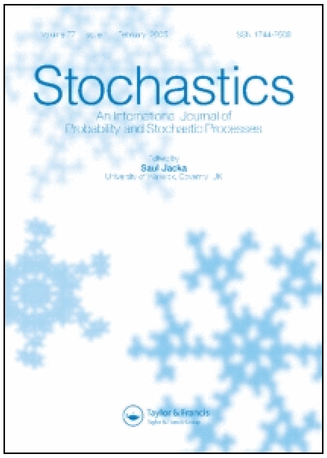

Stochastics An International Journal of Probability and Stochastic Processes

formerly Stochastics and Stochastics Reports

Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713652481

The duality of optimal exercise and domineering claims: a Doob-Meyer decomposition approach to the Snell envelope

To link to this article: DOI: 10.1080/17442500601051914
URL: http://dx.doi.org/10.1080/17442500601051914

Stochastics: An International Journal of Probability and Stochastics Processes,
Vol. 79, Nos. 1–2, February–April 2007, 27–60

The duality of optimal exercise and domineering claims: a Doob–Meyer decomposition approach to the Snell envelope

FARSHID JAMSHIDIAN*

Applied Mathematics, FELAB, University of Twente, Twente, The Netherlands

(Received 6 March 2006; revised 27 September 2006; in final form 4 October 2006)

We develop a concept of a "domineering claim" and apply it to the existence, uniqueness and properties of optimal stopping times in continuous time. The notion pinpoints a key observation of pathwise optimality implicit in Davis and Karatzas. It also ties in well with several formulations of a duality in optimal stopping theory, including the minimax duality pricing formula in Rogers and Haugh and Kogan for American and Bermudan options and its multiplicative version. We give a general formulation and proof that the Snell envelope is a supermartingale. Combined with the Doob–Meyer decomposition in different numeraire measures, this gives rise to (many) domineering claims. The multiplicative decomposition, for which a formula is derived, yields a uniquely invariant domineering numeraire. A pricing formula in Kim, Jacka, Carr *et al.* and Jamshidian are extended and related to the additive decomposition. The iterative construction of the Snell envelope in Chen and Glasserman is partially extended to continuous time. In Bermudan case, it is complemented with construction of stopping times converging to the optimal one, reminiscent of Kolodko and Schoemakers. The perpetual American put is treated by incorporating an approach of Beibel and Lerche. Assuming smooth pasting, the jump-diffusion setting of Chiarella and Ziogas is extend based on the Itô–Meyer formula.

Keywords: Duality; Optimal stopping time; Snell envelope; Supermartingale; Doob–Meyer decomposition; Multiplicative decomposition

MS Classification 60G40, 60H30.

1. Introduction

This paper develops further a duality method in optimal stopping based on the Doob–Meyer decomposition of the Snell envelope. It presents known results too, citing reference best we can; and it uses duality arguments in all proofs. The known results serve as motivation for related new results or extensions, or else are included to ease the development. For example, we present a duality result of [31] and independently [15] as motivation for our related multiplicative results, while we state and prove a suitable form of the well-known supermartingale property of the Snell envelope to ensure the conditions that meet such applications as our extension of a known result on the existence of optimal stopping times.

The duality method arises in many contexts, e.g. with constraints in the incomplete market settings of [13] or [27]. Especially relevant here is the formula in [8] which almost provides a

---

Version 4-October-2006. For possible future updates visit www.home.math.utwente.nl/˜jamshidianf.

*Email: f.jamshidian@ewi.utwente.nl

Stochastics: An International Journal of Probability and Stochastics Processes
ISSN 1744-2508 print/ISSN 1744-2516 online
http://www.tandf.co.uk/journals
DOI: 10.1080/17442500601051914

pathwise characterization of optimality, implicit though more evident in [31] too. The Doob–Meyer decomposition of the Snell envelope is also related to the Markovian PDE optimal-boundary literature such as the earlier work [3,18,23], particularly to the original free-boundary formula of [29] for the American put and its "early exercise premium" simplification in [5,16,24], not least to the "delayed exercise compensation" technique of [19] and its recent extensions in [7]. Our approach also bears resemblance to a multiplicative method in [2] that helps solve several classical optimal stopping problems.

The minimax formulation of the duality in [15,31] and its multiplicative version are among several contenders for numerical valuation of American options by Monte-Carlo simulation when the traditional dynamic-programming based PDE and binomial techniques break due to high dimensionality†. By itself, minimax duality produces only upper bound estimates in simulation, because only proxies for "domineering claims" are prescribable algorithmically. With the help of some optimization as in [31] or [22], the estimates may still turn out quite accurate in some important practical situations.

Minimax duality can be effectively combined through a convergent feedback loop with ingenuous optimal exercise time approximations that produce tight lower bounds, as in [1,14,15,25]. An interesting recent development in this direction is the simulation within simulation iterative construction of [26], proved convergent to the optimal exercise time in a finite number of steps, while reportedly requiring only very few iterations in practice for outstanding high-dimensional problems such as pricing of Bermudan swaptions.

Another new development is the "supersolution" iteration of [6] converging to the Snell envelope. It improves the proxy claim and upper bound at each iteration and is applied to compare the bias, variance and numerical performance of the additive versus multiplicative minimax duality. (See also [4] for the variance of multiplicative method from an importance sampling perspective.) The two methods are also studied and interpreted in [21].

1.1 Domineering claims

We build on a novel concept of "domineering claim", basically a martingale that lies above the underlying payoff (reward) process, but such that on each path the payoff process reaches arbitrary close to it at some point and even touches it for a large class of payoff processes (at which point is then optimal to exercise). The Doob–Meyer decomposition of the Snell envelope furnishes many domineering claims when looked at from viewpoints of different numeraire measures relativistically through the Bayes' rule.

A special domineering claim, with a unique numeraire-invariance property and financial interpretation, arises from the multiplicative rather than additive decomposition. Unlike the additive compensator, the multiplicative compensator (interpreted as a number of shares in the current American option) is independent of the choice of numeraire.

The multiplicative decomposition was first established by Itô‡. Previously, it has been applied in mathematical finance to extract interest rates from the state price density and

---

†We refer to [14] and references therein for other Monte-Carlo methods such as the stochastic mesh method, optimal exercise policy approximation and regression on basis functions.

‡I thank F. Delbaen for clarifying this point at the Symposium.

employed in [9] to analyze passport options. It is treated in the new edition of the treatise [17]. We derive a different (but equivalent) formula for it than appears there.

Domineering claims give much information about optimal stopping times and help their characterization. We give an easy proof of the known result that an optimal stopping time exists if the payoff process is quasi-left continuous¶. We also show a similar but weaker condition suffices. Our main result in this regard is a formula that basically provides necessary and sufficient conditions for the existence of an optimal stopping time.

Our most technical result proves the Snell envelope is a right-continuous supermartingale of class D. The supermartingale property, alongside the Doob–Meyer decomposition, is the cornerstone of the duality approach. It is a classical result of optimal stopping theory, long known with various assumptions on the underlying payoff process. A similar statement to ours, but with additional assumptions of positivity and existence of left limits, is made in [30] with proof referred to [10]. Our formulation implies that a right-continuous adapted process is of class D if and only if it is dominated by a uniformly integrable martingale.

We see the additive (multiplicative) compensator of the Snell envelope equals the minimal future (relative) distance between the payoff process and the martingale part of Snell envelope. This relates to a continuous-time formulation of the supersolution iteration of Ref. [6] converging to the Snell envelope. In the Bermudan case, we propose dual stopping times.

We follow [2] for the perpetual put, where Brownian motion asymptotics come into play.

We develop general path-dependent heuristics for the formula in [5,16,19,24]. The Snell envelope compensator turns out to be the "delayed exercise compensation" in [19], extended to jumps in [7]. We extend [7] probabilistically to general Markovian jump-diffusion.

The duality approach to optimal stopping has much intuitive appeal. A domineering claim is described by a self-financing trading strategy that invests in and rolls over one share of the newly issued American option plus invests in the base numeraire a number of shares equal the additive compensator. The multiplicative domineering numeraire invests only in the current American option, with number of shares equal the multiplicative compensator.

1.2 Finite versus infinite horizon

Most practical applications of optimal stopping in finance are of finite time horizon $t \in [0, m]$ for some maturity $m < \infty$. But, infinite horizon analysis is theoretically more unifying and general, while only slightly more technical. There are also interesting concrete instances with explicit solutions available only in the infinite horizon case $t \in [0, \infty)$, such as the perpetual American put, the Russian options in [32], or the (radar) detection problem in [33]. Using a unified method resembling the multiplicative minimax duality, as well as asymptotic properties of sample paths of Brownian motions, [2] provide definitive solutions to these and related infinite horizon problems.

We have opted for an exposition that treats finite and infinite horizons simultaneously.

Except for Section 7, we let the symbol $\mathbb{T}$ throughout stand for either the interval $[0, \infty)$ (infinite horizon) or the compact interval $[0, m]$ for some fixed $0 < m < \infty$ (finite horizon).

¶I thank A. Irle for informing me that this result conjectured at my talk has long been known for decades. I thank R. Dalang for pointing out to me the fallacy in an incorrect converse that I also conjectured.

1.3 Notation and terminology

We fix throughout a stochastic basis

$$(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P}),$$

and for simplicity, assume $\mathcal{F}_0$ is trivial, i.e. consists of the events of probability 0 or 1.

In the infinite horizon case $\mathbb{T} = [0, \infty)$, we let $\mathcal{T}$ stand for the set of all *finite* stopping times $T < \infty$ a.s. In the finite horizon case $\mathbb{T} = [0, m]$, we let $\mathcal{T}$ denote the set of all stopping times $T \leq m$ a.s. We employ the following financially motivated abbreviations:

A ***claim*** is a right-continuous, uniformly integrable martingale§. We denote their set $\mathcal{C}$.

In the infinite horizon case, if $C = (C_t)_{t \geq 0}$ is a claim, then C_t converges as $t \longrightarrow \infty$, a.s. and in L^1, to a random variable C_∞, denoted here also by the symbol C_m: we set $m := \infty$.

A ***numeraire*** is a claim B such that $B_m > 0$ a.s. We denote their set $\mathcal{C}^+$.

Clearly, a numeraire B is a positive claim, i.e. a.s. $B_t > 0$ for all $t \in \mathbb{T}$.

A ***payoff process*** is a right-continuous adapted process of class D‖.

As is well known, any claim is a payoff process. This implies a right-continuous adapted process Z is a payoff process if $|Z| \leq C$ for some claim C. We show the converse later.

Strictly speaking, such terminology is compatible with the finance literature only when interest rates and risk premia are zero. The definitions of a claim, etc. are suitably modified in the general case (as the *deflated* claim price being a martingale, etc.). Except for the final section, a financial context is not explicitly explored in this paper and the terms “claim”, “numeraire” and “payoff process” serve primarily as convenient abbreviations.

2. Domineering claims, optimal stopping, minimax duality

In this section, Z denotes an arbitrary payoff process if not already explicitly specified such.

2.1 Domineering claims and duality

The following application of the optional sampling theorem highlights the issues and motivates the notion of a domineering claim.

PROPOSITION 2.1. Let Z be a payoff process and $C \geq Z$ be a claim. Then $\sup_{T \in \mathcal{T}} \mathbb{E} Z_T \leq C_0$. Moreover, if the stopping time $T_* := \inf\{t \in \mathbb{T} : Z_t = C_t\}$ is finite a.s. then we have,

$$\sup_{T \in \mathcal{T}} \mathbb{E} Z_T = C_0 = \mathbb{E} Z_{T_*}.$$

Proof. As $Z \leq C$, by the optional sampling theorem $\mathbb{E} Z_T \leq \mathbb{E} C_T = C_0$ for all $T \in \mathcal{T}$. The first statement follows. As for the second, by right continuity of $Z - C$ we have, $Z_{T_*} = C_{T_*}$. Hence, using the first part and the optional sampling theorem we have, $\sup_{T \in \mathcal{T}} \mathbb{E} Z_T \leq C_0 = \mathbb{E} C_{T_*} = \mathbb{E} Z_{T_*} \leq \sup_{T \in \mathcal{T}} \mathbb{E} Z_T$, implying the desired equality. □

We say a claim C *domineers* a payoff process Z if $C \geq Z$ and $C_0 = \sup_{T \in \mathcal{T}} \mathbb{E} Z_T$.

---

§Of course in the finite horizon case $\mathbb{T} = [0, m]$, any martingale is automatically uniformly integrable.

‖A process $X = (X_t)_{t \in \mathbb{T}}$ is of class D if the family of random variables $(X_T)_{T \in \mathcal{T}}$ is uniformly integrable.

The martingale part of the Doob–Meyer decomposition of the Snell envelope, as utilized in [8,15,31] furnishes the main example of a domineering claim. In general, there are infinitely many domineering claims#. The mere existence of a domineering claim, as recorded next, is more relevant to our analysis than the choice of a specific one.

THEOREM 2.2. Let Z be a payoff process. Then there exists a claim that domineers Z.

Proof. For $t \in \mathbb{T}$, set $V_t := \text{ess.}\sup_{t \leq T \in \mathcal{T}} \mathbb{E}(Z_T | \mathcal{F}_t)$. Then (V_t) has a right-continuous version V, which is a class-D supermartingale. (See Theorem 3.1 below). Let $V = C - A$ be the Doob–Meyer decomposition of V, where C is a claim and A an increasing predictable process with $A_0 = 0$. Then C domineers Z because $C_0 = V_0$ and $C \geq V \geq Z$. □

We next present six corollaries of Theorem 2.2, starting with the now obvious duality,

COROLLARY 2.3. *Duality*: Let Z be a payoff process. Then

$$\sup_{T \in \mathcal{T}} \mathbb{E} Z_T = \min_{Z \leq C \in \mathcal{C}} C_0.$$

Moreover, the minimum is attained at a claim $C \geq Z$ if and only if C domineers Z**.

Proof. The "$\leq$" follows from Proposition 2.1 and "$\geq$" is immediate from Theorem 2.2. □

Remark. As emphasized throughout the symposium by L. Shepp, optimal stopping is intimately connected to linear-convex programming. The duality formula above furnishes such a connection, as it characterizes a domineering claim C as a solution to the following linear-convex programming problem on the space $\mathcal{C}$ of claims: *Minimize* C_0 *subject to* $C \geq Z$. The incomplete market settings in [13,27] impose the constraint on such minimization that C belong to the stable subspace generated by a given set of claims.

Remark. Theorem 3.1 below easily implies that $\sup_{T \in \mathcal{T}} \mathbb{E} Z_T = \min_{S \geq Z} S_0$, where the minimum is taken over all supermartingales $S \geq Z$. (Indeed the minimum is attained at the Snell envelope). But, we do not consider this a duality result for it does not utilize the Doob–Meyer decomposition theorem. It is a weaker result than the duality formula in Corollary 2.3 in that it is evident from the latter, whereas the converse implication becomes immediate only after applying the Doob–Meyer decomposition to the Snell envelope.

As mentioned before, the class-D condition is equivalent to being dominated by a claim:

COROLLARY 2.4. Let X be a right-continuous adapted process. Then X is of class D if and only if $|X| \leq C$ for some claim C.

#An exception is when Z is a submartingale; then there is only one domineering claim, namely $\mathbb{E}(Z_m | \mathbb{F})$.

**Replacing Z with $-Z$, one also concludes that $\inf_{T \in \mathcal{T}} \mathbb{E} Z_T = \max_{Z \geq C \in \mathcal{C}} C_0$.

Proof. Assume $|X| \le C$ for some claim C. Then X is class D because C is so[††]. Conversely, if X is of class D, then so is $|X|$. Therefore, by Theorem 2.2 applied to the payoff process $Z = |X|$, there exists a claim C that domineers $|X|$. In particular, $C \ge |X|$. □

The following pathwise characterization of domineering claims is basic to the development.

PROPOSITION 2.5. A claim C domineers a payoff process Z if and only if

$$\sup_{t\in\mathbb{T}} (Z_t - C_t) = 0.$$

Proof. Assume C domineers Z. Then, using the definition and optional sampling theorem,

$$0 = \sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T - C_0 = \sup_{T\in\mathcal{T}} \mathbb{E}(Z_T - C_T) \le \mathbb{E} \sup_{t\in\mathbb{T}} (Z_t - C_t).$$

Since $Z \le C$, this implies $\sup_{t\in\mathbb{T}} (Z_t - C_t) = 0$ a.s.

