = (h x_{n+1})*\mu_{(L,\mathcal{L}\zeta)}$ are P-special. Hence, by Proposition 1, $h*\mu$ is P^*-special, and by the equation in its proof, we have $h*\nu^Q = h*\nu + (h\psi)*\nu$. Finally, this and (44) imply (47). □

The terminal and spot measures each have their own advantages, although, the former is often more convenient. At any case, it is easy to move between them.

Corollary. The three equivalent conditions in Theorem 6 are equivalent to the three equivalent conditions in Theorem 7, provided both $|\phi_i\psi^*|*\nu < \infty$ and $|\phi_i\psi|*\nu < \infty$ for all i. In this case, the following relationships hold among the various quantities.

(48) $$\alpha = \alpha^* - \sum_{j=1}^{n}\frac{\delta_j\beta_j}{1+\delta_j L_{j-}}; \qquad \psi = (1+\psi^*)\prod_{j=1}^{n}(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}})^{-1} - 1;$$

(49) $$\alpha^* = \alpha + \sum_{j=1}^{n}\frac{\delta_j\beta_j}{1+\delta_j L_{j-}}; \qquad \psi^* = (1+\psi)\prod_{j=1}^{n}(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}}) - 1;$$

(50) $$\frac{dQ}{dP^*}\Big|_{\mathcal{F}_t} = \frac{B_{n+1}(t)}{B_{n+1}(0)B^*(t)}, \quad W^Q = W^{P^*} + \sum_{j=1}^{n}\frac{\delta_j\beta_j^{\mathrm{t}}}{1+\delta_j L_{j-}}, \quad \frac{d\nu^Q}{d\nu^{P^*}} = \frac{1+\psi}{1+\psi^*} = \prod_{j=1}^{n}(1+\frac{\delta_j\phi_j}{1+\delta_j L_{j-}})^{-1} .$$

Proof. Assume condition (i) of Theorem 6. Then Eq. (36), with $i = 1$, implies condition (i) of Theorem 7. The converse is similar. Eq. (48) follows by setting $i = n$ in (45), and Eq. (49) follows by setting $i = 1$ in (36). Eq. (50) follows by comparing (46) and (37), and using (48). □

The above theorems represent the core of our theoretical results. Let us now discuss how to adopt them in the applied context. In reality, bond prices and Libor rates are defined until a maturity, after which they cease to exist. That is, one is given a *term structure*, i.e., a sequence

$$0 = T_0 < T_1 < \cdots < T_{n+1} = T \ ,$$

with T_i representing the maturity of the i-th bond (whose price is B_i). As such, B_i and L_i are defined only on $[0,T_i]$. (Also, in practice $\delta_i \cong T_{i+1} - T_i$.) One further requires $B_i(T_i)=1$, but for now we leave this open and return to it later. In such a situation, our definition of a state price density ξ must be modified to state that each ξB_i is a martingale on $[0, T_i]$.

In this situation, it is clear that Theorem 4, 5, and 6, which are with respect to B_{n+1} as "numeraire", continue to hold with only a slight modification: all that is required is to qualify each formula involving B_i and L_i to hold only on $[0, T_i]$. But, such a modification to Theorem 7 makes it void past T_1, for all its statements refer to B_1, which is defined on only $[0, T_1]$.

Fortunately, all is not lost. As B_i is defined only $[0,T_i]$, no portfolio can include the i-th bond past T_i. Consequently, if we extend B_i on $(T_i,T]$ in any manner, subject only to it staying a semimartingale, the price of the portfolio does not change, as it has zero weight in B_i on $(T_i,T]$. This implies that for all practical purposes we may extend B_i to the whole of $[0,T]$ as we see fit. Once this is done, our former definition of state price density can be retained, and Theorems 4-7 continue to hold on all of $[0,T]$.

The choice of extension has no significant bearing on Theorems 4, 5, 6 because B_{n+1}, being already defined on $[0,T]$, is not affected by such extensions. However, Theorem 7 has different implications according to the chosen extension of B_i. One possible way to extend B_i is to make it proportional to B_{n+1} on $(T_i,T]$, i.e., to set $B_i(t) = B_{n+1}(t)B_i(T_i)/B_{n+1}(T_i)$ for $t > T_i$. But, this is not so useful, as Theorem 7 then becomes essentially identical to Theorem 6.

