

# Numeraire Invariance and application to Option Pricing and Hedging

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## 1. INTRODUCTORY SURVEY

### 1.1. Merton's (1973) extension of Black-Scholes.

Let  $C$  be call option on an asset  $A$  with expiry  $T$  payoff

$$C_T = (A_T - K)^+.$$

To *replicate*  $C$ , another asset is needed. Merton's idea is to take as the second asset the zero-coupon bond  $B$  with  $B_T = K$ , and express the payoff in terms of both assets:

$$C_T = (A_T - B_T)^+.$$

The payoff's *homogeneity* allows one to factor out  $B$ :

$$F_T = (X_T - 1)^+,$$

where

$$X := \frac{A}{B}, \quad F := \frac{C}{B},$$

are the *forward prices* of the asset and the option. He replicates the forward option by trading the forward asset:

$$dF_t = \delta_t dX_t.$$

Assuming  $F_t = f(t, X_t)$  for some  $f$ , by **Itô's formula** this is equivalent to a formula for  $\delta$  and a **PDE** for  $f$ :

$$\delta_t = \frac{\partial f}{\partial x}(t, X_t),$$

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_t^2 x^2 \frac{\partial^2 f}{\partial x^2} = 0, \quad f(T, x) = (x - 1)^+,$$

where  $\sigma_t$  is the deterministic **forward-price volatility**:

$$d[X]_t = \sigma_t^2 X_t^2 dt.$$

Thus by “factoring out” one asset, the problem with a stochastic interest rate reduces to a call option struck at 1 in the Black-Scholes model with *zero interest rate*.

More generally, when the asset pays dividends at a constant rate  $y$ , the above applies with the forward asset price

$$X_t = e^{-y(T-t)} \frac{A_t}{B_t}.$$

## 1.2. Margrabe's extension to Exchange Options.

Margrabe (1978) showed that Merton's argument extends to an option to exchange any two assets  $A$  and  $B$ . His idea was to replicate the exchange option according to the SDE

$$dC_t = \delta_t^A dA_t + \delta_t^B dB_t.$$

Assuming  $C_t = c(t, A_t, B_t)$  for some function  $c(t, a, b)$ , he noted that by *Itô's formula* this is equivalent to the system

$$\delta_t^A = \frac{\partial c}{\partial a}(t, A_t, B_t), \quad \delta_t^B = \frac{\partial c}{\partial b}(t, A_t, B_t),$$

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma_A^2 a^2 \frac{\partial^2 c}{\partial a^2} + \frac{1}{2}\sigma_B^2 b^2 \frac{\partial^2 c}{\partial b^2} + \sigma_A \sigma_B \rho ab \frac{\partial^2 c}{\partial a \partial b} = 0,$$

(at least when  $|\rho| \neq 1$ ) where  $\sigma_A$  and  $\sigma_B$  are the volatilities of  $A$  and  $B$  and  $\rho$  is their correlation. By *Euler's formula*, the **deltas** (hedge ratios) are **self financing** if and only if  $C(t, a, b)$  is **homogenous** of degree 1 in  $(a, b)$ . Then,

$$C_t = \delta_t^A A_t + \delta_t^B B_t. \quad (\text{by homogeneity})$$

As Merton had observed, the 2-dim PDE above is by homogeneity equivalent to the 1-dim PDE of Sec. 1.1, with

$$f(t, x) = c(t, a, b)/b = c(t, x, 1), \quad x = a/b,$$

and with volatility  $\sigma$  given by that of **asset ratio**  $A/B$ :

$$\sigma^2 = \sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho.$$

Coining the term **numeraire**, Margrabe supplemented this fact with a heuristic financial **invariance** argument. He proposed (acknowledging Stephen Ross) to measure both the underlying asset and the option in terms of asset  $B$ , as in a barter economy where  $B$  serves as the medium of exchange. This provided the intuition behind the reduction to the Black-Scholes model with a zero interest rate.

Significantly, the exchange option is replicated in this model by dynamic trading in only assets  $A$  and  $B$ . In particular, no reference to a finite variation “money market” numeraire of the form  $B_t^* = \exp(\int_0^t r_s ds)$  is required.