Conversely, assume $\sup_{t\in\mathbb{T}} (Z_t - C_t) = 0$. Then $Z \le C$, so $\mathbb{E}\, Z_T \le \mathbb{E} C_T = C_0$. Hence, $\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T \le C_0$. To prove the reverse inequality, it suffices to show that for any $\varepsilon > 0$, there is a stopping time $T \in \mathcal{T}$ such that $\mathbb{E}\, Z_T \ge C_0 - \varepsilon$. Let $\varepsilon > 0$. The assumption implies that (pathwise) the set $\{t \in \mathbb{T} : Z_t - C_t \ge -\varepsilon\}$ is nonempty a.s. Thus, the stopping time $T := \inf\{t \in \mathbb{T} : Z_t - C_t \ge -\varepsilon\}$ is finite and so in $\mathcal{T}$. Since $Z - C$ is right-continuous, we must have $Z_T - C_T \ge -\varepsilon$. Therefore, $\mathbb{E}\, Z_T \ge \mathbb{E} C_T - \varepsilon = C_0 - \varepsilon$, as desired. □

Remark. The pathwise formula $\sup_{t\in\mathbb{T}} (Z_t - C_t) = 0$ above does not appear but is implicit in [8,31]. For example, it easily implies and follows from the main formula of [8], which in our notation reads $\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T = \mathbb{E} \sup_{t\in\mathbb{T}} (Z_t - C_t + C_m)$.

The additive minimax duality formula below was obtained by [31] and independently in the Bermudan case by Ref. [15] and applied to Monte-Carlo pricing.

COROLLARY 2.6. *Additive minimax duality*: Let Z be a payoff process. Then

$$\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T = \min_{C\in\mathcal{C}} \left(C_0 + \mathbb{E} \sup_{t\in\mathbb{T}} (Z_t - C_t) \right).$$

The minimum is attained at any claim C that domineers Z (and at translations of C).

Proof. By the optional sampling theorem, for any claim C and stopping time $T \in \mathcal{T}$,

$$\mathbb{E}\, Z_T = C_0 + \mathbb{E}(Z_T - C_T) \le C_0 + \mathbb{E} \sup_{t\in\mathbb{T}} (Z_t - C_t).$$

Taking sup over T, we see “$\le$“ holds. Let C be a claim that domineers Z, which exists by Theorem 2.2. By the definition of domineering claim and Proposition 2.5,

---

[††] Indeed, $1_{|X|>n}|X| \le 1_{C>n}C$. Hence, $\sup_{T\in\mathcal{T}} \mathbb{E}(1_{|X_T|>n}|X_T|) \le \sup_{T\in\mathcal{T}} \mathbb{E}(1_{C_T>n}C_T) \to 0$ as $n \to \infty$.

$\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T = C_0 = C_0 + \mathbb{E}\sup_{t\in\mathbb{T}}(Z_t - C_t)$. Equality and the second statement thus follow. □

2.2 Existence of optimal times for quasi-left continuous payoffs

Let Z be a payoff process. A stopping time T^* is called *optimal* for Z if $T^* \in \mathcal{T}$ and $\mathbb{E}\, Z_{T^*} = \sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T$.

Let C be a claim that domineers Z. Define the increasing sequence of stopping times

$$T_n := \inf\left\{t \in \mathbb{T} : Z_t \geq C_t - \frac{1}{n}\right\}, \quad n \in \mathbb{N}.$$

Lemma 2.7. We have $T_n \in \mathcal{T}$, all n and $\lim_{n\to\infty} \mathbb{E}\, Z_{T_n} = \sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T$.

Proof. By Proposition 2.5, the set $\{t \in \mathbb{T} : Z_t \geq C_t - 1/n\}$ is nonempty a.s. Thus $T_n < \infty$ a.s., in fact, $T_n \in \mathcal{T}$. By the right continuity of $Z - C$ and the definition of T_n, we have $Z_{T_n} \geq C_{T_n} - 1/n$. Using the optional sampling theorem and that C domineers Z, we get

$$\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T \geq \mathbb{E}\, Z_{T_n} \geq \mathbb{E} C_{T_n} - \frac{1}{n} = C_0 - \frac{1}{n} = \sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T - \frac{1}{n}.$$

Hence $\mathbb{E}\, Z_{T_n}$ converges to $\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T$ as n approaches ∞. □

As (T_n) is increasing, its limit $T^\star := \lim_{n\to\infty} T_n$ exists and is a stopping time. Obviously, $T^\star \in \mathcal{T}$ in the finite horizon case (as $T_n \leq m$). But, this is not necessarily so in the infinite horizon case, e.g. $T^\star = \infty$ everywhere if $Z_t = 1 - \mathrm{e}^{-t}$ (since $C = 1$ in this case).

Corollary 2.8. Let Z be a quasi-left continuous payoff process‡‡. Then the stopping time $T^\star$ defined above is optimal in the finite horizon case and it is also optimal in the infinite horizon case provided it is finite.

Proof. Since Z is quasi-left continuous, $T_n \uparrow T^\star$ and $T^\star \in \mathcal{T}$, we have $Z_{T_n} \to Z_{T^\star}$ a.s. Since Z is of class D, this implies $\mathbb{E}\, Z_{T_n} \to \mathbb{E}\, Z_{T^\star}$. Hence $T^\star$ is optimal by Lemma 2.7. □

2.3 Multiplicative minimax duality

The multiplicative version requires a Bayes' rule formula related to choice of numeraire:

Proposition 2.9. Let Z be a payoff process, B be a numeraire and $T \in \mathcal{T}$. Then

$$\mathbb{E}\, Z_T = \mathbb{E}\left(B_m \frac{Z_T}{B_T}\right).$$

---

‡‡Recall this means Z has left limits and does not jump at any predictable time T, i.e. $1_{T<\infty}\Delta Z_T = 0$ a.s. It implies that $Z_{T_n} \to Z_T$ a.s., whenever $T_n \uparrow T$ with $T_n, T \in \mathcal{T}$. (See, e.g. Proposition I.2.26 in Ref. [17]).

Proof. Set $\mathcal{F}_T := \{\Lambda \in \mathcal{F} : \Lambda \cap \{T \leq t\} \in \mathcal{F}_t \forall t\}$, as usual. Then, iterating expectation,

$$\mathbb{E}\left(B_m \frac{Z_T}{B_T}\right) = \mathbb{E}\left(\mathbb{E}\left(B_m \frac{Z_T}{B_T} | \mathcal{F}_T\right)\right) = \mathbb{E}\left(\frac{Z_T}{B_T} \mathbb{E}(B_m | \mathcal{F}_T)\right) = \mathbb{E}\left(\frac{Z_T}{B_T} B_T\right) = \mathbb{E}\, Z_T,$$

where we also used the $\mathcal{F}_T$-measurability of Z_T/B_T and the optional sampling theorem. □

We also need a multiplicative version of the pathwise characterization in Proposition 2.5.

PROPOSITION 2.10. A numeraire B domineers a payoff process Z if and only if

$$\sup_{t \in \mathbb{T}} \frac{Z_t}{B_t} = 1.$$

Proof. Define the probability measure $\mathbb{P}^B$ by $(\mathrm{d}\mathbb{P}^B/\mathrm{d}\mathbb{P}) = (B_m/B_0)$. One easily verifies directly or as an immediate consequence Corollary 2.4 that Z/B is $\mathbb{P}^B$-class D and hence a $\mathbb{P}^B$-payoff process. Moreover, denoting $\mathbb{E}^B$ the expectation operator of $\mathbb{P}^B$, Proposition 2.9 implies that $\mathbb{E}\, Z_T = B_0\, \mathbb{E}^B(Z_T/B_T)$. It thus follows from the definition of a domineering claim that B domineers Z if and only if the constant process 1 domineers the $\mathbb{P}^B$-payoff process Z/B under the measure $\mathbb{P}^B$. But, Proposition 2.5 applied to Z/B and $\mathbb{P}^B$ shows that the latter condition is equivalent to $\sup_{t \in \mathbb{T}} ((Z_t/B_t) - 1) = 0$, which is the desired condition¶¶. □

The multiplicative formulae next for positive and nonnegative payoffs are from Ref. [20].

COROLLARY 2.11. *Positive minimax duality*: Let Z be a payoff process such that $Z_m > 0$ a.s. in the finite horizon case and $\overline{\lim}_{t \to \infty} Z_t > 0$ a.s. in infinite horizon. Then,

$$\sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T = \min_{B \in \mathcal{C}^+} \mathbb{E}\left(B_m \sup_{t \in \mathbb{T}} \frac{Z_t}{B_t}\right).$$

The minimum is attained at any numeraire B that domineers Z (and at multiples of B).

Proof. For any numeraire B and stopping time $T \in \mathcal{T}$, we have, using Proposition 2.9,

$$\mathbb{E}\, Z_T = \mathbb{E}\left(B_m \frac{Z_T}{B_T}\right) \leq \mathbb{E}\left(B_m \sup_{t \in \mathbb{T}} \frac{Z_t}{B_t}\right).$$

Taking sup over T, we see "$\leq$" holds. Let B be a claim that domineers Z. Then B is a numeraire by the positivity assumption on Z, as $B \geq Z$. Using the definition of domineering claim, the martingale property of B and Proposition 2.10, we have, $\sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T = B_0 = \mathbb{E}(B_m) = \mathbb{E}(B_m \sup_{t \in \mathbb{T}} (Z_t/B_t))$. Equality and the second statement thus follow. □

The above positive multiplicative minimax duality extends to the nonnegative case, the only difference being that the "min" gets replaced by an "inf":

¶¶An alternative argument uses the fact any numeraire B is pathwise bounded above and bounded below above zero. This easily implies that $\sup_{t \in \mathbb{T}}(Z_t/B_t) = 1$ if and only if $\sup_{t \in \mathbb{T}}(Z_t - B_t) = 0$. But, the latter condition is equivalent to B domineering Z by Proposition 2.5.

Corollary 2.12. *Nonnegative minimax duality*: Let $Z \geq 0$ be a payoff process. Then

$$\sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T = \inf_{B \in \mathcal{C}^+} \mathbb{E}\left(B_m \sup_{t \in \mathbb{T}} \frac{Z_t}{B_t} \right).$$

Proof. As in the proof of the previous corollary, "≤" holds. To show "≥", let $\varepsilon > 0$. Let C be a claim that domineers Z. Then $C \geq Z \geq 0$. Set $B^\varepsilon = C + \varepsilon$. Clearly, $B^\varepsilon \in \mathcal{C}^+$ domineers the payoff process $Z + \varepsilon$. Hence, by the second part of Corollary 2.11 applied to $Z + \varepsilon$,

$$\sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T + \varepsilon = \mathbb{E}\left(B_m^\varepsilon \sup_{t \in \mathbb{T}} \frac{Z_t + \varepsilon}{B_t^\varepsilon} \right) \geq \mathbb{E}\left(B_m^\varepsilon \sup_{t \in \mathbb{T}} \frac{Z_t}{B_t^\varepsilon} \right) \geq \inf_{B \in \mathcal{C}^+} \mathbb{E}\left(B_m \sup_{t \in \mathbb{T}} \frac{Z_t}{B_t} \right).$$

Since $\varepsilon > 0$ was arbitrary, it follows that "≥" holds. □

Remark. The "inf" above is attained at certain "semipositive options" defined in Ref. [20].

3. The Snell envelope and its Doob–Meyer decomposition

The ***Snell envelope*** process V of a payoff process Z is defined, for each t a.s. by

$$V_t := \operatorname{ess} \sup_{t \leq T \in \mathcal{T}} \mathbb{E}(Z_T | \mathcal{F}_t).$$

By a ***Superclaim***, we mean a right-continuous class-D supermartingale.

3.1 The superclaim property of the Snell envelope

We next show the Snell envelope is a superclaim. This property and the Doob–Meyer decomposition theorem are the two pillars of the duality approach to optimal stopping. Our statement is similar to the (first) Theorem in Section IV.A of Ref. [30] (p. 28). Equation (3.1) below, known as *the principle of dynamic programming*, is usually stated with $\alpha = 1$. It plays a key role in the proof.

Theorem 3.1. Let Z be a payoff process. Then its Snell envelope V has a unique right-continuous version, denoted also V, which is a superclaim. Moreover, V is the smallest superclaim satisfying $V \geq Z$, i.e. if $V' \geq Z$ is another superclaim, then $V \leq V'$.

Furthermore, for any two stopping times $S \leq \tau \in \mathcal{T}$ and nonnegative bounded $\mathcal{F}_\tau$-measurable random variable α, we have

$$\mathbb{E}(\alpha V_\tau | \mathcal{F}_S) = \operatorname{ess} \sup_{\tau \leq T \in \mathcal{T}} \mathbb{E}(\alpha Z_T | \mathcal{F}_S). \tag{3.1}$$

The proof is given after six lemmas. We first prove a version of equation (3.1) for deterministic τ and use it to show V is a uniformly integrable, right-continuous supermartingale. Using this, we next show that $V_\tau = \operatorname{ess.} \sup_{t \leq T \in \mathcal{T}} \mathbb{E}(Z_T | \mathcal{F}_\tau)$ for all stopping times $\tau \in \mathcal{T}$ (which holds by definition for deterministic τ). This then implies equation (3.1) with general τ, which is in turn used (with a general α) to prove that V is of class D.

Lemma 3.2. Let $S \le \tau \in \mathcal{T}$ and α be as above. Set $V'_\tau := \operatorname{ess.sup}_{\tau \le T \in \mathcal{T}} \mathbb{E}(Z_T|\mathcal{F}_\tau)$. Then

$$\mathbb{E}(\alpha V'_\tau|\mathcal{F}_S) = \operatorname*{ess\,sup}_{\tau \le T \in \mathcal{T}} \mathbb{E}(\alpha Z_T|\mathcal{F}_S).$$

In particular, equation (3.1) holds when τ is deterministic (for then $V'_\tau = V_\tau$ by definition).

Proof. If $T \in T$ and $T \ge \tau$, then since $\mathbb{E}(Z_T|\mathcal{F}_\tau) \le V'_\tau$, we have

$$\mathbb{E}(\alpha Z_T|\mathcal{F}_S) = \mathbb{E}(\mathbb{E}(\alpha Z_T|\mathcal{F}_\tau)|\mathcal{F}_S) = \mathbb{E}(\alpha\mathbb{E}(Z_T|\mathcal{F}_\tau)|\mathcal{F}_S) \le \mathbb{E}(\alpha V'_\tau|\mathcal{F}_S).$$

Hence "$\ge$" follows. To show "$\le$", let $(t_n)_{n=1}^\infty$ be a dense subset of $\mathbb{T}$, e.g. an enumeration of the rational numbers in $\mathbb{T}$. Set

$$\mathcal{T}_n := \{T \in \mathcal{T} : T(\Omega) \subset \{t_1, \ldots, t_n\}\}, \ \ n \in \mathbb{N}.$$

Note $\mathcal{T}_n \subset \mathcal{T}_{n+1} \subset \mathcal{T}$. Let $T \in \mathcal{T}$ and $T \ge \tau$. For $n \in \mathbb{N}$, define the stopping time $T_n \in \mathcal{T}_n$ by

$$T_n := \min\{t \in \{t_1, \ldots, t_n\} : t \ge T\} \in \mathcal{T}_n.$$

Clearly, $T_n \ge T_{n+1} \ge T$. As $(t_n)_{n=1}^\infty$ is dense, $T_n \downarrow T$. Since Z is right-continuous this implies $Z_{T_n} \to Z_T$. As Z is of class D, $\mathbb{E}(Z_T|\mathcal{F}_\tau) = \lim_{n\to\infty} \mathbb{E}(Z_{T_n}|\mathcal{F}_\tau)$ a.s. and in L^1.

It is well-known that for each n there is a stopping time $T_n^* \in \mathcal{T}_n$ such that $\mathbb{E}(Z_{T_n^*}|\mathcal{F}_\tau) \ge \mathbb{E}(Z_{T'}|\mathcal{F}_\tau)$ for all $T' \in \mathcal{T}_n$ with $T' \ge \tau$. (This is shown by (backward) induction (e.g. Refs. [14,20]) and amounts to saying that in the *Bermudan* case, given τ, there always exists an optimal stopping time $T_n^* \ge \tau$.) Moreover, $\mathbb{E}(Z_{T_{n+1}^*}|\mathcal{F}_\tau) \ge \mathbb{E}(Z_{T_n^*}|\mathcal{F}_\tau)$ as $\mathcal{T}_n \subset \mathcal{T}_{n+1}$.