A more fruitful way is to extend B_i by making it proportional to B_{i+1} on (T_i,T_{i+1}), and proportional to B_{i+2} on (T_{i+1},T_{i+2}), and so on. More precisely, we extend B_i as follows

$$B_i(t) = B_{j+1}(t)\prod_{k=i}^{j}\frac{B_k(T_k)}{B_{k+1}(T_k)}, \quad t \in [T_j, T_{j+1}), \quad 1 \le i \le j \le n, \quad B_i(T_{n+1}) = B_{n+1}(T_{n+1})\prod_{k=i}^{n}\frac{B_k(T_k)}{B_{k+1}(T_k)}.$$

This (right-continuous) extension makes B_i a semimartingale on all of $[0,T]$. By design,

$$L_i(t) = L_i(T_i), \quad t \ge T_i.$$

In other words, this extension is uniquely characterized by making L_i constant after its maturity T_i . This implies B_{i+1}/B_1 is constant on $[T_i,T]$. Further, B_1 is now given globally on [0,T] by

(51) $$B_1(t) = B_{i^*(t)}(t)\prod_{j=1}^{i^*(t)-1}\frac{B_j(T_j)}{B_{j+1}(T_j)} = B_{i^*(t)}(t)\prod_{j=1}^{i^*(t)-1}(1+\delta_j L_j(T_j)), \qquad 0 < t \le T = T_{n+1},$$

where $i^*(t) = \min\{i : t \le T_i\}$. Note, the function $i^* : [0,T] \to \{0,1,\ldots,n+1\}$ is left continuous and

$$T_{i^*(t)-1} < t \le T_{i^*(t)} = 0, \qquad i^*(T_i) = i \ .$$

Thus, the extended B_1 is interpreted as the value of a bond from investing a given amount $B_1(0)$ a time 0 at spot Libor rate $L_i(0)$, and at T_1 reinvesting the principal and interest at the prevailing spot Libor rate $L_1(T_1)$, and so on. It is the analogue of the continuously compounded "money market" numeraire encountered in models with a continuum of bonds. But, in the present context, the latter is not necessarily defined, and even if it were, it arguably does not play a significant role. To recap, with this extension, B_1 represents the value of a "money market" fund, rolled over and reinvested each term, at the spot Libor rate. The extension of all the other B_i admit a similar interpretation, because, as noted above, B_i is proportional to B_1 after its maturity T_i. (in fact already after T_{i-1}).

With this particular extension of B_i, Theorem 7 becomes useful. The *spot-Libor measure* P^* there, defined by $dP^*/dP = \xi(T)B^*(T)$, where $B^* = B_1/B_1(0)$, represents the appropriate counterpart of the *risk-neutral measure* encountered in the continuously compounded framework. In fact, the risk-neutral measure does not necessarily exist in the present setting. It exists only when the (or a) state-price density ξ is a special semimartingale. In that case, $\xi = M/R$ for some local martingale $M>0$ and a predictable process $R > 0$. If further M is a martingale, then the measure associated with it is the (or a) risk-neutral measure. Even then, there may not exist any self-financing portfolio with price R. But, the spot and forward measures do not require any such assumptions on ξ: they exist provided only ξ exists. In the special case that B^* is predictable, the risk-neutral measure exists (provided ξ exists), and in fact coincides with the spot measure (simply take $R = B^*$, $M = \xi B^*$.) But, the assumption that B^* is predictable is by no means necessary to the development of this theory. In fact, it is not even required that B^* (or ξ) be special. This indicates that the spot-Libor measure P^* is a more natural concept than the risk-neutral measure in this finite-dimensional theory.

Returning to Theorem 7, note Eq. (41) can now be written in differential form as

$$(41)' \quad dL_i = \sum_{j=i^*(t)}^{i} \frac{\delta_j \beta_i \beta_j^{t}}{1+\delta_j L_{j-}} dt - d(\phi_i(\prod_{j=i^*(t)}^{i} (1+\frac{\delta_j \phi_j}{1+\delta_j L_{j-}})^{-1} - 1)) * \nu^{P^*} + \beta_i dW^* + d\phi_i * (\mu - \nu^{P^*}).$$