**1.3. Equivalent martingale measures.** Harrison and Kreps (1979) and Harrison and Pliska (1981) pioneered the application of martingale theory in option pricing.

They showed that ***no-arbitrage*** in the sense of no “free lunches” is essentially equivalent to the existence of an equivalent measure under which discounted prices are martingales. (See [2] for the general theory.) Options can thus be priced by computing the *discounted payoff expectation*.

For discounting, they utilized the money market numeraire  $\exp(\int_0^t r_s ds)$ . With the advent of the ***forward measure***, the expectations were simplified by discounting with a zero-coupon bond despite its infinite variation (e.g., [7] and [4]).

Another useful numeraire was one by Neuberger (1990) to price interest-rate swaptions; it serves as the industry standard to this date. Eventually, El-Karoui et. al. (1995) showed that one can basically ***change numeraire*** to any asset and associate to it an equivalent martingale measure.

## 2. THE PRINCIPLE OF NUMERAIRE INVARIANCE

We fix a stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$  with a finite time horizon  $t \in [0, T]$ . We denote the stochastic integral of a locally bounded predictable integrand  $\theta = (\theta^1, \dots, \theta^n)$  against a (vector) semimartingale  $X = (X^1, \dots, X^n)$  by

$$\theta \cdot X = \sum_{i=1}^n \int_0^\cdot \theta_t^i dX_t^i.$$

In what follows,  $A$  will denote a vector semimartingale:

$$A = (A^1, \dots, A^m). \quad (m \geq 2)$$

Each  $A^i$  represents the observable price process of a traded *zero-dividend* asset. When  $A^m, A_-^m > 0$ , we will set

$$X^i := \frac{A^i}{A^m},$$

and

$$X := (X^1, \dots, X^n), \quad n := m - 1.$$

**2.1. Self-financing trading strategies (SFTS).** A SFTS  $\delta$  for a semimartingale  $A = (A^1, \dots, A^m)$  is a locally bounded predictable process  $\delta = (\delta^1, \dots, \delta^m)$  such that

$$(2.1) \quad \sum_{i=1}^m \delta^i A^i = \sum_{i=1}^m \delta_0^i A_0^i + \delta \cdot A.$$

This is equivalent to saying that  $C = C_0 + \delta \cdot A$ , i.e.,

$$(2.2) \quad dC = \sum_{i=1}^m \delta^i dA^i,$$

where  $C$  is the SFTS *price process* defined by

$$(2.3) \quad C := \sum_{i=1}^m \delta^i A^i.$$

Clearly  $C$  is then a semimartingale,  $\Delta C = \sum_i \delta^i \Delta A_i$ , thus

$$C_- = \sum_{i=1}^m \delta^i A_-^i.$$

**2.2. Numeraire invariance.** *Let  $\delta$  be a SFTS for  $A$  and  $S$  be any (scalar) semimartingale. Then  $\delta$  is also a SFTS for  $SA = (SA^1, \dots, SA^n)$ , i.e.,*

$$(2.4) \quad d(SC) = \sum_{i=1}^m \delta^i d(SA^i),$$

where, as before,  $C := \sum_i \delta^i A^i = C_0 + \delta \cdot A$ .

*Proof.* By Itô's product rule, then substituting for  $dC$  and  $C_-$  and regrouping, followed by Itô's product rule again,

$$\begin{aligned} d(SC) &= S_- dC + C_- dS + d[S, C] \\ &= S_- \sum_{i=1}^m \delta^i dA^i + \sum_{i=1}^m \delta^i A_-^i dS + \sum_{i=1}^m \delta^i d[S, A^i] \\ &= \sum_{i=1}^m \delta^i (S_- dA^i + A_-^i dS + d[S, A^i]) = \sum_{i=1}^m \delta^i d(SA^i). \quad \square \end{aligned}$$

- To our best knowledge, this result first appeared in the 1992 edition of Duffie [3], where it is called ***numeraire invariance***.
- Interpreting  $S$  as an exchange rate, it means the self-financing property is independent of the currency.