In particular, $\mathbb{E}(Z_{T_n^*}|\mathcal{F}_\tau) \ge \mathbb{E}(Z_{T_n}|\mathcal{F}_\tau)$. Hence $\underline{\lim}_{n\to\infty} \mathbb{E}(Z_{T_n^*}|\mathcal{F}_\tau) \ge \mathbb{E}(Z_T|\mathcal{F}_\tau)$. This implies $\underline{\lim}_{n\to\infty} \mathbb{E}(Z_{T_n^*}|\mathcal{F}_\tau) \ge V'_\tau$. Since $\mathbb{E}(Z_{T_n^*}|\mathcal{F}_\tau) \le V'_\tau$, we conclude $\mathbb{E}(Z_{T_n^*}|\mathcal{F}_\tau) \uparrow V'_\tau$ a.s. Since α is $\mathcal{F}_\tau$ measurable, this implies $\mathbb{E}(\alpha Z_{T_n^*}|\mathcal{F}_\tau) \uparrow \alpha V'_\tau$ a.s. Therefore, applying the monotone convergence theorem and then iterating expectation, we have $\mathbb{E}(\alpha V'_\tau|\mathcal{F}_S) = \lim_{n\to\infty} \mathbb{E}(\alpha Z_{T_n^*}|\mathcal{F}_S)$. Hence $\mathbb{E}(\alpha V'_\tau|\mathcal{F}_S) \le \sup_{\tau \le T \in \mathcal{T}} \mathbb{E}(\alpha Z_T|\mathcal{F}_S)$, as desired. (Note, it also follows that V is adapted because, as we showed, $\mathbb{E}(Z_{T_n^*}|\mathcal{F}_t) \uparrow V'_t = V_t$ for $t \in \mathbb{T}$.) □

Lemma 3.3. The Snell envelope V is a supermartingale.

Proof. The proof of Lemma 3.2 showed V is adapted. Let $s \le t \in \mathbb{T}$. By Lemma 3.2, equation (3.1) holds when $\tau = t$. Applying it with $\alpha = 1$, $S = s$ and then using $s \le t$, we get

$$\mathbb{E}(V_t|\mathcal{F}_s) = \operatorname*{ess\,sup}_{t \le T \in \mathcal{T}} \mathbb{E}(Z_T|\mathcal{F}_s) \le \operatorname*{ess\,sup}_{s \le T \in \mathcal{T}} \mathbb{E}(Z_T|\mathcal{F}_s) =: V_s.$$

Hence V is a supermartingale. □

Lemma 3.4. The Snell envelope V has a unique right-continuous version.

Proof. Since V is a supermartingale, it suffices to show $\mathbb{E}V_t$ is right-continuous. As $\mathbb{E}V_t$ is decreasing, this follows if we show for every $t \ge 0$, there is a sequence of stopping times $(T_n^t)_{n=1}^\infty$ in $\mathcal{T}$ such that $T_n^t \ge t + (1/n)$ and $\mathbb{E}\,Z_{T_n^t} \to \mathbb{E}V_t$, for then $\mathbb{E}V_{T_n^t} \to \mathbb{E}V_t$ too, as $V \ge Z$.

Let $t \geq 0$. By equation (3.1) (with $\tau = t$, $\alpha = 1$), $\mathbb{E}V_t = \sup_{t \leq T \in \mathcal{T}} \mathbb{E}\, Z_T$. Hence, there is a sequence $T_n \in \mathcal{T}$ such that $T_n \geq t$ and $\mathbb{E}\, Z_{T_n} \longrightarrow \mathbb{E}V_t$ as $n \longrightarrow \infty$. Define $T_n^t := T_n \vee (t + (1/n))$.

The sequence $W_n := 1_{\{T_n < t+(1/n)\}}(Z_{T_n^t} - Z_{t+(1/n)})$ converges to 0 a.s. by the right continuity of Z and is uniformly integrable because Z is of class D; so $\mathbb{E}W_n \longrightarrow 0$. Thus, as $n \longrightarrow \infty$,

$$\mathbb{E}\, Z_{T_n^t} = \mathbb{E}\, Z_{T_n} + \mathbb{E}(Z_{T_n^t} - Z_{T_n}) = \mathbb{E}\, Z_{T_n} + \mathbb{E}(1_{\{T_n < t+(1/n)\}}(Z_{T_n^t} - Z_{t+(1/n)})) \longrightarrow \mathbb{E}V_t,$$

since $\mathbb{E}\, Z_{T_n} \longrightarrow \mathbb{E}V_t$ and $\mathbb{E}W_n \longrightarrow 0$. Thus the sequence (T_n^t) has the requisite properties.

□

Lemma 3.5. The Snell envelope V is uniformly integrable.

Proof. First, we may assume $Z \geq 0$, for if true in this case, then the Snell envelope W of the payoff process $|Z|$ is uniformly integrable, which implies V is uniformly integrable, since clearly $|V| \leq W$. So, we assume $Z \geq 0$, implying $V \geq 0$.

We must show for every $\varepsilon > 0$, there is $K > 0$ such that $\mathbb{E}(1_{\{V_t > K\}} V_t) < \varepsilon$ for all $t \in \mathbb{T}$.

Let $\varepsilon > 0$. Since Z is of class D, there exist $\delta > 0$ such that $\mathbb{E}(1_\Lambda Z_T) < \varepsilon$ for all $T \in \mathcal{T}$ if $\Lambda \in \mathcal{F}$ satisfies $\mathbb{P}\Lambda < \delta$. Choose $K > (V_0/\delta)$. Then

$$\mathbb{P}\{V_t > K\} \leq \frac{\mathbb{E}V_t}{K} \leq \frac{V_0}{K} < \delta.$$

(Here we used, $\mathbb{E}X = \mathbb{E}(1_{X>K}X) + \mathbb{E}(1_{X \leq K}X) \geq K\mathbb{P}\{X > K\}$, with $X = V_t$). By Lemma 3.2, equation (3.1) holds for $\tau = t$. Applying it with $\alpha = 1_{\{V_t > K\}}$, $S = 0$, we conclude

$$\mathbb{E}(1_{\{V_t > K\}} V_t) = \sup_{t \leq T \in \mathcal{T}} \mathbb{E}(1_{\{V_t > K\}} Z_T) < \varepsilon,$$

as desired. □

From now on, we always take the *right-continuous version* of the Snell envelope V.

Lemma 3.6. If either $V \geq 0$ or V is of class D, then $V_\tau = \operatorname{ess.sup}_{\tau \leq T \in \mathcal{T}} \mathbb{E}(Z_T | \mathcal{F}_\tau) =: V'_\tau$ for all $\tau \in \mathcal{T}$. Hence equation (3.1) holds in either of these two cases by Lemma 3.2.

Proof. Let $T \geq \tau$ be in $\mathcal{T}$. By the previous three lemmas, the optional sampling theorem is applicable to V. Hence, $V_\tau \geq \mathbb{E}(V_T | \mathcal{F}_\tau) \geq \mathbb{E}(Z_T | \mathcal{F}_\tau)$. Therefore, $V_\tau \geq V'_\tau$.

Let $(t_n)_{n=1}^{\infty}$ be a dense subset of $\mathbb{T}$ and define the sets $\mathcal{T}_n$ as in the proof of Lemma 3.2.

Assume first that $\tau \in \mathcal{T}_n$ for some n. Then, for each integer $1 \leq i \leq n$, we have,

$$\begin{aligned} 1_{\{\tau = t_i\}} V_\tau &= 1_{\{\tau = t_i\}} V_{t_i} = 1_{\{\tau = t_i\}} \text{ess.} \sup_{t_i \leq T \in \mathcal{T}} \mathbb{E}(Z_T | \mathcal{F}_{t_i}) = \text{ess} \sup_{t_i \leq T \in \mathcal{T}} \mathbb{E}(1_{\{\tau = t_i\}} Z_T | \mathcal{F}_{t_i}) \\ &\leq \text{ess} \sup_{\tau \leq T \in \mathcal{T}} \mathbb{E}(1_{\{\tau = t_i\}} Z_T | \mathcal{F}_{t_i}) = \text{ess} \sup_{\tau \leq T \in \mathcal{T}} \mathbb{E}(1_{\{\tau = t_i\}} Z_T | \mathcal{F}_\tau) \\ &= 1_{\{\tau = t_i\}} \text{ess.} \sup_{\tau \leq T \in \mathcal{T}} \mathbb{E}(Z_T | \mathcal{F}_\tau). \end{aligned}$$

(The inequality above follows because for each stopping time $T \geq t_i$, there is a stopping time $T' \geq \tau$ such that $1_{\{\tau = t_i\}} Z_{T'} = 1_{\{\tau = t_i\}} Z_T$, e.g. $T' = 1_{\{\tau = t_i\}} T + 1_{\{\tau \neq t_i\}} t_n$.) Hence $V_\tau \leq V'_\tau$. Thus $V_\tau = V'_\tau$. So, equation (3.1) holds by Lemma 3.2, implying $\mathbb{E}V_\tau = \sup_{\tau \leq T \in \mathcal{T}} \mathbb{E}\, Z_T$.

Now, let $\tau \in \mathcal{T}$ be general. Define the sequence $\tau_n := \min\{t \in \{t_1, \ldots, t_n\} : t \geq \tau\}$. Then, $\tau_n \in \mathcal{T}_n$ and $\tau_n \downarrow \tau$. If $V \geq 0$, then this, the right continuity of V and Fatou's Lemma imply that $\mathbb{E}V_\tau \leq \underline{\lim}\, \mathbb{E}V_{\tau_n}$; whereas, if V is class D, then more strongly, $\mathbb{E}V_\tau = \lim \mathbb{E}V_{\tau_n}$. Therefore either way, using the formula $\mathbb{E}V_{\tau_n} = \sup_{\tau_n \leq T \in \mathcal{T}} \mathbb{E}Z_T$ shown above, we get

$$\mathbb{E}V_\tau \leq \underline{\lim}_{n\to\infty} \mathbb{E}V_{\tau_n} = \underline{\lim}_{n\to\infty} \sup_{\tau_n \leq T \in \mathcal{T}} \mathbb{E}Z_T \leq \sup_{\tau \leq T \in \mathcal{T}} \mathbb{E}Z_T = \sup_{\tau \leq T \in \mathcal{T}} \mathbb{E}(\mathbb{E}(Z_T|\mathcal{F}_\tau))$$

$$\leq \mathbb{E}(\operatorname{ess} \sup_{\tau \leq T \in \mathcal{T}} \mathbb{E}(Z_T|\mathcal{F}_\tau)).$$

Hence, $\mathbb{E}V_\tau \leq \mathbb{E}V'_\tau$. Since as shown earlier $V_\tau \geq V'_\tau$, it follows that $V_\tau = V'_\tau$. □

LEMMA 3.7. The Snell envelope V is of class D.

Proof. First, we may assume $Z \geq 0$, for if true in this case, then the Snell envelope W of the payoff process $|Z|$ is class D, which implies V is of class D, since clearly $|V| \leq W$. So, we assume $Z \geq 0$, implying $V \geq 0$.

We must show for every $\varepsilon > 0$, there is $K > 0$ such that $\mathbb{E}(1_{\{V_\tau > K\}} V_\tau) < \varepsilon$ for all $\tau \in \mathcal{T}$.

Let $\varepsilon > 0$. Since Z is of class D, there exist $\delta > 0$ such that $\mathbb{E}(1_\Lambda Z_T) < \varepsilon$ for all $T \in \mathcal{T}$ if $\Lambda \in \mathcal{F}$ satisfies $\mathbb{P}\Lambda < \delta$. Choose $K > (V_0/\delta)$. Then

$$\mathbb{P}\{V_\tau > K\} \leq \frac{\mathbb{E}V_\tau}{K} \leq \frac{V_0}{K} < \delta.$$

(Here we used, $\mathbb{E}X = \mathbb{E}(1_{X>K}X) + \mathbb{E}(1_{X\leq K}X) \geq K\mathbb{P}\{X > K\}$, with $X = V_\tau$). Since $V \geq 0$, equation (3.1) holds by Lemma 3.6. Applying it with $\alpha = 1_{\{V_\tau > K\}}$ and $S = 0$ gives,

$$\mathbb{E}(1_{\{V_\tau > K\}} V_\tau) = \sup_{\tau \leq T \in \mathcal{T}} \mathbb{E}(1_{\{V_\tau > K\}} Z_T) < \varepsilon,$$

as desired. □

Proof of Theorem 3.1. Lemmas 3.3, 3.4 and 3.7 imply that V is a superclaim. As for the second statement, let $V' \geq Z$ be another superclaim. Let $t \in \mathbb{T}$ and $T \in \mathcal{T}$ with $T \geq t$. By the optional sampling theorem, $V'_t \geq \mathbb{E}(V'_T|\mathcal{F}_t) \geq \mathbb{E}(Z_T|\mathcal{F}_t)$. Hence $V'_t \geq V_t$, as desired. The final statement, i.e. equation (3.1), follows from Lemma 3.6, for as shown, V is of class D. □

Remark. Even when Z is continuous, the Snell envelope V may have jumps. This can happen when Z is driven by a Brownian motion but its "volatility" has a jump not adapted to the Brownian filtration, for instance, an American option on a stock with continuous price but a volatility that jumps at a totally inaccessible time T, causing V to jump at T.

In what follows, the compensator of a special semimartingale X is denoted X^p and we set $X^m := X - X^p$. So, X^p is the unique predictable finite-variation process such that $X^p_0 = 0$ and X^m is a local martingale. As such, $X = X^p + X^m$ is the *canonical decomposition* of X. The compensator X^p is decreasing if X is a local supermartingale. The *Doob–Meyer decomposition* states a superclaim X has such a decomposition with X^m a claim and X^p decreasing and integrable (and in infinite horizon, $\lim_{t\to\infty} X^p_t \in L^1$).

Corollary 3.8. A claim C domineers Z if and only if $C_0 = V_0$ and $C \geq V$. In particular, the claim V^m domineers Z. □

3.2 Domineering claims at τ and implications

Let Z be a payoff process and $\tau \in \mathcal{T}$. We say a claim C *domineers Z at τ* if $C_\tau = V_\tau$ and $C \geq Z$ after τ, i.e. on the stochastic interval $[[\tau, \infty[[$ (or $[[\tau, m]]$ if finite horizon). By the optional sampling theorem then $C \geq V$ after τ. An elementary but fruitful pathwise characterization is

Theorem 3.9. Let C be a claim and $\tau \in \mathcal{T}$. Then C domineers Z at τ if and only if

$$\sup_{\tau \leq t \in \mathbb{T}} (Z_t - C_t) = 0.$$

Proof. Assume C domineers Z at τ. Since $C_\tau = \mathbb{E}(C_T | \mathcal{F}_\tau)$ for $\tau \leq T \in \mathbb{T}$, we have

$$0 = V_\tau - C_\tau = \operatorname{ess} \sup_{\tau \leq T \in \mathcal{T}} \mathbb{E}(Z_T - C_T | \mathcal{F}_\tau) \leq \mathbb{E}(\operatorname{ess} \sup_{\tau \leq t \in \mathbb{T}} (Z_t - C_t) | \mathcal{F}_\tau).$$

As $Z \leq C$ after τ, this implies $\sup_{\tau \leq t \in \mathbb{T}} (Z_t - C_t) = 0$ a.s. Conversely, assume $\sup_{\tau \leq t \in \mathbb{T}} (Z_t - C_t) = 0$. Then $Z \leq C$ after τ, so $\mathbb{E}(Z_T | \mathcal{F}_\tau) \leq \mathbb{E}(C_T | \mathcal{F}_\tau) = C_\tau$. Hence, $\sup_{T \geq \tau} \mathbb{E}(Z_T | \mathcal{F}_\tau) \leq C_\tau$. To prove the reverse inequality, it suffices to show that for any $\varepsilon > 0$, there is a stopping time $\tau \leq T \in \mathcal{T}$ such that $\mathbb{E}(Z_T | \mathcal{F}_\tau) \geq C_\tau - \varepsilon$. Let $\varepsilon > 0$. The assumption implies that (pathwise) the set $\{\tau \leq t \in \mathbb{T} : Z_t - C_t \geq -\varepsilon\}$ is nonempty a.s. Therefore, the stopping time $T := \inf\{\tau \geq t \in \mathbb{T} : Z_t - C_t \geq -\varepsilon\}$ is finite. By right continuity, $Z_T - C_T \geq -\varepsilon$. Thus, $\mathbb{E}(Z_T | \mathcal{F}_\tau) \geq \mathbb{E}(C_T - \varepsilon | \mathcal{F}_\tau) = C_\tau - \varepsilon$, as desired. □

For any integrable random variable ξ, we denote by $\mathbb{E}(\xi | \mathbb{F})$ the unique claim whose value at each $t \in \mathbb{T}$ equals $\mathbb{E}(\xi | \mathcal{F}_t)$ a.s. The fact that V^m domineers Z easily implies

Proposition 3.10. Let $\tau \in \mathcal{T}$. Then the claim $C := V^m + \mathbb{E}(V^p_\tau | \mathbb{F})$ domineers Z at τ.