The point here is that the lower index of the sum and the product have been changed from $j = 1$ to $j = i^*(t)$. This is made possible by the assumed extension of B_i . Indeed, for $j < i^*(t)$, L_j is constant on a neighbourhood of t, so $\beta_j(t) = \phi_j(t) = 0$. Therefore, indices $j < i^*(t)$ do not contribute to the sum or the product. We can rewrite Eq. (41) in yet another form:

$$(41)'' \quad L_i(t) = L_i(0) + \sum_{k=1}^{i^*(t)-1} (\sum_{j=1}^{k} \int_{T_{k-1}}^{T_k} \frac{\delta_j \beta_i \beta_j^{t}}{1+\delta_j L_{j-}} ds - \int_{(T_{k-1},T_{k-1}]\times E} \phi_i(\prod_{j=1}^{k} (1+\frac{\delta_j \phi_j}{1+\delta_j L_{j-}})^{-1} - 1)\nu^{P^*}(ds,dx)) +$$

$$\sum_{j=i^*(t)}^{i} \int_{T_{i^*(t)-1}}^{t} \frac{\delta_j \beta_i \beta_j^{t}}{1+\delta_j L_{j-}} ds - \int_{(T_{i^*(t)-1},t]\times E} \phi_i(\prod_{j=i^*(t)}^{i} (1+\frac{\delta_j \phi_j}{1+\delta_j L_{j-}})^{-1} - 1)\nu^{P^*}(ds,dx)) + \int_0^t \beta_i dW^* + \phi_i * (\mu - \nu^{P^*})_t$$

The other equations in Theorem 7 can be written in similar forms. The advantage of (41)′ and (41)″ over (41) is that, even when the B_j are defined only over $[0, T_j]$, they make sense as statements for $t \le T_i$. Moreover, B^* can be defined by the right hand side of (51), even when

B_j are defined only over [0, T_j]. Thus, Theorem 7, stated in either form $(41)'$ or $(41)''$ does not require any extension of B_i, and as such becomes a true counterpart of Theorem 6. However, these forms have disadvantages too. While comprehensive, $(41)''$ is clearly too long. And $(41)'$, as short hand for $(41)''$, masks its true complications, and so may be misleading.

To summarize, given that B_i is defined over [0, T_i], the exposition for the spot measure encounters inevitable complications. One has three choices. (i) Extend B_i to [0, T] as above, and use the form in Theorem 7; (ii) use the differential form $(41)'$ without any extension; (iii) use the integral form $(41)''$ without any extension. Here, we opted for (i) for the main exposition, and presented (ii) and (iii) more briefly. Although choice (i) relies on a particular extension of B_i, this extension is natural, and the convenience gained makes it a good choice.

5. Miscellaneous Properties and Discussion

1. Positivity. One often requires L_i to be positive. Note, if $\beta_i = L_{i-}\lambda_i$ for some bounded predictable process λ_i, and $\phi_i = L_{i-}\gamma_i$ for some $\gamma_i \in G_{\text{loc}}(\mu)$, then Eq. (32) is equivalent to

$$(32)' \quad L_i = \mathcal{E}(-\int \sum_{j=i+1}^{n} \frac{\delta_j L_{j-}\lambda_i\lambda_j^{\mathrm{t}}}{1+\delta_j L_{j-}} dt - (\gamma_i (\prod_{j=i+1}^{n} (1+\frac{\delta_j L_{j-}\gamma_j}{1+\delta_j L_{j-}}) - 1)) * \nu^Q + \int \lambda_i dW^Q + \gamma_i * (\mu - \nu^Q)).$$

So, for L_i to be positive it is sufficient that if $\gamma_i > -1$. We can similarly rewrite the other equations in Theorems 6 and 7 in terms of β_i and γ_i, e.g., Eq. (41) becomes

$$L_i = \mathcal{E}(\int \sum_{j=1}^{i} \frac{\delta_j L_{j-}\lambda_i\lambda_j^{\mathrm{t}}}{1+\delta_j L_{j-}} dt - (\gamma_i (\prod_{j=1}^{i} (1+\frac{\delta_j L_{j-}\gamma_j}{1+\delta_j L_{j-}})^{-1} - 1)) * \nu^{P*} + \int \lambda_i dW^{P*} + \gamma_i * (\mu - \nu^{P*})).$$