If  $S, S_- > 0$ , the result applied to  $1/S$  implies that:

*$\delta$  is a SFTS for  $A$  if and only if it is one for  $SA$ . Thus, if (2.3) holds then (2.2) and (2.4) are equivalent.*

Assume now  $A^m, A_-^m > 0$ . Apply to  $S = 1/A^m$ . It follows:

*$\delta$  is a SFTS for  $A$  if and only if it is a SFTS for  $A/A^m = (X, 1)$ , i.e., if and only if  $F := C/A^m$  satisfies  $F = F_0 + \delta' \cdot X$  where  $\delta' := (\delta^1, \dots, \delta^n)$ . Clearly then*

$$\delta^m = F - \sum_{i=1}^n \delta^i X^i = F_- - \sum_{i=1}^n \delta^i X_-^i. \quad (F := \frac{C}{A^m})$$

Conversely, given  $\delta' = (\delta^1, \dots, \delta^n)$  and an  $F_0$ , then with  $\delta^m$  as above,  $\delta = (\delta', \delta^m)$  is a SFTS for  $(X, 1)$  with price process  $F := F_0 + \delta' \cdot X$ . Hence by numeraire invariance  $\delta$  is a SFTS for  $A$  with price process  $C = A^m F$ .

Numeraire invariance thus reduces dimensionality by one:

- *In order to find a SFTS  $\delta$  with a given time- $T$  payoff  $C_T$ , it is sufficient to find a process  $\delta'$  and an  $F_0$  such that  $F_T = C_T/A_T^m$ , where  $F = F_0 + \delta' \cdot X$ .*

Since  $\delta^m = F - \sum_{i=1}^n \delta^i X^i$ , the  $m$ -th delta  $\delta^m$  is like  $F$  determined by  $\delta'$  and  $F_0$ . As such, one interprets the  $m$ -th asset as the **numeraire asset** chosen to finance an otherwise arbitrary trading strategy  $\delta'$  in the other assets, post an initial investment of  $C_0 = A_0^m F_0$ .

**2.3. Application to unique pricing.** One calls  $A$  *arbitrage free* if there exists a *state price density*, i.e., semimartingale  $S > 0$  such that  $SA^i$  are martingales, all  $i$ .

The (bounded) *law of one price* then holds:

- If  $A$  is arbitrage free and  $\delta$  is a **bounded SFTS** for  $A$  then  $SC$  is a martingale where  $C := \sum_{i=1}^m \delta^i A^i$ ; consequently  $C = 0$  if  $C_T = 0$ .

*Proof.* By numeraire invariance,  $d(SC) = \sum_{i=1}^m \delta^i d(SA^i)$ . Thus  $SC$  is a local martingale. But since  $\delta$  is bounded,  $SC$  is dominated by a martingale. So  $SC$  is a martingale.  $\square$

By a simple and well-known argument:

- If  $A^m, A_-^m > 0$  then  $A$  is arbitrage free if and only if there exists an equivalent probability measure  $\mathbb{Q}$  such that  $X^i := A^i/A^m$  are  $\mathbb{Q}$ -martingales for all  $i$ .

The equivalent martingale measure  $\mathbb{Q}$  is related to  $S$  by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{S_T A_T^m}{\mathbb{E}[S_0 A_0^m]}.$$

- If  $\delta$  is a **bounded SFTS**, then  $C/A^m$  is a  $\mathbb{Q}$ -martingale, where  $C := \sum_{i=1}^m \delta^i A^i$ ; hence

$$C_t = A_t^m \mathbb{E}^{\mathbb{Q}} \left[ \frac{C_T}{A_T^m} \mid \mathcal{F}_t \right].$$

*Proof.* By numeraire invariance  $d(C/A^m) = \sum_{i=1}^m \delta^i d(A^i/A^m)$ . So  $C/A^m$  is a local martingale. Since  $\delta$  is bounded,  $C/A^m$  is dominated by martingale. Thus  $C/A^m$  is a martingale.  $\square$