Proof. Set $M := \mathbb{E}(V^p_\tau | \mathbb{F})$. Clearly $M_\tau = V^p_\tau$. So, $C_\tau = V^m_\tau + V^p_\tau = V_\tau$. Next, $1_{t \geq \tau} V^p_\tau$ is $\mathcal{F}_t$-measurable for any t, so $1_{t \geq \tau} M_t = \mathbb{E}(1_{t \geq \tau} V^p_\tau | \mathcal{F}_t) = 1_{t \geq \tau} V^p_\tau$. Hence, $M_t = V^p_\tau$ on $\{t \geq \tau\}$. Thus, $C_t = V^m_t + V^p_\tau \geq V_t \geq Z_t$ on $\{t \geq \tau\}$. Therefore C domineers Z at τ. □

A first consequence is the following formula for the compensator of the Snell envelope.

Corollary 3.11. Let V be the Snell envelope of Z and $\tau \in \mathcal{T}$ be a stopping time. Then,

$$V^p_\tau = \sup_{\tau \leq t \in \mathbb{T}} (Z_t - V^m_t), \quad \text{i.e.} \quad \sup_{\tau \leq t \in \mathbb{T}} (Z_t - V_t + V^p_t - V^p_\tau) = 0.$$

In particular, as V^p is decreasing, we also have $\sup_{\tau \leq t \in \mathbb{T}} (Z_t - V_t) = 0$ and $1_{\{Z<V\}} \, \mathrm{d}V^p = 0$.

Proof. By Proposition 3.10, $C := V^m + \mathbb{E}(V^p_\tau | \mathbb{F})$ domineers Z at τ. Theorem 3.7 yields, $0 = \sup_{\tau \leq t \in \mathbb{T}} (Z_t - C_t) = \sup_{\tau \leq t \in \mathbb{T}} (Z_t - V^m_t - V^p_\tau)$, since $C_t = V^m_t + V_\tau$ for $t \geq \tau$. □

Given a stopping time τ, the following result (which is obvious when $\tau = 0$) shows that there is a sequence of stopping times $T_n \geq \tau$ such that $\mathbb{E}(Z_{T_n}|\mathcal{F}_\tau) \to V_\tau$ a.s. *uniformly* over Ω. A similar result appears in Theorem 13 of [28].

Corollary 3.12. *Conditional uniformly ε-optimal stopping*: Let $\tau \in \mathcal{T}$ and $\varepsilon > 0$. Then there exists a stopping time $T^\varepsilon \in \mathcal{T}$ such that $T^\varepsilon \geq \tau$ and $\mathbb{E}(Z_{T^\varepsilon}|\mathcal{F}_\tau) \geq V_\tau - \varepsilon$.

Proof. Set $T^\varepsilon := \inf\{\tau \leq t \in \mathbb{T} : Z_t + \varepsilon \geq C_t\}$, where C is any claim that domineers Z at τ. Theorem 3.9 implies that $T^\varepsilon \in \mathcal{T}$ (is finite). By right continuity of Z and C, we have $Z_{T^\varepsilon} + \varepsilon \geq C_{T^\varepsilon}$. Therefore, by the optional sampling theorem, for any $\mathcal{T} \ni T \geq \tau$,

$$\mathbb{E}(Z_{T^\varepsilon}|\mathcal{F}_\tau) + \varepsilon \geq \mathbb{E}(C_{T^\varepsilon}|\mathcal{F}_\tau) = C_\tau = \mathbb{E}(C_T|\mathcal{F}_\tau) \geq \mathbb{E}(Z_T|\mathcal{F}_\tau),$$

because C domineers Z at τ. Taking “sup” over T, we get $\mathbb{E}(Z_{T^\varepsilon}|\mathcal{F}_\tau) + \varepsilon \geq V_\tau$. □

3.3 Supersolution iterative construction of the Snell envelope

Let Z be a payoff process. Following [6], we call a superclaim W a *supersolution* if $W \geq Z$.

As in Ref. [6], for any supersolution W we define its *additive improvement* process W' by

$$W'_t := W^m_t + \mathbb{E}\left(\sup_{t \leq s \in \mathbb{T}} \left(Z_s - W^m_s\right)|\mathcal{F}_t\right).$$

The following two results extend to continuous time similar Bermudan results in Ref. [6].

Corollary 3.13. Let W be a supersolution, $t \in \mathbb{T}$. Then $W_t = V_t$ if and only if $W'_t = W_t$.

Proof. By Corollary 3.11, $V_t = V^m_t + \sup_{t \leq s \in \mathbb{T}}(Z_s - V^m_s)$. Taking expectation conditioned on t gives $V_t = V'_t$. Conversely, suppose $W'_t = W_t$. Then, $\mathbb{E}(\sup_{t \leq s \in \mathbb{T}}(Z_s - W_s + W^p_s - W^p_t)|\mathcal{F}_t) = 0$. But, since $Z \leq W$ and W^p is decreasing, $\sup_{t \leq s \in \mathbb{T}}(Z_s - W_s + W^p_s - W^p_t)$ is nonpositive; thus it must equals 0 a.s. Hence, $\sup_{t \leq s \in \mathbb{T}}(Z_s - C_s) = 0$, where $C := W^m + \mathbb{E}(W^p_t|\mathbb{F})$. Theorem 3.9 implies the claim C domineers Z at t. Thus, $V_t = C_t = W^m_t + W^p_t = W_t$. □

Proposition 3.14. Let W be a supersolution. Then W' is a supersolution and $W' \leq W$.

Proof. Clearly, $W' \geq Z$. Also, $W' \leq W$. Indeed, since $Z \leq W$ and W^p is decreasing, $W'_t = W_t + \mathbb{E}(\sup_{t \leq s \in \mathbb{T}}(Z_s - W_s + W^p_s - W^p_t)|\mathcal{F}_t) \leq W_t$. Moreover, W' is a superclaim. Indeed, W' is class D for it is bounded above and below by class D processes W and Z. It is also a supermartingale as (W^m is a claim and) $\mathbb{E}(\sup_{t \leq s \in \mathbb{T}}(Z_s - W^m_s)|\mathcal{F}_t)$ is so:

$$\begin{aligned}\mathbb{E}\left(\mathbb{E}\left(\sup_{t \leq s \in \mathbb{T}}\left(Z_s - W^m_s\right)|\mathcal{F}_t\right)|\mathcal{F}_u\right) &= \mathbb{E}\left(\sup_{t \leq s \in \mathbb{T}}\left(Z_s - W^m_s\right)|\mathcal{F}_u\right) \\ &\leq \mathbb{E}\left(\sup_{u \leq s \in \mathbb{T}}\left(Z_s - W^m_s\right)|\mathcal{F}_u\right).\end{aligned}$$

Finally, we claim W' has a unique right-continuous version. It suffices to show $\mathbb{E}W'_t$ is right-continuous. This follows by iterating expectation once we show $\mathbb{E}\sup_{t\le s\in\mathbb{T}}(Z_s - W^m_s)$ is so. But, this is so by the monotone convergence theorem since $\sup_{t\le s\in\mathbb{T}}(Z_s - W^m_s)$ is right-continuous and decreasing. Hence, (the right-continuous version of) W' is a superclaim. □

Proposition 3.14 too implies $V' = V$: since V is the smallest supersolution, $V \le V' \le V$.

Conjecture. Let $W^{(1)}$ be a supersolution. Define inductively the decreasing sequence $W^{(n+1)} := W^{(n)'}$. We conjecture that $W^{(n)}_t \downarrow V_t$ uniformly in t as $n \longrightarrow \infty$.

This conjecture is supported by the convergence in finite number of iterations in the Bermudan case as shown by [6] (see Corollary 7.3). The monotone convergence theorem easily implies $W^{(n)}$ converges to some supermartingale W. To conclude $W = V$ is more difficult.

3.4 Doob–Meyer decomposition in arbitrary numeraire

Given any numeraire B, let $\mathbb{P}^B$ denote the numeraire measure defined by $\mathrm{d}\mathbb{P}^B/\mathrm{d}\mathbb{P} := B_m/B_0$ and $\mathbb{E}^B$ denote its expectation operator. If Z is a payoff process, then by the Bayes' rule, Z/B is a $\mathbb{P}^B$-payoff process.

We can rewrite the multiplicative minimax duality formula in Corollary 2.11: for all $\tau \in \mathcal{T}$,

$$V_\tau = \min_{B\in\mathcal{C}^+} B_\tau\,\mathbb{E}^B\left(\sup_{\tau\le t\in\mathbb{T}} \frac{Z_t}{B_t}|\mathcal{F}_\tau\right).$$

Proposition 3.15. Let V be a superclaim and B be a numeraire. Then there exist a unique claim C with $C_0 = V_0$ and an increasing predictable process A with $A_0 = 0$ such that

$$V = C - AB.$$

Moreover, $A = -\int \frac{\mathrm{d}V^p}{B_-}$ and $C = V^m + \int A_-\mathrm{d}B + [A, B]$.

Proof. By the Bayes' rule, V/B is a right-continuous $\mathbb{P}^B$ class-D supermartingale. The desired decomposition $V = C - AB$ follows by applying the Doob–Meyer decomposition to V/B in the measure $\mathbb{P}^B$. Next, set $A' = -\int \mathrm{d}V^p/B_-$ and $C' = V^m + \int A'_-\mathrm{d}B + [A', B]$. Clearly, A' is predictable and increasing. Moreover, $[A', B]$ is a local martingale by Proposition I.4.49 (c) in [17] (since B is so and A' is predictable and of finite variation), implying C' is also a local martingale. But, Itô calculus easily yields $V/B = (C'/B) - A'$. By the uniqueness of canonical decomposition under $\mathbb{P}^B$, we thus conclude $C' = C$ and $A' = A$. □

We will see that the domineering claim C constructed above admits a financial interpretation as a self-financing trading strategy with initial value of $V_0 = \sup_{T\in\mathcal{T}}\mathbb{E}Z_T$ that dynamically invests in one share of current American option and A shares of numeraire B.

Proposition 3.16. Let Z be a payoff process, B be a numeraire and $V = C - AB$ be the decomposition of Snell envelope V of Z given by Proposition 3.15. Let $\tau \in \mathcal{T}$. Then the claim $C - M$ domineers Z at τ, where $M := \mathbb{E}(B_m A_\tau|\mathbb{F})$. Moreover, $M_t = A_\tau B_t$ on $\{t \ge \tau\}$.

Proof. Set $\hat{C} := C - M$. Clearly, $M_\tau = A_\tau B_\tau$. Hence, $\hat{C}_\tau = C_\tau - A_\tau B_\tau = V_\tau$. Next, since $1_{t\geq\tau} A_\tau$ is $\mathcal{F}_t$-measurable, $1_{t\geq\tau} M_t = \mathbb{E}(B_m 1_{t\geq\tau} A_\tau | \mathcal{F}_t) = 1_{t\geq\tau} A_\tau \mathbb{E}(B_m|\mathcal{F}_t) = 1_{t\geq\tau} A_\tau B_t$. So, M equals $A_\tau B$ after τ. Thus, $\hat{C} = C - A_\tau B \geq C - AB = V \geq Z$ after τ. (For the first inequality, we used A is increasing and $B > 0$.) Thus $\hat{C}$ domineers Z at τ. □

PROPOSITION 3.17. *Early exercise premium*: In finite horizon, with notation as above,

$$V_t - \mathbb{E}(Z_m|\mathcal{F}_t) = \mathbb{E}(B_m(A_m - A_t)|\mathcal{F}_t) = B_t\,\mathbb{E}^B(A_m - A_t|\mathcal{F}_t).$$

Proof. Using $\mathbb{E}(B_m A_t|\mathcal{F}_t) = A_t B_t$ and $C = \mathbb{E}(C_m|\mathbb{F})$ and $V_m = Z_m$, we have

$$V + \mathbb{E}(B_m A_\cdot|\mathbb{F}) = V + AB = C = \mathbb{E}(C_m|\mathbb{F}) = \mathbb{E}(V_m + A_m B_m|\mathbb{F}) = \mathbb{E}(Z_m|\mathbb{F}) + \mathbb{E}(A_m B_m|\mathbb{F}).$$

Rearranging gives the first equality. The second equality follows by the Bayes' rule. □

4. The multiplicative Doob–Meyer–Itô decomposition

For any *positive* payoff process Z, we show there is a decomposition $AV = B$ of the Snell envelope V, with A increasing, predictable, $A_0 = 1$ and B a positive local martingale with $B_0 = V_0$. Note, $B \geq V \geq Z$. So B domineers Z when it is an actual claim.

Recall, $X = X^p + X^m$ denotes the canonical decomposition, with X^p the compensator.

LEMMA 4.1. Let X be a special semimartingale with $X_0 = 0$. Assume $1 + \Delta X^p \neq 0$ everywhere. Then the process

$$\sum_{s\leq\cdot} \frac{\Delta X^p_s \Delta X^m_s}{1 + \Delta X^p_s} = \left[\sum_{s\leq\cdot} \frac{\Delta X^p_s}{1 + \Delta X^p_s}, X^m\right]$$

is a finite variation local martingale (the countable sums being absolutely convergent) and

$$\mathcal{E}(X) = \mathcal{E}(X^p)\mathcal{E}\left(X^m - \sum_{s\leq\cdot} \frac{\Delta X^p_s \Delta X^m_s}{1 + \Delta X^p_s}\right).$$

Proof. Writing $\sum_{s\leq\cdot} = \sum_{s\leq\cdot;\Delta X_s \geq -(1/2)} + \sum_{s\leq\cdot;\Delta X_s < -(1/2)}$ (second sum being finite), we have

$$\sum_{s\leq\cdot} \left|\frac{\Delta X^p_s}{1 + \Delta X^p_s}\right| \leq 2\sum_{s\leq\cdot} |\Delta X^p_s| + \sum_{s\leq\cdot} 1_{\{\Delta X_s < -\frac{1}{2}\}} \left|\frac{\Delta X^p_s}{1 + \Delta X^p_s}\right| < \infty,$$

since X^p is of finite variation and the second sum is finite. The absolutely convergent sum $\sum_{s\leq\cdot} \Delta X^p_s/(1 + \Delta X^p_s)$ is therefore a finite-variation, predictable process, implying its bracket with X^m is local martingale because X^m is so. Moreover, the bracket is clearly as given.