2. Heuristic Discussion of Continuous Compounding Limit. Here, we are dealing with a continuum of semimartingales $B^s > 0$ and continuously compounded instantaneous forward rates $L^s = -\partial \log(B^s)/\partial s$, indexed by $0 \le s \le T$. Take an integer n, and set $\delta = \delta_i = 1/(n+1)$, $T_i = i\delta$, $B_i = B^{T_i}$, $i = 0,\ldots,n+1$. For large n, we can approximate L^s by the forward Libor rates defined in Eq. (18). Proceeding at a formal level, the various sums and products we have encountered can, in the limit, be replaced respectively with integrals and exponential of integrals. For instance, consider the product $\prod_{j=i+1}^{n}(1+\delta_j\phi_j/(1+\delta_j L_{j-}))$ appearing in Eq. (32). Interestingly enough, it looks like simple compounding of interest. So, assuming a notation analogous to that in Theorem 6, we expect the product to converge to $\exp(\int_s^T \phi^u du)$. As such, Eq. (32) and (35), pertaining to the terminal measure Q, heuristically lead to

$$(32)^{\mathrm{c}} \qquad L^s = -\int \beta^s (\int_s^T \beta^u du)^{\mathrm{t}} dt - (\phi^s (\exp(\int_s^T \phi^u du) - 1)) * \nu^Q + \int \beta^s dW^Q + \phi^s * (\mu - \nu^Q).$$

$$(35)^{\mathrm{c}} \qquad \mathcal{L}(\frac{B^s B^T(0)}{B^s(0) B^T}) = \int (\int_s^T \beta^u du) dW^Q + (\exp(\int_s^T \phi^u du) - 1) * (\mu - \nu^Q);$$

The other equations in Theorem 6 admit similar analogues, as do all the equations in Theorem 7. A rigorous argument along these lines must address certain convergence difficulties. For example, consider a "lognormal-jump" model of type $(32)'$ above with deterministic λ_i and γ_i. As a stochastic system of differential equations, Eq. $(32)'$ has a unique solution L, because the terms $L_{j-}/(1+\delta_j L_{j-})$ are bounded functions of L_{j-}, so the linear growth condition for coefficients is satisfied in $(32)'$. But, in the limit $L_{j-}/(1+\delta_j L_{j-})$ tends to L^s_-, causing the coefficients to have quadratic growth. This causes the limit to explode and convergence to fail.

Extensions of Theorems 4 and 5 to continuous compounding face additional challenge. In the Libor case, μ_L is a random measure on $\mathbb{R}\times E$, with $E = \mathbb{R}^n$. But in the continuous-compounding case, E must first be identified as a subset of $\mathbb{R}^{[0,T]}$ with an appropriate topology, then μ_L defined as an appropriate random measure on $\mathbb{R}\times E$. ($\mathbb{R}^{[0,T]}$ denotes the set of real-valued functions on [0,T].) The subset of continuous functions with the sup norm seems an appropriate choice for E, as it corresponds to the continuity of jumps of L^s as a function of s. For $s \in [0,T]$, let x_s denote the function on $\mathbb{R}^{[0,T]}$ defined by $x_s(f) = f(s)$, and $\int_s^T x_u du$ denote the function on E whose value at f is $\int_s^T f(u)du$. Then, tentative analogues of Eq. (27) and (23) are

$$(27)^c \qquad L^s = -\int_s^T [L^{s^c}, L^{u^c}]du - (x_s(\exp(\int_s^T x_u du) - 1)) * \nu_L^Q + L^{s^{c,Q}} + x_s * (\mu_L - \nu_L^Q);$$

$$(23)^c \qquad \mathcal{L}\left(\frac{B^s B^T(0)}{B^s(0)B^T}\right) = \int_u^T L^{u^{c,Q}} du + (\exp(\int_s^T x_u du) - 1) * (\mu_L - \nu_L^Q).$$

<u>3. Market Prices of Risk.</u> From our point of view, the spot-Libor measure is more appropriate than the risk-neutral measure; as such, the quantities α^* and ψ^* in Theorem 7 (defined by $\mathcal{L}(\xi B^*) = \int \alpha^* dW + \psi^* * (\mu - \nu)$) can be advocated to be *market prices of risk*. But, if one pursues the traditional definition, then they can be derived as follows. Suppose

$$dB_i / B_{i-} = \mu_i dt + \sigma_i dW + d(\eta_i * (\mu - \nu)), \qquad d\xi / \xi_- = -r\,dt + \tilde{\alpha} dW + d(\tilde{\psi} * (\mu - \nu)).$$

Calculating ξB_i using the product rule, and then setting the drift to zero, we get

$$\mu_i = r - \tilde{\alpha}\sigma_i - d((\tilde{\psi}\eta_i) * \nu)/dt.$$

As such, $\tilde{\alpha}$ and $\tilde{\psi}$ are market prices of risk. As $B^* = B_1/B_1(0)$ is proportional to B_i on $[T_{i-1}, T_i)$,

$$dB^* / B^*_- = \mu_{i^*(t)} dt + \sigma_{i^*(t)} dW + d(\eta_{i^*(t)} * (\mu - \nu)).$$

Calculating ξB^* using the product rule, and equating terms, we get

$$\alpha^* = \tilde{\alpha} + \sigma_{i*(t)}, \qquad \psi^* = \tilde{\psi} + \eta_{i*(t)} + \tilde{\psi}\eta_{i*(t)}.$$

Substituting into various equations of Theorem 7, we obtain formulae in terms of $\tilde{\alpha}$ and $\tilde{\psi}$.