**Unique hedging.** Assume  $X$  is a  $\mathbb{Q}$ -square-integrable martingale and  $\delta^i$  are bounded,  $i \leq n := m-1$ . Then  $F := C/A^m$  is a  $\mathbb{Q}$ -square-integrable martingale since  $dF = \sum_{i=1}^n \delta^i dX^i$ . So,  $d\langle F \rangle^{\mathbb{Q}} = \sum_{i,j=1}^n \delta^i \delta^j d\langle X^i, X^j \rangle^{\mathbb{Q}}$ . Thus:

- If  $\langle X^i \rangle^{\mathbb{Q}}$  are absolutely continuous and the  $n \times n$  matrix  $(\frac{d}{dt} \langle X^i, X^j \rangle^{\mathbb{Q}})$  is **nonsingular**, then given any random variable  $R$ , there exists **at most one bounded SFTS**  $\delta$  for  $A$  such that  $\sum_{i=1}^m \delta_T^i A_T^i = R$ .

(Above,  $\langle X^i, X^j \rangle^{\mathbb{Q}}$  is the  $\mathbb{Q}$ -compensator of  $[X^i, X^j]$ .) When there are “redundant assets”, the matrix is singular, and hedging is *not* unique.

### 3. APPLICATION TO EXPONENTIAL POISSON MODEL

**3.1. Option to exchange two assets.** Let  $A$  and  $B$  denote the asset price processes. Assume  $A = BX$ , where

$$(3.1) \quad X_t = X_0 e^{\beta P_t - (e^\beta - 1)\lambda t}$$

for some constants  $\beta \neq 0$ ,  $\lambda > 0$  and semimartingale  $P$  such that  $[P] = P$  and  $P_0 = 0$  (so,  $P_t = \sum_{s \leq t} 1_{\Delta P_s \neq 0}$ ). Equivalently, by Itô's formula,  $X$  follows

$$(3.2) \quad dX_t = X_{t-} (e^\beta - 1) d(P_t - \lambda t).$$

Define the function  $f(t, x)$ ,  $x > 0$  by

$$f(t, x) := \sum_{n=0}^{\infty} (x e^{\beta n - (e^\beta - 1)\lambda(T-t)} - 1)^+ \frac{\lambda^n}{n!} (T - t)^n e^{-\lambda(T-t)},$$

Clearly  $f(T, x) = (x - 1)^+$ . Define

$$\delta_t^A := \delta_A(t, X_{t-}), \quad \delta_A(t, x) := \frac{f(t, e^\beta x) - f(t, x)}{(e^\beta - 1)x}.$$

Then, using (3.4) below, one can show,

$$(3.3) \quad dF = \delta^A dX, \quad F_t := f(t, X_t).$$

Therefore by *numeraire invariance*,  $(\delta^A, \delta^B)$  is a SFTS for  $A$  with price process  $C = BF$ , where

$$\delta^B := F - \delta^A X = F_- - \delta^A X_-.$$

Further,  $C_T = (A_T - B_T)^+$  since  $F_T = (X_T - 1)^+$ .

Also, this is a *bounded* SFTS. In fact,

$$0 \leq \delta^A \leq 1, \quad -1 \leq \delta^B \leq 0.$$

*Remark:* In contrast to Margrabe's model,

- The **deltas** are given by *partial differences*.
- $f(t, x)$  satisfies a *partial difference equation*:

$$(3.4) \quad \frac{\partial u}{\partial t}(t, p) + \lambda(u(t, p + 1) - u(t, p)) = 0,$$

where  $u(t, p) := f(t, X_0 e^{\beta p - (e^\beta - 1)\lambda t})$ .

**3.2. Multivariate exponential Poisson model.** Let  $A > 0$  be an  $m$ -dimensional semimartingale with  $A_- > 0$ . Set  $X := (A^i/A^m)_{i=1}^n$ ,  $n := m - 1$ . Assume

$$X_t^i := X_0^i \exp\left(\sum_{j=1}^k (\beta_{ij} P_t^j - (e^{\beta_{ij}} - 1)\lambda_j t)\right),$$

( $1 \leq k \leq n$ ) or equivalently,

$$dX_t^i = X_{t-}^i \sum_{j=1}^k (e^{\beta_{ij}} - 1)(dP_t^j - \lambda_j dt),$$

where,  $\beta_{ij}$  are constants with the  $n \times k$  matrix  $(e^{\beta_{ij}} - 1)$  of full rank,  $\lambda_j > 0$  are constants, and  $P^j$  are semimartingales such that  $[P^j] = P^j$ ,  $P_0^j = 0$  and  $[P^j, P^l] = 0$  for  $j \neq l$ .