As for the second formula, set $W := \sum_{s\leq\cdot}(\Delta X^p_s \Delta X^m_s/(1 + \Delta X^p_s))$ and $Y := X^m - W$. We must show $\mathcal{E}(X) = \mathcal{E}(X^p)\mathcal{E}(Y)$. (As usual, $\mathcal{E}(\cdot)$ denotes the Doléans–Dade stochastic exponential.) It suffices to show $X = X^p + Y + [X^p, Y]$, or equivalently, $W + [X^p, W] =$

$[X^p, X^m]$. But this is so as W is sum of its jumps and by its definition, $\Delta W + \Delta X^p \Delta W = \Delta X^p \Delta X^m$. □

The lemma gives the multiplicative decomposition of the stochastic exponential $\mathcal{E}(X)$. The factor $\mathcal{E}(X^p)$ is predictable and of finite variation as X^p is so and the second factor is a local martingale as $X^m - \sum_{s\leq\cdot}(\Delta X^p_s \Delta X^m_s/(1+\Delta X^p_s))$ is so. Inverting $Y = \mathcal{E}(X)$ leads to

THEOREM 4.2. Let Y be a special semimartingale such that $Y > 0$ and $Y_- > 0$. Then there is a unique decomposition $Y = AM$, where M is a local martingale and A is a predictable, finite-variation process with $A_0 = 1$. Moreover, $A > 0$, $M > 0$, $Y_- + \Delta Y^p > 0$ and

$$A = \mathcal{E}\left(\int \frac{\mathrm{d}Y^p}{Y_-}\right), \quad M = Y_0 \mathcal{E}\left(\int \frac{\mathrm{d}Y^m}{Y_-} - \sum_{s\leq\cdot} \frac{\Delta Y^p_s \Delta Y^m_s}{\left(Y_{s-} + \Delta Y^p_s\right) Y_{s-}}\right).$$

Proof. *Existence*: Set $X = \int \mathrm{d}Y/Y_-$, so that $Y = Y_0 \mathcal{E}(X)$. Since, $Y > 0$, $1 + \Delta X > 0$. Taking predictable projection, one shows this implies $1 + \Delta X^p > 0$. (Also then $Y_- + \Delta Y^p = Y_-(1+\Delta X^p) > 0$). Therefore Lemma 4.1 is applicable. Using $X^p = \int \mathrm{d}Y^p/Y_-$, $X^m = \int \mathrm{d}Y^m/Y_-$, $\Delta X^p = \Delta Y^p/Y_-$, $\Delta X^m = \Delta Y^m/Y_-$ and simplify the sum, we get $Y = AM$, with A, M as above.

Uniqueness: Suppose $Y = AM$ with the requisite properties for A and M. By Itô's product rule, $Y = Y_0 + \int M_- \mathrm{d}A + \int A_- \mathrm{d}M + [A, M]$. The latter two summands are local martingales and $\int M_- \mathrm{d}A$ is predictable and of finite variation. The uniqueness of the canonical decomposition thus implies $Y^p = \int M_- \mathrm{d}A$ and $Y^m = Y_0 + \int A_- \mathrm{d}M + [A, M]$. The first equation gives $\mathrm{d}Y^p = M_- \mathrm{d}A$. Hence dividing by Y_-, we get $\mathrm{d}Y^p/Y_- = \mathrm{d}A/A_-$. Thus, $A = \mathcal{E}(\int \mathrm{d}A/A_-) = \mathcal{E}(\int \mathrm{d}Y^p/Y_-)$, showing A is unique. Taking the continuous martingale part of the second equation, we obtain $Y^c = \int A_- \mathrm{d}M^c$ (since $[A, M]$ is purely discontinuous). Therefore $M^c = \int \mathrm{d}Y^c/A_-$, which is unique. It remains to show ΔM is unique. But, by the second equation, $\Delta Y^m = A_- \Delta M + \Delta A \Delta M = A \Delta M$. Hence, $\Delta M = \Delta Y^m/A$ is unique. □

Remark. The multiplicative formula in Theorem II.8.21 of [17] reads $Y = A'M'$, where

$$A' = 1/\mathcal{E}\left(-\int \frac{\mathrm{d}Y^p}{Y_- + \Delta Y^p}\right), \quad M' = Y_0 \mathcal{E}\left(\int \frac{\mathrm{d}Y^m}{Y_- + \Delta Y^p}\right).$$

We have $A' = A$ and $M' = M$[§§]. Note A is simpler than A', while M' is shorter than M.

Remark. If Y is a right-continuous local supermartingale, then Y has left limits by Theorem 4.4.3 of [12] and $Y_- > 0$ if $Y > 0$ by Theorem 4.4.16 there. In the latter case, the multiplicative compensator $A = \mathcal{E}(\int \mathrm{d}Y^p/Y_-)$ is decreasing as Y^p, the additive one, is so.

[§§] To show $A' = A$, it suffices to show $\int(\mathrm{d}Y^p/Y_-) - \int \mathrm{d}Y^p/(Y_- + \Delta Y^p) = [\int(\mathrm{d}Y^p/Y_-)Y_-, \int(\mathrm{d}Y^p/Y_- + \Delta Y^p)]$, which follows because both sides equal $\int(\Delta Y_p \mathrm{d}Y^p)/(Y_-(Y_- + \Delta Y^p))$. To show $M' = M$, we must show $\int \mathrm{d}Y^m/(Y_- + \Delta Y^p) = \int(\mathrm{d}Y^m/Y_-) - \sum_{s\leq\cdot}(\Delta Y^p_s \Delta Y^m_s)/((Y_{s-} + \Delta Y^p_s)Y_{s-})$. The jump process of the LHS is $\Delta Y^m/(Y_- + \Delta Y^p)$ and that of the RHS is $(\Delta Y^m/Y_-) - (\Delta Y^p \Delta Y^m)/((Y_- + \Delta Y^p)Y_-)$, which are the same. The continuous local martingale part of LHS is $\int \mathrm{d}Y^c/(Y_- + \Delta Y^p)$ and that of the RHS is $\int \mathrm{d}Y^c/Y_-$, which are equal, as presence of jump in integrand does not contribute to stochastic integral of a continuous semimartingale. Therefore, the LHS and RHS processes are equal.

Remark. Unlike the additive compensator, the multiplicative compensator is invariant under change of measure: if $Y = AM$ is the multiplicative decomposition of Y and B is any numeraire, then $Y/B = A(M/B)$ is that of Y/B under measure $\mathbb{P}^B$, as M/B is a $\mathbb{P}^B$-local martingale.

COROLLARY 4.3. Let Z be a positive payoff process and $V = AB$ be the multiplicative decomposition of its Snell envelope V. Assume B is a numeraire. Then,

$$A_t = \sup\nolimits_{t \le s \in \mathbb{T}} (Z_s/B_s).$$

Proof. Define the measure $\mathbb{P}^B$ by $\mathrm{d}\mathbb{P}^B/\mathrm{d}\mathbb{P} = B^m/B_0$. By the Bayes' rule, $A = V/B$ is the Snell envelope of the $\mathbb{P}^B$-payoff process Z/B under $\mathbb{P}^B$. Under any measure, $A^p = A - 1$ and $A^m = 1$. Corollary 3.11 applied to the $\mathbb{P}^B$-payoff process Z/B in measure $\mathbb{P}^B$ thus yields $A_t - 1 = \sup_{t \le s \in \mathbb{T}} ((Z_s/B_s) - 1)$. □

In the continuous (more generally, quasi-left continuous) case the multiplicative decomposition simplifies as the jump term in the local martingale part M vanishes. Furthermore, the Novikov's condition furnishes a criteria for M to be an actual martingale. Whence,

COROLLARY 4.4. Let Y be a positive continuous semimartingale. Then $Y = Y_0\, \mathrm{e}^{\int \mathrm{d}Y^p/Y} \mathcal{E}(\int \mathrm{d}Y^m/Y)$. Moreover, $\mathcal{E}(\int \mathrm{d}Y^m/Y)$ is a claim if $\mathbb{E} \exp((1/2)\int_0^m \mathrm{d}[Y_t]/Y_t^2) < \infty$.

5. Optimal stopping times

Henceforth, Z will denote a payoff process, often explicitly specified as such.

Recall, a finite stopping time T^* is called *optimal* if $\mathbb{E}\, Z_{T^*} = \sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T$.

5.1 Some properties of optimal stopping times

The following characterization of optimal stopping times in terms of domineering claims has useful consequences.

PROPOSITION 5.1. A stopping time T^* is optimal for a payoff process Z if and only if $Z_{T^*} = C_{T^*}$ for some, hence all, claims C that domineer Z.

Proof. Assume T^* is optimal for Z. Let C be a claim that domineers Z. By the optional sampling theorem, followed by the domineering property, next optimality of T^*,

$$\mathbb{E} C_{T^*} = C_0 = \sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T = \mathbb{E}\, Z_{T^*}.$$

Since $Z \le C$, this implies $Z_{T^*} = C_{T^*}$. Conversely, T^* is optimal if C satisfies $Z_{T^*} = C_{T^*}$, because then $\mathbb{E}\, Z_{T^*} = \mathbb{E} C_{T^*} = C_0 = \sup_{T \in \mathcal{T}} \mathbb{E}\, Z_T$. □

Corollary 5.2. If T^* is optimal for Z then $V_{T^*} = Z_{T^*}$ where V is the Snell envelope of Z.

Proof. Set $C := V^m$. Since $Z \le V \le C$, the result follows from Proposition 5.1. □

Proposition 5.1 also implies that domineering claims are unique in the "continuation region":

Corollary 5.3. Let C be a claim that domineers Z and V be the Snell envelope of Z. Assume T^* is an optimal stopping time. Then $C = V$ on the stochastic interval $[[0, T^*]] := \{(t, \omega)\} : t \le T^*(\omega)\}$. In particular, $V^p = 0$ on $[[0, T^*]]$ and $C = V^m$ on $[[0, T^*]]$.

Proof. This is equivalent to showing the stopped processes C^{T^*} and V^{T^*} are equal (indistinguishable). (Here $X_t^T := X_{t\wedge T}$.) This is so because (a) $V^{T^*} \le C^{T^*}$, (b) by the optional sampling theorem C^{T^*} is a martingale and V^{T^*} is a supermartingale and (c) by Proposition 5.1 and Corollary 5.2, they have the same terminal condition: $C_m^{T^*} = C_{T^*} = Z_{T^*} = V_{T^*} = V_m^{T^*}$. □

5.2 Existence and the shortest stopping time

The following furthers Proposition 2.1.

Corollary 5.4. A payoff process Z has an optimal stopping time if and only if for some, hence all, domineering claims C, the stopping time

$$T_* := \inf\{t \in \mathbb{T} : Z_t = C_t\}$$

is finite. In this case, T_* is optimal and independent of choice of C and $T_* \le T^*$ for all optimal stopping times T^* and $T_* = \inf\{t \in \mathbb{T} : Z_t = V_t\}$, where V is the Snell envelope.

Proof. If an optimal stopping time T^* exists, then by Proposition 5.1, $Z_{T^*} = C_{T^*}$. Therefore the set $\inf\{t \in \mathbb{T} : Z_t = C_t\}$ is pathwise nonempty a.s. Thus $T_* < \infty$. Conversely, if $T_* < \infty$, then by the definition of T_* and the right continuity of C and Z, we have $Z_{T_*} = C_{T_*}$, hence T_* is optimal by Proposition 5.1. Next, if T^* is any optimal stopping time, then $T_* \le T^*$ since $Z_{T^*} = C_{T^*}$ and $T_* := \inf\{t \in \mathbb{T} : Z_t = C_t\}$. Corollary 5.3 implies that T_* equals $\inf\{t \in \mathbb{T} : Z_t = V_t\}$ when it is optimal. This implies the independence on choice of C. □

We now return to the stopping $T^\star$ defined in Section 2.2. Recall,

$$T_n := \inf\left\{t \in \mathbb{T} : Z_t \ge C_t - \frac{1}{n}\right\}, \quad T^\star := \lim_{n\to\infty} T_n,$$

given a domineering claim C. Recall $T_n \in \mathcal{T}$ and in the finite horizon case, $T^\star \in \mathcal{T}$.

The following provides necessary and sufficient condition for $T^\star$ to be optimal and shows that then $T^\star = T_* = \inf\{t \in \mathbb{T} : Z_t = V_t\}$ is independent of choice of domineering claim C.

Theorem 5.5. Let Z be a càdlàg payoff process and T_n and $T^\star$ be as above. Assume $T^\star \in \mathcal{T}$. Then, $\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T = \mathbb{E}\, Z_{T^\star} - \mathbb{E}(1_\Lambda \Delta Z_{T^\star})$, where $\Lambda := \bigcap_{n=1}^{\infty}\{T_n < T^\star\}$. Moreover, $T^\star$ is optimal if and only if $\mathbb{E}(1_\Lambda \Delta Z_{T^\star}) = 0$ and in this case, $T^\star = T_*$, with T_* as in Corollary 5.4.

Proof. On Λ, clearly $Z_{T_n} \to Z_{T^\star_-}$. On Λ^c, we have $T_n = T^\star$ for all but finitely many n (pathwise); hence $Z_{T_n} \to Z_{T^\star}$ on Λ^c. Therefore, $Z_{T_n} \to 1_\Lambda Z_{T^\star_-} + 1_{\Lambda^c} Z_{T^\star} = Z_{T^\star} - 1_\Lambda \Delta Z_{T^\star}$, implying $\mathbb{E}\, Z_{T_n} \to \mathbb{E}\, Z_{T^\star} - \mathbb{E}(1_\Lambda \Delta Z_{T^\star})$ as Z is class D. But, $\mathbb{E}\, Z_{T_n} \to \sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T$ by Lemma 2.7. Hence, $\sup_{T\in\mathcal{T}} \mathbb{E}\, Z_T = \mathbb{E}\, Z_{T^\star} - \mathbb{E}(1_\Lambda \Delta Z_{T^\star})$. Hence $\mathbb{E}(1_\Lambda \Delta Z_{T^\star}) \le 0$ and $T^\star$ is optimal if and only if $\mathbb{E}(1_\Lambda \Delta Z_{T^\star}) = 0$. Finally, in general $T^\star \le T_*$ because clearly $T_n \le T_*$. But, if $T^\star$ is optimal then $T_* \le T^\star$ by Corollary 5.4., implying thus $T^\star = T_*$. □

A similar but weaker condition than quasi-left continuity is sufficient for existence:

Corollary 5.6. Assume the payoff process Z has left limits, $T^\star \in \mathcal{T}$ and $\mathbb{E}(1_{T<\infty} \Delta Z_T) = 0$ for all predictable stopping stopping times T. Then the stopping time $T^\star$ is optimal.

Proof. Define the predictable stopping time $T^\star_\Lambda$ by $T^\star_\Lambda(\omega) = T^\star(\omega)$ if $\omega \in \Lambda$ and $T^\star_\Lambda(\omega) = \infty$ otherwise. Then, $1_\Lambda \Delta Z_{T^\star} = 1_{T^\star_\Lambda < \infty} \Delta Z_{T^\star_\Lambda}$. The result now follows from Theorem 5.5. □

5.3 Uniqueness and the longest optimal stopping time

Normally, the optimal stopping time is unique, i.e. equals the shortest one $T_* = \inf\{t \in \mathbb{T} : Z_t = V_t\}$ in Corollary 5.4. An exception is when Z is a claim—then all finite stopping times are optimal. Define

$$T_0 := \inf\{t \in \mathbb{T} : V^p_t < 0\} = \inf\{t \in \mathbb{T} : V_t < V^m_t\}.$$

Clearly, $1_{T_0<\infty} V^p_{T_0-} = 0$. Also, $T_0 \ge T^*$ for any optimal stopping T^* because $V^p = 0$ on $[[0, T^*]]$ by Corollary 5.3. So, if T_0 is optimal, then it is the longest optimal stopping time.

Proposition 5.7. The stopping time T_0 is optimal if and only if T_0 is finite and $V^p_{T_0} = 0$.

Proof. If T_0 is optimal then $V_{T_0} = V^m_{T_0}$ by Corollary 5.2.; so $V^p_{T_0} = 0$. Conversely, assume $V^p_{T_0} = 0$. Since V^p is decreasing, $V^p_t < 0$ on the set $\{t > T_0\}$. By Corollary 3.11, $V^p_t = \sup_{t\le s\in\mathbb{T}} (Z_s - V^m_s)$. Hence, $\sup_{T_0\le s\in\mathbb{T}} (Z_s - V^m_s) = 0$ a.s. while, for any t, $\sup_{t\le s\in\mathbb{T}}(Z_s - V^m_s) < 0$ on $\{t > T_0\}$. This obviously implies $Z_{T_0} = V^m_{T_0}$. Hence, $\mathbb{E}\, Z_{T_0} = \mathbb{E} V^m_{T_0} = V^m_0 = V_0$, as desired. □

In finite horizon, the same argument yields a similar conclusion without assuming $T_0 < \infty$:

Proposition 5.8. Assume finite horizon $\mathbb{T} = [0, m]$. Then $V^p_{m\wedge T_0-} = 0$. Moreover $m \wedge T_0$ is optimal if and only $V^p_{m\wedge T_0} = 0$. In particular, $m \wedge T_0$ is optimal if V^p is continuous.

For any domineering claim C, $\{V < C\} = \{V^p < 0\}$; so $T_0 = \inf\{t \in \mathbb{T} : V_t < C_t\}$.

5.4 Simple counterexamples

A process less than 1 but with left limit of 1 at $t = 1$ has no optimal stopping time.