4. Model Construction. Recall L_i has the form $L_i = L_i(0) + A_i + \int \beta_i dW + \phi_i * (\mu - \nu)$. In practice one is given β_i, ϕ_i, μ, and wishes to construct the system L in the terminal (or spot) measure, in such a way that condition (ii) of Theorem 6 (or 7) holds, i.e., Eq. (32) (or 41) is satisfied. A difficulty not present in the continuous semimartingale case is that while β_i and ν are in principle observable, ν^Q, given by $d\nu^Q = (1+\psi)d\nu$, is not. This is best handled by parameterization, and estimation of parameters by calibration to liquid derivatives such as liquid caps, floors and European swaptions. Such a procedure can also be applied to β_i.

The problem is posed as a n-dimensional system of stochastic differential equation (SDE), that is, it is assumed that β_i, ϕ_i, ψ are given functions of the form $\beta_i(t,L)$, $\phi_i(t,L,x)$, $\psi(t,L,x) > -1$, rather than simply given as exogenous processes. So, one actually has to solve a SDE, rather than just take a stochastic integral. The coefficients $\beta_i(t,L)$, $\phi_i(t,L,x)$, $\psi(t,L,x)$ are defined (or have non-zero values) only on $\{L : L_i > -1/\delta_i\}$, because forward prices must be positive. As Eq. (35) shows, we further need to have $\phi_i(t,L,x) > - L_i - 1/\delta_i$ to ensure the solution to the SDE will stay in the domain $\{L : L_i > -1/\delta_i\}$ at all times.

Except for possible applications to the continuously compounded limit, there seems to be no reason not to require that L_i be positive. Then $\beta_i(t,L)$, $\phi_i(t,L,x)$, $\psi(t,L,x)$ are only defined (or non-zero) for $L > 0$. The appropriate functions to model are coefficients of $\mathcal{L}(L_i)$, i.e., $\lambda_i(t,L,x) = \beta_i(t,L)/L_i$ and $\gamma_i(t,L,x) = \phi_i(t,L,x)/L_i$, as in Eq. (32)′. To ensure the solution $L(t)$ stays positive, one requires $\gamma_i(t,L,x) > -1$. Additionally, the functions should be Lipschitz and appropriately bounded. In practice, a good policy may have the functions λ_i, γ_i independent of L, and possibly also independent of t. A parsimonious model for μ is a Poisson random measure.

5. Unity Constraint $B_i(T_i) = 1$. Once the system L is constructed in the forward (or spot) measure, there is no need to construct the numeraire B_{n+1}, or any other B_i, for determination of a contingent claim price C which is "homogenous", in the sense that C/B_{n+1} is adapted to the filtration generated by L. Indeed, since C/B_{n+1} is determined by L alone, so is $C(0)$, because $B_{n+1}(0)$ is known. Practically all Libor and swap derivative prices are homogenous as above. So, once L is modelled, nothing about bond prices is required for their valuation, beyond $B_{n+1}(0)$. So, in these cases, the question of whether $B_i(T_i) = 1$ does not arise in the first place.

Yet, it is conceivable that one is pricing a non-homogenous payoff (e.g., the payoff $(B_{n+1}(T_n))^2$ at T). At any rate, one may ask whether the constructed system L is consistent with existence of a price system $B > 0$ which (a) satisfies $B_i(T_i) = 1$, (b) has a price density, (c) has the given L as its forward Libor system (i.e., satisfies (18)). The answer is clearly yes. Given L, take any semimartingale $B_{n+1} > 0$, and define $B_i = B_{n+1}\prod_{j=i}^{n}(1+\delta_j L_j)$. Then (c) is satisfied. Also, as L is presumed to satisfy condition (ii) of Theorem 6, (b) follows. To ensure (a), simply choose B_{n+1} such that $B_{n+1}(T_{n+1}) = 1$, and for $i \le n$, $B_{n+1}(T_i) = \prod_{j=i}^{n} 1/(1+\delta_j L_j(T_i))$. Although this choice is not unique, it does not affect evaluation of contingent claims.

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