Let  $h(a)$ ,  $a \in \mathbb{R}_+^m$  be a given payoff function, assumed **homogenous** of degree 1 and of linear growth in  $a$ .

Define  $g(x) := h(x, 1)$ ,  $x \in \mathbb{R}_+^n$ ,  $n := m - 1$ . Define

$$f(t, x) := \sum_{q_1, \dots, q_n=0}^{\infty} g(x_1 e^{\sum_{j=1}^n (\beta_{1j} q_j - (e^{\beta_{1j}} - 1) \lambda_j (T-t))}, \dots, x_n e^{\sum_{j=1}^n (\beta_{nj} q_j - (e^{\beta_{nj}} - 1) \lambda_j (T-t))}) \prod_{i=1}^n \frac{\lambda_i^{q_i}}{q_i!} (T - t)^{q_i} e^{-\lambda_i (T-t)}.$$

Let  $\alpha = (\alpha_{ij})$  be any  $n \times k$  matrix such that for  $1 \leq j, l \leq k$ ,  $\sum_{i=1}^n (e^{\beta_{il}} - 1) \alpha_{ij} = 1$  if  $j = l$  and 0 otherwise.

Define

$$(3.5) \quad \delta_t^i := \delta_i(t, X_{t-}), \quad (1 \leq i \leq n)$$

where

$$\delta_i(t, x) := \frac{1}{x_i} \sum_{j=1}^k \alpha_{ij} (f(t, e^{\beta_{1j}} x_1, \dots, e^{\beta_{nj}} x_n) - f(t, x)).$$

Then one can show

$$(3.6) \quad dF = \sum_{i=1}^n \delta^i dX^i, \quad F_t := f(t, X_t).$$

Hence by *numeraire invariance*,  $\delta = (\delta^1, \dots, \delta^n, \delta^m)$  is a SFTS for  $A$ , where  $\delta^m := F - \sum_{i=1}^n \delta^i X^i$ . Its price process  $C = \sum_1^m \delta^i A^i = C_0 + \delta \cdot A$  is clearly given by  $A^m F$ :

$$(3.7) \quad C_t = A_t^m f(t, X_t).$$

Further,  $C_T = h(A_T)$  because  $h(a)$  is homogenous of degree 1 and  $f(T, x) = g(x) := h(x, 1)$ .

Moreover,  $\delta^i$  are bounded if  $\gamma_i(x)$  are bounded, where

$$\gamma_i(x) := \frac{1}{x_i} \sum_{j=1}^k \alpha_{ij} (g(e^{\beta_{1j}} x_1, \dots, e^{\beta_{nj}} x_n) - g(x)),$$

and  $\gamma_m(x) := g(x) - \sum_{i=1}^n \gamma_i(x) x_i$ .

**3.3. Relation to Poisson predictable representation.** Let  $P = (P^1, \dots, P^k)$  be a vector of independent Poisson processes  $P^i$  with intensities  $\lambda_i > 0$ . Let  $v(p)$ ,  $p \in \mathbb{R}^k$ , be a function of exponential linear growth. Then, one has the following representation:

$$v(P_T) = \sum_{q_1, \dots, q_k=0}^{\infty} v(q_1, \dots, q_k) \prod_{i=1}^k \frac{\lambda_i^{q_i}}{q_i!} T^{q_i} e^{-\lambda_i T} \\ + \sum_{i=1}^k \int_0^T \Delta_i u(t, P_{t-}) d(P_t^i - \lambda_i t),$$

where  $\Delta_i u(t, p) := u(t, p_1, \dots, p_i + 1, \dots, p_n) - u(t, p)$  and

$$u(t, p) := \sum_{q_1, \dots, q_k=0}^{\infty} v(p + q) \prod_{i=1}^k \frac{\lambda_i^{q_i}}{q_i!} (T - t)^{q_i} e^{-\lambda_i (T-t)}.$$