As an example of a non-quasi-left continuous payoff process that has an optimal stopping time, fix $0 < s \leq 1$ and let R be a nonzero $\mathcal{F}_s$-measurable random variable with $\mathbb{E}(R|\mathcal{F}_{s-}) = 0$. Define the payoff process $Z_t = t$ for $t < 1$ and $Z_t = 1 + R$ for $t \geq 1$. Then $T^\star = 1$, Z jumps at 1 and Z is not quasi-left continuous. But, it is easy to see that $T^\star$ is optimal if $s = 1$. (Z has no optimal stopping time if $s < 1$.)

It is possible (though pathological) that T_* be optimal but $T^\star$ not be optimal, e.g. a payoff process Z such that $Z_2 = 1, Z_t < 1$ for $t \neq 2$ and Z has left limit of 1 at $t = 1$ (where it jumps down). In this case $T^\star = 1$ but $T_* = 2$.

6. The perpetual American put

The perpetual American put problem can be solved by an explicit construction of a "domineering martingale" M which is a geometric Brownian motion, a direct construction as in Ref. [2] that bypasses the Snell envelope and its Doob–Meyer decomposition. As a geometric Brownian motion is not uniformly integrable in infinite horizon, the optional sampling theorem does not apply directly to M. Still, for certain range of parameters, it turns out that the stopping time $T_* := \inf\{0 \leq t < \infty : Z_t = M_t\}$ is optimal and $\mathbb{E}\,Z_{T_*} = M_0$.

The following is similar to Proposition 2.1, but weakens the uniform integrability assumption.

Proposition 6.1. Let Z be a payoff process and $M \geq Z$ be a positive local martingale. Assume the stopping time $T_* := \inf\{0 \leq t < \infty : Z_t = M_t\}$ is finite a.s. and the stopped local martingale $M^{T_*} := M_{\cdot\wedge T^*}$ is a claim. Then T_* is optimal and $\mathbb{E}\,Z_{T_*} = M_0$.

Proof. A standard argument gives $\mathbb{E}M_T \leq M_0$ for all finite stopping times T[||||]. This, followed by using that M^{T_*} is a claim and that by right continuity $Z_{T_*} = M_{T_*}$, gives

$$\sup_{0\leq T<\infty} \mathbb{E}\,Z_T \leq \sup_{0\leq T<\infty} \mathbb{E}M_T \leq M_0 = \mathbb{E}M^{T_*}_\infty = \mathbb{E}M_{T_*} = \mathbb{E}\,Z_{T_*} \leq \sup_{0\leq T<\infty} \mathbb{E}\,Z_T.$$

Hence, $\sup_{0\leq T<\infty} \mathbb{E}\,Z_T = M_0 = \mathbb{E}\,Z_{T_*}$, as desired. □

For M^{T_*} above to be a claim it is sufficient that $\mathbb{E}[M]_{T_*} < \infty$. Although M itself may not be a claim, it always equals the Snell envelope and any domineering claim on $[[0, T_*]]$.

The following is for the most part based on Ref. [2], where X below is a Brownian motion.

[||||] Let $(T_n)_{n=1}^\infty$ be a localizing sequence for M. Applying the optional sampling theorem to the claim M^{T_n} gives, $\mathbb{E}M_{T_n\wedge T} = \mathbb{E}M^{T_n}_T = M^{T_n}_0 = M_0$. But, $M_{T_n\wedge T} \rightarrow M_T$ a.s. Thus by Fatou's lemma, $\mathbb{E}M_T \leq M_0$.

THEOREM 6.2. Let X be a continuous local martingale with $X_0 = 0$ such that $[X]_t \to \infty$ a.s. as $t \to \infty$. Let $0 < \alpha < 1$ and $0 < K < 1 + (1/\alpha)$. Define the (bounded) payoff process

$$Z := \left(K\,\mathrm{e}^{-\frac{1}{2}\alpha[X]} - \mathcal{E}(X)\right)^{+}.$$

Then, $\sup_{0\le T<\infty} \mathbb{E}\, Z_T = (\alpha^{\alpha} K^{1+\alpha}/(1+\alpha)^{1+\alpha})$. In fact, define the positive local martingale

$$M := \frac{\alpha^{\alpha} K^{1+\alpha}}{(1+\alpha)^{1+\alpha}}\mathcal{E}(-\alpha X).$$

Then, $Z \le M$, the stopping time $T_* := \inf\{0 \le t < \infty : Z_t = M_t\}$ is finite a.s. and the stopped process M^{T_*} is a claim. Therefore by Proposition 6.1,

$$\sup_{0\le T<\infty} \mathbb{E}\, Z_T = \mathbb{E}\, Z_{T_*} = M_0 = \frac{\alpha^{\alpha} K^{1+\alpha}}{(1+\alpha)^{1+\alpha}}. \tag{6.1}$$

Moreover, M^{T_*} is bounded and

$$T_* = \inf\left\{0 \le t < \infty : S_t = \frac{\alpha K}{1+\alpha}\right\}, \quad \text{where } S := \mathrm{e}^{(\alpha/2)[X]}\mathcal{E}(X). \tag{6.2}$$

Proof. Set $S := \mathrm{e}^{(\alpha/2)[X]}\mathcal{E}(X)$ and $N := \mathcal{E}(-\alpha X)$. We easily have $S^{\alpha} N = \mathrm{e}^{-(\alpha/2)[X]}\mathcal{E}(X)$. Thus,

$$Z = (K - S)^{+} S^{\alpha} N. \tag{6.3}$$

The maximum of the function $s \mapsto (K - s)^{+} s^{\alpha}$ equals $\alpha^{\alpha} K^{1+\alpha}/(1+\alpha)^{1+\alpha}$ (attained at $\alpha K/1+\alpha$). Hence, $Z \le \alpha^{\alpha} K^{1+\alpha}/((1+\alpha)^{1+\alpha})N = M$. Equation (6.3) also yields $\{Z = M\} = \{S = \alpha K/1+\alpha\}$, implying equation (6.2).

The assumption implies X is a time-changed Brownian motion. Hence, $\underline{\lim}_{t\to\infty} X_t = -\infty$ a.s. So, $\underline{\lim}_{t\to\infty} \mathrm{e}^{X_t} = 0$ a.s. But, $S = \mathrm{e}^{X-(1-\alpha)[X]/2} \le \mathrm{e}^{X}$, as $\alpha < 1$. Thus, $\underline{\lim}_{t\to\infty} S_t = 0$. a.s. Since S is continuous, $S_0 = 1$ and $\alpha K/(1+\alpha) < 1$, it follows using equation (6.2) that $T_* < \infty$ a.s.

As $S_0 = 1 > \alpha K/1+\alpha$, equation (6.2) also implies $S \ge \alpha K/1+\alpha$ on $[[0, T_*]]$. Hence, $N = \mathrm{e}^{-(1/2)\alpha[X]} S^{-\alpha} \le \mathrm{e}^{-(1/2)\alpha[X]}((1+\alpha)/\alpha K)^{\alpha}$ on $[[0, T_*]]$. This shows N^{T_*}, hence M^{T_*}, are bounded. (In fact, $M^{T_*} < 1/\alpha$##.) Therefore, the local martingale M^{T_*} is a claim. Equation (6.1) now follows from Proposition 6.1. □

Remark. For the American put, $r := (\alpha/2)(\mathrm{d}[X]/\mathrm{d}t)$ is interpreted as the short rate and $S = \mathrm{e}^{\int r\,\mathrm{d}t}\mathcal{E}(X)$ as the stock price (zero-dividend here). Note then, $Z = \mathrm{e}^{-\int r\,\mathrm{d}t}(K - S)^{+}$.

Remark. If $\alpha > 1$ and $K > 1 + 1/\alpha$, we still have T_* is finite, $Z \le M$ and $Z_{T_*} = M_{T_*}$. But, we cannot conclude T_* is optimal, for in this case M^{T_*} is not a claim because $\mathbb{E}M_{T_*} < M_0$.

7. Iterative constructions in the Bermudan case

In this section, we consider the finite Bermudan case with $\mathbb{T} = \{0, 1, \ldots, m\}$ and define a stopping time at each iteration of the m-step iterative construction of the Snell envelope in [6], so that they converge in m-steps to the optimal stopping time, similarly to [26].

##Also, $[M]_{T_*} < 1$ because, using $N := \varepsilon(-\alpha X) = 1 - \alpha\int N\,\mathrm{d}X$ and the estimate for N on $[[0, T]]$, $[N]_{T_*} = \alpha^2\int_0^{T_*} N_t^2\,\mathrm{d}[X]_t \le \alpha^2((1+\alpha)/\alpha K)^{2\alpha}\int_0^{T_*}\mathrm{e}^{-\alpha[X]_t}\mathrm{d}[X]_t = \alpha((1+\alpha)/\alpha K)^{2\alpha}(1 - \mathrm{e}^{-\alpha[X]_{T_*}}) < \alpha((1+\alpha)/\alpha K)^{2\alpha}$.

In the Bermudan case, a (super) claim is a (super) martingale, a payoff process is an adapted integrable process and a predictable process is a previsible process. The canonical decomposition $X = X^p + X^m$ of a payoff process X is characterized inductively by

$$X^m_{t+1} - X^m_t = X_{t+1} - \mathbb{E}(X_{t+1}|\mathcal{F}_t), \quad X^m_0 = X_0; \tag{7.1}$$

$$X^p_{t+1} - X^p_t = \mathbb{E}(X_{t+1} - X_t|\mathcal{F}_t), \quad X^p_0 = 0. \tag{7.2}$$

Let Z be a payoff process. Following Ref. [6], we call a supermartingale W a *supersolution* if $W \geq Z$ and for any supersolution W define its *additive improvement* process W' by

$$W'_t := W^m_t + \mathbb{E}\left(\sup_{t\leq s\leq m}\left(Z_s - W^m_s\right)|\mathcal{F}_t\right).$$

Proposition 7.1 and Corollary 7.3 below are contained in [6], our proofs being slightly different.

PROPOSITION 7.1. Let W be a supersolution. Then W' is also a supersolution and $W' \leq W$.

Proof. Clearly, $W' \geq Z$ and $W'_m = Z_m$. Also, $\mathbb{E}(\sup_{t\leq s\leq m}(Z_s - W^m_s)|\mathcal{F}_t)$ is a superclaim:

$$\mathbb{E}\left(\mathbb{E}\left(\sup_{t\leq s\leq m}\left(Z_s - W^m_s\right)|\mathcal{F}_t\right)|\mathcal{F}_u\right) = \mathbb{E}\left(\sup_{t\leq s\leq m}\left(Z_s - W^m_s\right)|\mathcal{F}_u\right)$$
$$\leq \mathbb{E}\left(\sup_{u\leq s\leq m}\left(Z_s - W^m_s\right)|\mathcal{F}_u\right).$$

So, W' is a superclaim too. Moreover $W' \leq W$. Indeed, since $Z \leq W$ and W^p is decreasing, we have $W'_t = W_t + \mathbb{E}(\sup_{t\leq s\leq m}(Z_s - W_s + W^p_s - W^p_t)|\mathcal{F}_t) \leq W_t$. □

LEMMA 7.2. Let V be the Snell envelope (i.e. the smallest supersolution) and W be any supersolution. Let $0 \leq t \leq m-1$ and assume $W_s = V_s$ for $s > t$. Then $W'_s = V_s$ for $s \geq t$.

Proof. If $s > t$, then $V_s \leq W'_s \leq W_s = V_s$. So, it suffices to show $W'_t = V_t$. Set $X = W - V$. Since $X_s = 0$ for $s > t$, equation (7.1) for X^m yields $X^m_s = X^m_t$ for $s \geq t$. Hence,

$$W'_t = \mathbb{E}\left(\sup_{t\leq s\leq m}\left(Z_s - W^m_s + W^m_t\right)|\mathcal{F}_t\right) = \mathbb{E}\left(\sup_{t\geq s\in\mathbb{T}}\left(Z_s - V^m_s + V^m_t\right)|\mathcal{F}_t\right) = V_t.$$

(Last equality says $V' = V$; it holds since Proposition 7.1 applied to V gives $V \leq V' \leq V$.) □

The lemma and an induction immediately yield the additive supersolution iteration of Ref. [6]:

COROLLARY 7.3. Let $W^{(0)}$ be a supersolution with $W^{(0)}_m = Z_m$, e.g. $W^{(0)}_t = \mathbb{E}(\sup_{t\leq s\leq m} Z_s|\mathcal{F}_t)$. Set inductively $W^{(n+1)} := W^{(n)'}$. Then $W^{(m)} = V$. More generally, $W^{(n)}_t = V_t$ for $t \geq m - n$.

The following is the Bermudan version of Corollary 3.11, using an alternative argument.

PROPOSITION 7.4. Let V be the Snell envelope of Z. Then, $V_t^p = \sup_{t \le s \le m}(Z_s - V_s^m)$.

Proof. By Proposition 7.1, $V' \le V$. But, $V \le V'$ since V is the smallest supersolution. Hence $V = V'$, or equivalently $\mathbb{E}(\sup_{t \le s \le m}(Z_s - V_s + V_s^p - V_t^p)|\mathcal{F}_t) = 0$. But, since $Z \le V$ and V^p is decreasing, $\sup_{t \le s \le m}(Z_s - V_s + V_s^p - V_t^p) \le 0$. Hence, $\sup_{t \le s \le m}(Z_s - V_s + V_s^p - V_t^p) = 0$, which, using $V = V^p + V^m$, is equivalent to $\sup_{t \le s \le m}(Z_s - V_s^m) - V_t^p = 0$, as desired. □

Remark. If $V = AB$ is the multiplicative decomposition of the Snell envelope V, then, applying the above to payoff Z/B in measure $\mathbb{P}^B$ as in Corollary 4.3, it follows, $A_t = \sup_{t \le s \le m}(Z_s/B_s)$.

Let V be the Snell envelope of Z. Since V^p is previsible, T_0 is a stopping time, where

$$T_0 \coloneqq m \wedge \min\{0 \le t \le m-1 : V_{t+1}^p < 0\}.$$

PROPOSITION 7.5. The stopping time T_0 above and $T_* \coloneqq \min\{0 \le t \le m : Z_t = V_t\}$ are optimal and $T_* \le T^* \le T_0$ for all optimal stopping times T^*. Moreover, $V_{T_0}^p = 0$.

Proof. Clearly, $V_{T_0}^p \ge 0$. But $V^p \le 0$ since $V^p = 0$ and V^p is decreasing. Hence $V_{T_0}^p = 0$. Also, $V_{T_0+1}^p < 0$ on the set $\{T_0 < m\}$ by the definition of T_0. Hence Proposition 7.4 implies, $\max_{T_0 \le s \le m}(Z_s - V_s^m) = 0$ while $\max_{T_0+1 \le s \le m}(Z_s - V_s^m) < 0$ on the set $\{T_0 < m\}$. This obviously implies $Z_{T_0} = V_{T_0}^m$ on $\{T_0 < m\}$ and hence everywhere as $Z_m = V_m$. Hence, $\mathbb{E}Z_{T_0} = \mathbb{E}V_{T_0}^m = V_0^m = V_0$ and thus T_0 is optimal. Let T^* be another optimal stopping time. Then $Z_{T^*} = V_{T^*} = V_{T^*}^m$ since $V_0 = \mathbb{E}Z_{T^*} \le \mathbb{E}V_{T^*} \le \mathbb{E}V_{T^*}^m = V_0$. So, $V_{T^*}^p = 0$, which implies $T^* \le T_0$. As $Z_{T^*} = V_{T^*}$, it follows $T_* \le T^*$. That T_* is optimal is well known. □

For any supersolution W, set $T_m^W \coloneqq m$ and for $0 \le t \le m-1$, define the stopping time

$$T_t^W \coloneqq m \wedge \min\{t \le s \le m-1 : W_{s+1}^p < W_t^p\}.$$

Clearly, $T_0 = T_0^V$. Also, $W_{T_t^W}^p = W_t^p$; hence $\mathbb{E}(W_{T_t^W}|\mathcal{F}_t) = \mathbb{E}(W_{T_t^W}^m|\mathcal{F}_t) + W_t^p = W_t$.

PROPOSITION 7.6. Let $0 \le t \le m$. Then $\mathbb{E}(Z_{T_t^V}|\mathcal{F}_t) = V_t$.

Proof. Set $T_t \coloneqq T_t^V$. Clearly, $V_{T_t}^p \ge V_t^p$. But $T_t \ge t$ and V^p is decreasing. Hence $V_{T_t}^p = V_t^p$. Also, $V_{T_t+1}^p < V_t^p$ on the set $\{T_t < m\}$ by the definition of T_t. Hence Proposition 7.4 implies, $\max_{T_t \le s \le m}(Z_s - V_s^m) = V_t^p$ while $\max_{T_t+1 \le s \le m}(Z_s - V_s^m) < V_t^p$ on $\{T_t < m\}$. This obviously implies $Z_{T_t} = V_{T_t}^m + V_t^p$. Hence, $\mathbb{E}(Z_{T_t}|\mathcal{F}_t) = \mathbb{E}(V_{T_t}^m + V_t^p|\mathcal{F}_t) = V_t^m + V_t^p = V_t$. □

LEMMA 7.7. Let $0 \le t \le m$ and W be a supersolution such that $W_s = V_s$ for $s \ge t$. Then $W_s^p - W_t^p = V_s^p - V_t^p$ for all $s \ge t$ and hence $T_t^W = T_t^V$.

Proof. Set $X = W - V$. Since $X_s = 0$ for $s \geq t$, equation (7.2) for X^p shows that $X_s^p = X_t^p$ for $s \geq t$. Hence, $W_s^p - W_t^p = V_s^p - V_t^p$ for $s \geq t$, which obviously implies $T_t^W = T_t^V$. □

The main result of this section complements the iteration of Ref. [6] in Corollary 7.3. We obtain, $\mathbb{E}(Z_{T_0^{W^{(n)}}}) \leq V_0 \leq W_0^{(n)}$ for all $0 \leq n \leq m$, with the duality gap narrowing as n increases:

THEOREM 7.8. Let $W^{(0)}$ be a supersolution with $W_m^{(0)} = Z_m$, e.g. $W_t^{(0)} = \mathbb{E}(\sup_{t \leq s \leq m} Z_s | \mathcal{F}_t)$. Define inductively, $W^{(n+1)} := W^{(n)'}$. Then, $\mathbb{E}(Z_{T_t^{W^{(n)}}} | \mathcal{F}_t) = V_t$ for $t \geq m - n$, $0 \leq n \leq m$.

Proof. Let $0 \leq n \leq m$. By Corollary 7.3, we have $W_t^{(n)} = V_t$ for $t \geq m - n$. Hence by Lemma 7.7, $T_t^{W^{(n)}} = T_t^V$ for $t \geq m - n$. Therefore by Proposition 7.6, $\mathbb{E}(Z_{T_t^{W^{(n)}}} | \mathcal{F}_t) = V_t$ for $t \geq m - n$. □

Remark. As in Ref. [6], Proposition 7.1 and Corollary 7.3 continue to hold if W' is defined as the *multiplicative improvement* $W_t' := \mathbb{E}(B_m \sup_{t \leq s \leq m} (Z_s / B_s) | \mathcal{F}_t) = B_t \, \mathbb{E}^B (\sup_{t \leq s \leq m} (Z_s / B_s) | \mathcal{F}_t)$, where $W = AB$ is the multiplicative decomposition of W. So will Proposition 7.6 and Theorem 7.8, with $T_t^W := m \wedge \min\{t \leq s \leq m - 1 : A_{s+1} < A_t\}$. (See the argument in Corollary 4.3.)

8. Financial interpretation and heuristics

The exposition in this final section is in part heuristic rather than rigorous, nevertheless useful we hope. We implicitly assume finite horizon $\mathbb{T} = [0, m]$, regularity, etc. as needed.

All the interpretations below are based on viewing the Snell envelope V_t as the time-t price of the *current* American option (issued at time t) on a given payoff process Z.

8.1 The multiplicative decomposition

Let us interpret the multiplicative decomposition of the Snell envelope, expressed in the form $AV = C$, where A is an increasing predictable process with $A_0 = 1$ and C is a claim (that domineers Z). Relying on the right continuity of Z, we approximate the American option by a Bermudan option that can be exercised at one of only finitely many dates $0 \leq t_1 < \cdots < t_n$, receiving payoff Z_{t_i}.

Start with V_0 capital and with it at time $t = 0$ buy $A_{t_1} = 1$ share of this Bermudan option. At time t_1, sell the option and with the proceeds buy A_{t_2} units of the Bermudan option with exercise dates $t_2, \ldots, t_n$. At time t_2, sell all A_{t_2} units of this Bermudan option and with the proceeds buy A_{t_3} units of Bermudan option with exercise dates $t_3, \ldots, t_n$. Continue this trading strategy at time t_3 and subsequent times t_i until t_n.

Clearly this is a *self-financing* trading strategy. Therefore its price (portfolio value) is a claim, which we denote by C. By its definition C_{t_i} equals the net value of Bermudan options sold at time t_i. Since A_{t_i} units of Bermudan options with exercise date $t_i, \ldots, t_n$ are sold, each unit of which is priced at V_{t_i}, we conclude that $C_{t_i} = A_{t_i} V_{t_i}$.

Moreover, the number of share (units) A is previsible by construction. It is also increasing. Indeed, if it is not optimal to exercise at t_i, then $A_{t_{i+1}} = A_{t_i}$ because the sold Bermudan options at time t_i are identical to the ones next bought at t_i. And otherwise, if it is optimal to exercise at t_i, then $A_{t_{i+1}} > A_{t_i}$ because unit price of the sold options is Z_{t_i}, which is by optimality greater than unit price $\mathbb{E}(V_{t_{i+1}}|\mathcal{F}_t)$ of bought options.

8.2 *Additive decompositions and domineering claims*

Given a numeraire B, let us interpret the invariant Doob–Meyer decomposition $V = C - AB$ of Proposition 3.15, where C is a domineering claim and A is an increasing and predictable process with $A_0 = 0$.

As in Section 8.1, we discretize and at time 0 buy one unit of the Bermudan option with exercise dates $t_1, \ldots, t_n$. At time t_1, sell it and with the proceeds buy one unit of the Bermudan option with exercise dates $t_2, \ldots, t_n$ and a_1 units of the numeraire B. (Clearly $a_1 \geq 0$.) At time t_2, sell the Bermudan options bought at t_1 and with the proceeds buy one unit of the Bermudan option with exercise dates $t_3, \ldots, t_n$ and $a_2 \geq 0$ units of the numeraire B. Continue this self-financing trading strategy at t_3 and subsequent times t_i until t_n.

Define the claim C to be the price process of this self-financing portfolio. Set $A_{t_1} = 0$ and for $i \geq 2$, define $A_{t_i} = a_1 + \cdots + a_{i-1}$. Clearly, A is previsible and increasing. By construction, A_{t_i} is the number of shares of the numeraire B held in the portfolio at time t_i before any transaction (that is the number of shares brought forward from time t_{i-1}).

The value C_{t_i} of the portfolio, looked at time t_i just before any transaction, equals the value of one unit of the Bermudan option carried forward, i.e. V_{t_i}, plus the value of the A_{t_i} units of numeraire B brought forward, i.e. $A_{t_i}B_{t_i}$. Therefore, $C = V + AB$.

8.3 *Invariance of the multiplicative decomposition*

As the above shows, investments in the (generalized) additive decomposition $V = C - AB$ are rolled over both the numeraire B and (one unit of) the newly issued American option V (in the case of $V = C - A$, the reference numeraire is implicitly $B = 1$), whereas in the multiplicative case all investment is rolled over V. As such, in contrast to the additive decomposition, *the multiplicative decomposition does not rest on any reference numeraire for its existence*.

8.4 *A formula for the compensator of the Snell envelope (pure diffusion)*

Consider the (decreasing) compensator V^p of V. From Corollary 3.11 or Section 5.3, we know V^p is constant while $Z < V$, that is, V^p does not decrease while it is not optimal to exercise. So, $\mathrm{d}V^p = 1_{\{Z=V\}}\,\mathrm{d}V^p$. Assuming Z and V are continuous, on the stopping region $\{Z = V\}$ we expect $\mathrm{d}V^p = \mathrm{d}Z^p$. This indicates that in the continuous case we have,

$$\mathrm{d}V^p = 1_{\{Z=V\}}\,\mathrm{d}Z^p.$$

Since V is a supermartingale, $\mathrm{d}V^p \leq 0$. Hence the formula $\mathrm{d}V^p = 1_{\{Z=V\}}\,\mathrm{d}Z^p$ implies $\mathrm{d}Z^p \leq 0$ on the stopping region $\{Z = V\}$, that is Z is a supermartingale there.

8.5 The American call

The payoff process of a standard American call option with expiration m and strike $K \in \mathbb{R}$ on an asset with price process S is $Z = (S - K)^+$. Clearly, on the stopping region $\{Z = V\}$, we have $S \geq K$, usually $S > K$. Therefore on this region, $Z = S - K$, indicating $\mathrm{d}Z^p = \mathrm{d}S^p$. Hence, for the American call in the diffusion case,

$$\mathrm{d}V^p = 1_{\{V=S-K\}}\,\mathrm{d}S^p.$$

If S is a submartingale, the stopping region is empty and the above gives, $V^p = 0$, i.e. V is a martingale, as expected. The stopping region will be nonempty if $\mathrm{d}S^p < 0$.

8.6 A formula for the Snell envelope (pure-diffusion)

As $V_m = Z_m$, we have $V_t^m = \mathbb{E}(V_m^m|\mathcal{F}_t) = \mathbb{E}(Z_m - V_m^p|\mathcal{F}_t)$. Hence, in general (see also Proposition 3.17),

$$V_t = \mathbb{E}\left(Z_m - \int_t^m \mathrm{d}V_s^p|\mathcal{F}_t\right).$$

In the case of pure diffusion, we can substitute $\mathrm{d}V^p = 1_{\{Z=V\}}\mathrm{d}Z^p$ and thus get

$$V_t = \mathbb{E}\left(Z_m - \int_t^m 1_{\{Z_s=V_s\}}\,\mathrm{d}Z_s^p|\mathcal{F}_t\right).$$

For an American call payoff process $Z = (S - K)^+$, this and the previous section implies

$$V_t = \mathbb{E}\left((S_m - K)^+ - \int_t^m 1_{\{V_s=S_s-K\}}\,\mathrm{d}S_s^p|\mathcal{F}_t\right).$$

(We stress again $1_{\{Z_s=V_s\}}\,\mathrm{d}Z_s^p = 1_{\{Z_s=V_s\}}\,\mathrm{d}V^p \leq 0$.) More generally, for any numeraire B,

$$V_t = B_t\,\mathbb{E}^B\left(\frac{(S_m - K)^+}{B_m} - \int_t^m 1_{\{V_s=S_s-K\}}\frac{\mathrm{d}S_s^p}{B_s}|\mathcal{F}_t\right).$$

8.7 Pure diffusion with interest rates

Relaxing previous setting, let r denote the *instantaneous interest rate process*. Now, V denotes the smallest process such that $V \geq Z$ and $\mathrm{e}^{-\int r\,\mathrm{d}t}V$ is a supermartingale: $V_t :=$ ess. $\sup_{t\leq T\in\mathcal{T}} \mathbb{E}(\mathrm{e}^{-\int_t^T r_s\,\mathrm{d}s}Z_T|\mathcal{F}_t)$. By Section 8.6,

$$\begin{aligned}\mathrm{e}^{-\int_0^t r_s\,\mathrm{d}s}V_t &= \mathbb{E}\left(\mathrm{e}^{-\int_0^m r_s\,\mathrm{d}s}Z_m - \int_t^m 1_{\{Z_s=V_s\}}\,\mathrm{d}\left(\mathrm{e}^{-\int_0^s r_u\,\mathrm{d}u}Z_s\right)^p|\mathcal{F}_t\right)\\ &= \mathbb{E}\left(\mathrm{e}^{-\int_0^m r_s\,\mathrm{d}s}Z_m - \int_t^m 1_{\{Z_s=V_s\}}\,\mathrm{e}^{-\int_0^s r_u\,\mathrm{d}u}\left(\mathrm{d}Z_s^p - r_sZ_s\,\mathrm{d}s\right)|\mathcal{F}_t\right).\end{aligned}$$

Thus, we arrive at the general formula in the pure diffusion case:

$$V_t = \mathbb{E}\left(\mathrm{e}^{-\int_t^m r_s\,\mathrm{d}s}Z_m|\mathcal{F}_t\right) + \mathbb{E}\left(\int_t^m \mathrm{e}^{-\int_t^s r_u\,\mathrm{d}u}1_{\{V_s=Z_s\}}\left(r_sZ_s\,\mathrm{d}s - \mathrm{d}Z_s^p\right)|\mathcal{F}_t\right). \tag{8.1}$$

In particular, for a call $Z = (S - K)^+$ struck at K on a continuous semimartingale S, letting y denote the *dividend yield process* defined via $\mathrm{d}S^p = (r - y)S\,\mathrm{d}t$, we get

$$V_t = \mathbb{E}\Big(\mathrm{e}^{-\int_t^m r_s\,\mathrm{d}s}(S_m - K)^+ | \mathcal{F}_t\Big) + \mathbb{E}\left(\int_t^m \mathrm{e}^{-\int_t^s r_u\,\mathrm{d}u} 1_{\{V_s = S_s - K\}}(y_s S_s - r_s K)\,\mathrm{d}s | \mathcal{F}_t\right).$$

Other payoffs can also be considered, such as the American option on the minimum two assets studied in Ref. [11]. As a simpler example, consider an *American swaption* payoff $Z = (S - K)^+$, just like a call, but K a semimartingale too rather than a constant. Then,

$$\begin{aligned} V_t &= \mathbb{E}\Big(\mathrm{e}^{-\int_t^m r_s\,\mathrm{d}s}(S_m - K_m)^+ | \mathcal{F}_t\Big) \\ &\quad + \mathbb{E}\left(\int_t^m \mathrm{e}^{-\int_t^s r_u\,\mathrm{d}u} 1_{\{V_s = S_s - K_s\}}\big(r_s(S_s - K_s)\,\mathrm{d}s - \mathrm{d}S_s^p + \mathrm{d}K_s^p\big) | \mathcal{F}_t\right). \end{aligned}$$

8.8 *Delayed exercise compensation*

The above results have an intuitive financial interpretation, originally arrived at in Ref. [19] (1989) we reckon. Commonly, $\int rS\,\mathrm{d}t - S^p$ is interpreted as the cumulative dividend of any process S. So, the process y defined by $\mathrm{d}S^p = (r - y)S\,\mathrm{d}t$ represents the (continuous) dividend yield. By the call pricing formula following equation (8.1), the price V of the American call equals the European call price plus the present value of the nonnegative cashflow $1_{\{V=S-K\}}(yS - rK)\,\mathrm{d}s$.

It follows that *an American call option is equivalent to the European call option plus the nonnegative continuous cashflow* $1_{\{V=S-K\}}(yS - rK)$.

A supporting arbitrage argument is that whenever it is optimal to exercise the call, we have $S > K$. If we delay optimal exercise for an infinitesimal amount of time $\mathrm{d}t$, we still can exercise at $t + \mathrm{d}t$. All we loose by this delay is the dividend $yS\,\mathrm{d}t$ paid by the stock. But, we gain the interest $rK\,\mathrm{d}t$ on the strike. Because of continuity, we gain no further. So, if we were compensated for this dividend less the interest, we could delay optimal exercise without incurring any net gain or loss. As we continue delaying and being compensated, the underlying asset price may go down so that it is no longer optimal to exercise. Naturally, we will not exercise then, but will receive no compensation either. If subsequently asset price rises and it becomes optimal again to exercise, we continue delaying optimal exercise and again receive instead the dividend compensation less interest on strike and so on.

In this way, we never exercise early and are left in the end with the European call payoff $(S_m - K)^+$, plus the cumulative delayed exercise compensation $\int_0^m 1_{\{V=S-K\}}(yS - rK)\,\mathrm{d}t$.

8.9 *Single-state variable diffusion case by inhomogeneous PDE*

In 1989, four working papers and sequels on an "early exercise premium" formula appeared, eventually published as [5,16,19,24]. By a complex integration by parts calculation, [24] initially simplified the free-boundary American option pricing formula of [29]. Soon after, using specialized identities for the normal distribution, [5] further improved this calculation. In short order, based on the high-contact condition in [29] but not on the formula there, [19] provided an easy derivation of the simplified/improved formula, presented its interpretation and extended it to dividends and bond options. Independently, [16] arrived at the formula by probabilistic methods. Some of the authors also presented other derivations and further characterizations in various revisions subsequent to the initial iteration.