Also,  $u(t, p)$  satisfies the partial difference equation and the SDE

$$\frac{\partial u}{\partial t}(t, p) + \sum_{i=1}^k \lambda_i \Delta_i u(t, p) = 0; \\ du(t, P_t) = \sum_{i=1}^k \Delta_i u(t, P_{t-}) d(P_t^i - \lambda_i t).$$

## 4. APPLICATION TO DIFFUSION PROCESSES

**4.1. The role of homogeneity.** Let  $A$  be continuous semimartingale and  $\delta$  be a SFTS. Assume  $C_t = c(t, A_t)$  for a  $C^{1,2}$  function  $c$ . Since  $dC = \sum \delta^i dA^i$ , by Itô's formula,

$$(4.1) \quad \begin{aligned} & \frac{\partial c}{\partial t}(t, A_t)dt + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t) d[A^i, A^j]_t \\ &= \sum_{i=1}^m (\delta_t^i - \frac{\partial c}{\partial a_i}(t, A_t)) dA_t^i. \end{aligned}$$

In general,  $\sum_{i,j} (\delta^i - \frac{\partial c}{\partial a_i})(\delta^j - \frac{\partial c}{\partial a_j}) d[A^i, A^j] = 0$  since the (left so) right hand side of (4.1) has finite variation. Thus, if  $[A^i]$  are absolutely continuous and the  $m \times m$  matrix  $(\frac{d}{dt}[A^i, A^j])$  is *nonsingular*, then  $\delta_t^i = \frac{\partial c}{\partial a_i}(t, A_t)$ , so by (4.1)

$$(4.2) \quad \frac{\partial c}{\partial t}(t, A_t)dt + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t) d[A^i, A^j]_t = 0.$$

Moreover, then by (2.3),  $c(t, A_t) = \sum_i \frac{\partial c}{\partial a_i}(t, A_t) A_t^i$ . So, if the support of  $A_t$  is a cone, then it follows that  $c(t, a)$  is *homogenous of degree 1* in  $a$  on that cone.

But, in some degenerate cases, e.g., the Black-Scholes or 1-factor Markovian short-rate models, there also exist many nonhomogeneous functions  $c(t, a)$  satisfying  $C_t = c(t, A_t)$ .

Assume  $M^i := e^{-\int_0^t r_t dt} A^i$  are local martingales under an equivalent measure for some predictable process  $r$ . Then  $dA^i = rA^i dt + e^{\int r dt} dM^i$ . Therefore by (2.2) and (4.1),

$$\begin{aligned}
 (4.3) \quad & \frac{\partial c}{\partial t}(t, A_t) dt + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 c}{\partial a_i \partial a_j}(t, A_t) d[A^i, A^j]_t \\
 & = r_t (C_t - \sum_{i=1}^m \frac{\partial c}{\partial a_i}(t, A_t) A_t^i) dt.
 \end{aligned}$$

**4.2. Pricing and hedging.** Let  $A = (A^1, \dots, A^m)$  be a semi-martingale with  $A, A_- > 0$  such that  $X^i := A^i/A^m$  follows the SDE

$$dX_t^i = X_t^i \sum_{j=1}^k \varphi_{ij}(t, X_t) (dZ_t^j + \phi^j dt), \quad (i = 1, \dots, n := m - 1)$$

where  $Z^j$  are independent Brownian motions,  $\varphi_{ij}(t, x)$  are *bounded* continuous functions, and  $\mathbb{E} e^{\frac{1}{2} \sum_j \int_0^T (\phi_t^j)^2 dt} < \infty$ . Define the martingale

$$M := \mathcal{E}\left(-\sum_{j=1}^k \int \phi^j dZ^j\right) = e^{-\sum_{j=1}^k \left(\int \phi^j dZ^j + \frac{1}{2} \int (\phi^j)^2 dt\right)},$$

and the measure  $\mathbb{Q}$  by  $d\mathbb{Q} = M_T d\mathbb{P}$ . Then  $W^i := Z^i + \int \phi^i dt$  are  $\mathbb{Q}$ -Brownian motions and are  $\mathbb{Q}$ -independent since  $[W^k, W^l] = 0$  for  $k \neq l$ . The  $X^i$  are  $\mathbb{Q}$ -martingales since

$$(4.4) \quad dX_t^i = X_t^i \sum_{j=1}^k \varphi_{ij}(t, X_t) dW_t^j,$$

and  $\varphi_{ij}(t, x)$  are bounded. *Thus  $A$  is **arbitrage-free**.*

Further, for each  $s \leq T$  and  $x \in \mathbb{R}_+^n$ , there is a unique continuous positive  $\mathbb{Q}$ -square-integrable martingale  $X^{s,x} = (X_t^{s,x})$  on  $[s, T]$  with  $X_s^{s,x} = x$  satisfying this SDE, and we have  $X = X^{0, X_0}$ .

Let  $h(a)$ ,  $a \in \mathbb{R}_+^m > 0$ , be a homogenous Borel function of linear growth. Define  $g(x) := h(x, 1)$ ,  $x \in \mathbb{R}_+^n$ . Define the function

$$(4.5) \quad f(t, x) := \mathbb{E}^{\mathbb{Q}} g(X_T^{t,x}).$$

Note  $f(T, x) = g(x)$ . (Intuitively,  $f(t, x) = \mathbb{E}[g(X_T) | X_t = x]$ .) Also,

$$(4.6) \quad F_t := f(t, X_t) = \mathbb{E}^{\mathbb{Q}}(g(X_T) | \mathcal{F}_t).$$

Thus  $F = (f(t, X_t))$  is a  $\mathbb{Q}$ -martingale, and since  $X^i$  are too, assuming that  $f(t, x)$  is  $C^{1,2}$ , then Itô's formula yields,

$$(4.7) \quad dF_t = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t, X_t) dX_t^i,$$

and

$$(4.8) \quad \frac{\partial f}{\partial t}(t, X_t) dt + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t) d[X^i, X^j]_t = 0.$$

By (4.7) and **numeraire invariance**,  $\delta$  is a SFTS for  $A$ , where

$$(4.9) \quad \delta_t^i := \frac{\partial f}{\partial x_i}(t, X_t), \quad i \leq n, \quad \delta^m := F - \sum_{i=1}^n \delta^i X_i.$$

Note, the price process is  $C = A^m F$  (by the definition of  $\delta^m$ ). Moreover,  $C_T = h(A_T)$  since  $F_T = g(X_T)$  and  $h(a)$  is homogenous.

By (4.8), on the support  $X$ ,  $f(t, x)$  satisfies the PDE

$$(4.10) \quad \frac{\partial f}{\partial t} + \frac{1}{2} \sum_{i,j=1}^n x_i x_j \sigma_{ij}(t, x) \frac{\partial^2 f}{\partial x_i \partial x_j} = 0,$$

where

$$\sigma_{ij}(t, x) := \sum_{l=1}^k \varphi_{il}(t, x) \varphi_{jl}(t, x).$$

**4.3. Deterministic volatility example.** Assume  $\varphi_{ij}$ , hence  $\sigma_{ij}$ , are independent of  $x$ . Then simply  $X_T^{t,x} = x X_T / X_t$ . Hence by (4.5),

$$(4.11) \quad f(t, x) := \mathbb{E}^{\mathbb{Q}} \left[ g \left( x_1 \frac{X_T^1}{X_t^1}, \dots, x_n \frac{X_T^n}{X_t^n} \right) \right].$$

If  $\partial g / \partial x_i$  and  $g(x) - \sum x_i \partial g / \partial x_i$  are bounded, then so is  $\delta$ , since

$$\frac{\partial f}{\partial x_i}(t, x) = \mathbb{E}^{\mathbb{Q}} \left[ \frac{X_T^i}{X_t^i} \frac{\partial g}{\partial x_i} \left( x_1 \frac{X_T^1}{X_t^1}, \dots, x_n \frac{X_T^n}{X_t^n} \right) \right].$$

Conditioned on  $\mathcal{F}_t$  and unconditionally,  $X_T / X_t$  is  $\mathbb{Q}$ -*multivariately lognormally distributed*, with mean  $(1, \dots, 1)$  and log-covariances  $\int_t^T \sigma_{ij}(s) ds$ . Let  $P(t, T, z)$ , denote its distribution function. Thus

$$(4.12) \quad f(t, x) = \int_{\mathbb{R}_+^n} g(x_1 z_1, \dots, x_n z_n) P(t, T, dz).$$

4.4. **Extension to dividends.** Consider  $m$  assets with positive price processes  $\hat{A}^i$  and continuous dividend yields  $y_t^i$ . When there exist traded or replicable zero-dividend assets  $A^i$  such that  $A_T^i = \hat{A}_T^i$ , the problem reduces to pricing and hedging (European) options on the  $A^i$ . It is required that the  $2m$  assets  $A^i$  and  $\tilde{A}^i$  be arbitrage free, where

$$\tilde{A}_t^i := e^{\int_0^t y_s^i ds} \hat{A}_t^i$$

is the price of the zero-dividend asset that initially buys one share of  $\hat{A}$  and thereon continually reinvests all dividends in  $\hat{A}$  itself. (When  $y^i$  is deterministic, this requires  $A_t^i = e^{-\int_t^T y_s^i ds} \hat{A}_t^i$ .)

For instance, consider an exchange option ( $m = 2$ ). Say  $\hat{A}$  and  $\hat{B}$  are the yen/dollar and yen/Euro exchange rates viewed as yen-denominated dividend assets. Then  $A$  is the yen-value of the U.S.  $T$ -maturity zero-coupon bond and  $\tilde{A}$  is the yen-value of the U.S. money market asset. This exchange option is equivalent to a Euro-denominated call struck at 1 on the Euro/dollar exchange rate  $\hat{A}/\hat{B}$ . The ratio  $A/B$  is the *forward* Euro/dollar exchange rate. If it has deterministic volatility, we are as in a setting of [7] with results similar to Sec. 4.3.

**4.5. Change of numeraire.** For the exchange options, one has to calculate  $\mathbb{E}(X - Y)^+$  for some random variables  $X$  and  $Y > 0$ . Such expectations often become more tractable by a change of measure as in [4]. Define the equivalent probability measure  $\mathbb{Q}$  by  $\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{Y}{\mathbb{E}Y}$ . Clearly,

$$(4.13) \quad \mathbb{E}^{\mathbb{Q}}\left(\frac{X}{Y}\right) = \frac{\mathbb{E}(X)}{\mathbb{E}(Y)}. \quad \left(\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{Y}{\mathbb{E}(Y)}\right)$$

Replacing  $X$  by  $(X - Y)^+$  in (4.13) and using the homogeneity,

$$(4.14) \quad \mathbb{E}(X - Y)^+ = \mathbb{E}(Y)\mathbb{E}^{\mathbb{Q}}\left(\frac{X}{Y} - 1\right)^+.$$

If  $X/Y$  is  $\mathbb{Q}$ -lognormally distributed then (4.13) and (4.14) yield,

$$(4.15) \quad \mathbb{E}(X - Y)^+ = \mathbb{E}(X)N\left(\frac{\log(\mathbb{E}X/\mathbb{E}Y)}{\sqrt{\nu^{\mathbb{Q}}}} + \frac{\sqrt{\nu^{\mathbb{Q}}}}{2}\right) \\ - \mathbb{E}(Y)N\left(\frac{\log(\mathbb{E}X/\mathbb{E}Y)}{\sqrt{\nu^{\mathbb{Q}}}} - \frac{\sqrt{\nu^{\mathbb{Q}}}}{2}\right),$$

where  $\nu^{\mathbb{Q}} := \text{var}^{\mathbb{Q}}[\log(X/Y)]$ . If  $X$  and  $Y$  are bivariate lognormally distributed, as in Merton's and Margrabe's models, then  $X/Y$  is lognormally distributed in both  $\mathbb{P}$  and  $\mathbb{Q}$  with the same log-variance  $\nu^{\mathbb{Q}} = \nu := \text{var}[\log(X/Y)]$ . Then  $\nu^{\mathbb{Q}}$  can be replaced with  $\nu$  in (4.15).

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