The method of [19] (1989) views the Snell envelope drift ($\mathrm{d}V^p/\mathrm{d}t$) as an inhomogeneous term entering the fundamental PDE. The solution was represented by a discounted expectation, similar to formula (8.2) below, with the inhomogeneous term serving as a continuous cash flow. As such, in contrast to the normal-distribution formulae and manipulations predating it, this technique furnished directly a Markovian version of the expectation formula (8.1). Here, we rework it in a form simultaneously applicable to stocks and bonds, but more importantly, we point out the formula's connection to the Doob–Meyer decomposition.

In this setting, asset prices P are functions $P_t = P(t, X_t)$ of time t and a state variable X that follows an Itô process $\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t$, with W a Brownian motion under the risk-neutral measure $\mathbb{P}$. The instantaneous interest rate process r is similarly of form $r_t = r(t, X_t)$. The discounted price $\mathrm{e}^{-\int r\,\mathrm{d}t}P$ of any asset with zero continuous dividends (e.g. a European call) is assumed to be a $\mathbb{P}$-martingale, implying $\mathrm{d}P^p = rP\,\mathrm{d}t$. More generally, the price process P of a derivative with a continuous dividend $c_t = c(t, X_t)$ (cumulative dividend $\int c\,\mathrm{d}t$) satisfies $\mathrm{d}P^p = (rP - c)\,\mathrm{d}t$. Applying Itô's formula to the discounted process $\mathrm{e}^{-\int r\,\mathrm{d}t}P$, we see that $LP + c = 0$, where L is the differential operator

$$Lf = \frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2} - rf, \quad f = f(t, x).$$

We first consider an American call struck at K on a stock or bond with price process $S_t = S(t, X_t) > 0$. As above, $rS - (\mathrm{d}S^p/\mathrm{d}t)$ is interpreted as continuous dividend (or coupon). So, $y = y(t, X) := r - (1/S)(\mathrm{d}S^p/\mathrm{d}t)$ is the dividend yield: $\mathrm{d}S^p = (r - y)S\,\mathrm{d}t$. For stocks, normally $S(t, x) = x$ and r and y assumed nonnegative constants. For bonds, usually $r(t, x) = x$.

In general, $LS + yS = 0$ by the above. We assume $S(t, x)$ is monotone in x, say increasing.

As is well known (e.g. [3,18,23]), there exists an "optimal boundary" $x^*(t)$, above which the American call is optimally exercised and below which it is kept. As American call $V_t = V(t, X_t)$ equals $S_t - K$ at or above $x^*(t)$, it follows $LV = LS - LK = -(yS - rK) \leq 0$ on the (t, x) region $x > x^*(t)$. Below $x^*(t)$, the American call is not exercised and behaves like any asset with zero continuous dividend. So, $LV = 0$ on the (t, x) region $x < x^*(t)$.

The function $V(t, x)$ is continuous everywhere and smooth outside the optimal boundary $\{(t, x^*(t)) : t \in [0, m]\}$. In order to make sense of LV as an L^1_{loc} function on the whole (t, x) region (including the boundary), we invoke the following smooth fit assumption: V is $\mathrm{d}t$-absolutely continuous and $(\partial V/\partial x)$ exists and is $\mathrm{d}x$-absolutely continuous, so that $(\partial V/\partial t)$ and $(\partial^2 V/\partial x^2)$ are well-defined as L^1_{loc} functions (in both Radon–Nykodym and distributional sense). Since $LV + (yS - rK) = 0$ above the optimal boundary and $LV = 0$ below, we conclude

$$LV + 1_{x > x^*(t)}(yS - rK) = 0, \quad V(m, x) = (S(m, x) - K)^+.$$

By the Feynman–Kac formula, the solution V to this *inhomogeneous* parabolic PDE is

$$V_t = \mathbb{E}\Big(\mathrm{e}^{-\int_t^m r_s\,\mathrm{d}s}(S_m - K)^+|X_t\Big) + \mathbb{E}\bigg(\int_t^m \mathrm{e}^{-\int_t^s r_u\,\mathrm{d}u}1_{\{X_s > x^*(s)\}}(y_s S_s - r_s K)\,\mathrm{d}s|X_t\bigg). \tag{8.2}$$

This is a special case of equation (8.1). The first term is the European option price and the second the early exercise premium. Equation (8.2) evaluated at $X_t = x^*(t)$ yields an integral equation for the optimal boundary $x^*(t)$, as the left hand side becomes $S(t, x^*(t)) - K$.

Equation (8.2) yields the Doob–Meyer decomposition of $\mathrm{e}^{-\int r\,\mathrm{d}t}V$, for it can be rewritten as

$$\mathrm{e}^{-\int r\,\mathrm{d}t}V = M - A, \tag{8.3}$$

where M is the martingale

$$M_t = \mathbb{E}\left(\mathrm{e}^{-\int_0^m r_s\,\mathrm{d}s}(S_m - K)^+ + \int_0^m \mathrm{e}^{-\int_0^s r_u\,\mathrm{d}u} 1_{\{X_s > x^*(s)\}}(y_s S_s - r_s K)\,\mathrm{d}s | X_t\right),$$

and A is the increasing continuous process,

$$A_t = \int_0^t \mathrm{e}^{-\int_0^s r_u\,\mathrm{d}u} 1_{\{X_s > x^*(s)\}}(y_s S_s - r_s K)\,\mathrm{d}s.$$

Similar arguments yield similar formulae for American puts where $Z = (K - S)^+$, e.g.

$$V_t = \mathbb{E}\left(\mathrm{e}^{-\int_t^m r_s\,\mathrm{d}s}(K - S_m)^+ | X_t\right) + \mathbb{E}\left(\int_t^m \mathrm{e}^{-\int_t^s r_u\,\mathrm{d}u} 1_{\{X_s < x^*(s)\}}(r_s K - y_s S_s)\,\mathrm{d}s | X_t\right).$$

To calculate the expectations for bond options, the additional step of changing to the forward numeraire measure (for m and each s) is to be taken, as in [19], where a general 1-factor short-rate model is developed then applied to get explicit formulae in the Vasicek and the CIR models (and in the general one-factor Gaussian model in the 1989 version.)

8.10 Extension to Markovian jump-diffusion

Recently, [7] have extended the inhomogeneous PDE approach of [19] to (stochastic volatility and to) an inhomogeneous IPDE for the case of exponential Lévy underlyers (e.g. Merton's lognormal jump-diffusion model). The IPDE is solved explicitly using the Fourier transform, resulting in a coupled 2 × 2 system of integral equations for the American call price V and the optimal boundary S^*. They extend the financial interpretation as follows. In addition to instantaneously compensating for dividend and penalizing for interest on strike as in pure diffusion, the infinitesimal expectation to jump down at the next instant from the stopping region into the continuation region is penalized too, because then the option price depreciates less than if it had been exercised. Here, we propose a general formulation based on the Itô–Meyer formula, while still assuming rather than proving smooth paste and any needed regularity.

As already seen several times, in the zero-interest rates case, we have in general,

$$V_t = \mathbb{E}\left(Z_m - \int_t^m \mathrm{d}V_s^p | \mathcal{F}_t\right). \tag{8.4}$$

To calculate $\mathrm{d}V^p$ (which equals $1_{\{V_s = S_s - K\}}\,\mathrm{d}V^p$), we assume $Z = (S - K)^+$ for some positive special semimartingale S and constant K. We further assume $V_t = V(t, S_t)$ for some C^1 function $V(t, S)$ such that as a distribution $(\partial^2 V / \partial S^2)$ is locally integrable on $[0, m] \times (0, \infty)$. In particular, V is a difference of two convex functions in S. As such, (a time-dependent version of) Itô–Meyer formula is applicable. Applying it, then taking compensator,

$$\mathrm{d}V^p = \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial S}\mathrm{d}S^p + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\mathrm{d}[S^c] + \int_{y\neq 0}\left(V(\cdot,S_- + y) - V(\cdot,S_-) - y\frac{\partial V}{\partial S}(\cdot,S_-)\right)\hat{\nu}(\mathrm{d}t,\mathrm{d}y),$$

where $\hat{\nu} = \hat{\nu}(\mathrm{d}t,\mathrm{d}y)$ is the compensator measure of S (the jump-size intensity). We are interested in this formula in the stopping region $\{V = S - K\}$, as it is zero outside it. We assume a continuous optimal boundary curve $t \mapsto S^*(t)$, meaning $\{V_t = S_t - K\} = \{S_t \geq S^*(t)\}$. So, $\mathrm{d}V^p = 1_{\{S \geq S^*\}}\mathrm{d}V^p$. We further assume $\mathrm{d}S^p$, $\mathrm{d}[S^c]$, $\nu(\mathrm{d}t,\mathrm{d}x)$ and hence $\mathrm{d}V^p$ are $\mathrm{d}t$-absolutely continuous. Then $\mathrm{d}V^p = 1_{\{S_- > S^*\}}\mathrm{d}V^p$. On the region $\{S_- > S^*\}$, we have $V(\cdot,S_-) = S_- - K$, so, $(\partial V/\partial t) = (\partial^2 V/\partial S^2) = 0$ and $(\partial V/\partial S) = 1$ (evaluated at (t,S_{t-})). Thus,

$$\mathrm{d}V^p = 1_{\{S_- > S^*\}}\left(\mathrm{d}S^p + \int_{y\neq 0}(V(\cdot,S_- + y) - (S_- + y - K))\hat{\nu}(\mathrm{d}t,\mathrm{d}y)\right).$$

Equating the last two expressions for $\mathrm{d}V^p$ gives an IPDE for V, as addressed shortly. But, it more usual to model the jumps of $X \coloneqq \log(S)$. Let $\nu = \nu(\mathrm{d}t,\mathrm{d}x)$ denote its compensator measure. Interestingly then, the jump integral transforms multiplicatively:

$$\mathrm{d}V^p = \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial S}\mathrm{d}S^p + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\mathrm{d}[S^c] + \int_{x\neq 0}\left(V(\cdot,S_-\,\mathrm{e}^x) - V(\cdot,S_-) - (\mathrm{e}^x - 1)S_-\frac{\partial V}{\partial S}(\cdot,S_-)\right)\nu(\mathrm{d}t,\mathrm{d}x).$$

(This follows by a similar application of Itô–Meyer formula to the function $v(t,X) = V(t,S)$, where $S = \mathrm{e}^X$. Then one manipulates, especially using that by Itô's formula $\mathrm{d}S^p = S_-\,\mathrm{d}X^p + (1/2)S_-\mathrm{d}[X^c] + \int_{x\neq 0}\mathrm{e}^{X_-}(\mathrm{e}^x - 1 - x)\nu(\mathrm{d}t,\mathrm{d}x)$.) Now, $\mathrm{d}V^p = 1_{\{S_- > S^*\}}\mathrm{d}V^p$ and as before, on the region $\{S_- > S^*\}$, we have $V(\cdot,S_-) = S_- - K$, $(\partial V/\partial S) = 1$, etc. Thus,

$$\mathrm{d}V^p = 1_{\{S_- > S^*\}}\left(\mathrm{d}S^p + \int_{x\neq 0}(V(\cdot,S_-\,\mathrm{e}^x) - S_-\,\mathrm{e}^x + K)\nu(\mathrm{d}t,\mathrm{d}x)\right).$$

Since $V(t,S_{t-}\,\mathrm{e}^x) - S_{t-}\,\mathrm{e}^x + K = 0$ for $S_{t-}\,\mathrm{e}^x \geq S^*(t)$, we can also write the integral as

$$\mathrm{d}V^p = 1_{\{S_- > S^*\}}\left(\mathrm{d}S^p + \int_{x=-\infty}^{\log(S^*/S_-)}(V(\cdot,S_-\,\mathrm{e}^x) - S_-\,\mathrm{e}^x + K)\nu(\mathrm{d}t,\mathrm{d}x)\right). \tag{8.5}$$

(Note $\log(S^*/S_-) < 0$ here.) Plugging into equation (8.4), we get, for the case of zero interest rates,

$$V_t = \mathbb{E}\left((S_m - K)^+ - \int_t^m 1_{\{S_{s-} > S^*(s)\}}\left(\mathrm{d}S_s^p + \int_{x=-\infty}^{\log(S^*(s)/S_{s-})}(V(s,S_{s-}\,\mathrm{e}^x) - S_{s-}\,\mathrm{e}^x + K)\nu(\mathrm{d}s,\mathrm{d}x)\right)|S_t\right).$$

Now consider an integrable spot rate process r. Let S, V denote undiscounted processes as before. Then, applying the above to the payoff process $\mathrm{e}^{-\int r\,\mathrm{d}t}(S-K)^+$ finalizes to

$$\begin{aligned} V_t =\mathbb{E}\Bigg(&\mathrm{e}^{-\int_t^m r_s\,\mathrm{d}s}(S_m-K)^+ + \int_t^m 1_{\{S_{s-}>S^*(s)\}}\,\mathrm{e}^{-\int_t^s r_u\,\mathrm{d}u}\Big((y_sS_s-r_sK)\,\mathrm{d}s \\ &-\int_{x=-\infty}^{\log(S^*(s)/S_{s-})}(V(s,S_{s-}\,\mathrm{e}^x)-S_{s-}\,\mathrm{e}^x+K)\,\nu(\mathrm{d}s,\mathrm{d}x)\Big)\,|S_t\Bigg). \end{aligned} \tag{8.6}$$

Here, we also took the liberty to express in terms of dividend yield process y ($\mathrm{d}S^p =: (r-y)S_-\,\mathrm{d}t$), while reversing a sign. To derive the IPDE, we simply equate equation (8.5) with the one before its preceding (the one with partials) and divide by $\mathrm{d}t$—ditto for nonzero r. It yields, denoting σ the volatility process ($\mathrm{d}[S^c] =: \sigma^2S_-^2\,\mathrm{d}t$) and $\tilde{\nu}(t,\mathrm{d}x)\,\mathrm{d}t := \nu(\mathrm{d}t,\mathrm{d}x)$,

$$\begin{aligned} &\frac{\partial V}{\partial t} + (r-y)S_-\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S_-^2\frac{\partial^2 V}{\partial S^2} - rV \\ &\quad + \int_{x\neq 0}\left(V(t,S_-\,\mathrm{e}^x) - V(t,S_-) - (\mathrm{e}^x-1)S_-\frac{\partial V}{\partial S}(\cdot,S_-)\right)\tilde{\nu}(t,\mathrm{d}x) \\ &\quad + 1_{\{S_->S^*\}}\left(yS_- - rK - \int_{x=-\infty}^{\log(S^*/S_-)}(V(t,S_-\,\mathrm{e}^x) - S_-\,\mathrm{e}^x + K)\tilde{\nu}(t,\mathrm{d}x)\right) = 0. \end{aligned}$$

This is general, e.g. $\int_{|x|<1} x^2\tilde{\nu}(t,\mathrm{d}x) < \infty$. When $r = r(t,S)$, $\sigma = \sigma(t,S)$, $y = y(t,S)$, $\tilde{\nu}(t,\mathrm{d}x) = \tilde{\nu}(t,S_t,\mathrm{d}x)$ are functions of (t,S), it results in an IPDE. All terms (even really r,y,σ) are then functions of S_-, which as a dummy variable is replacable by S. For constant $r,y,\sigma/S$ and $\nu(\mathrm{d}t,\mathrm{d}x) = \lambda\,\mathrm{d}t\bar{\nu}(\mathrm{d}x)$ with finite activity $\int_{x\neq 0}\bar{\nu}(\mathrm{d}x) < \infty$, [7] solve the IPDE in closed-form (by Fourier inversion) as an infinite sum of double integrals. Evaluating at $S_t = S^*(t)$ gives an integral equation for $S^*(t)$ (in terms of $V(\cdot,S)$), which 2×2 integral equation system (V,S^*) they solve numerically by an iterative Volterra equation algorithm.

8.11 An American deal no one should buy

We conclude with the observation that no reasonable investment value can be attached to an "American option" whose underlying payoff process has no optimal stopping time. The argument works in general, but we assume the Black–Scholes setting with constant rate $r \geq 0$ and a zero-dividend stock with geometric Brownian motion price process S such that $(\mathrm{e}^{-rt}S_t)_{t=0}^m$ is a martingale.

Consider the (undiscounted) payoff process $Z_t = (S_t-1)^+$ for $t<m$ and $Z_m = (S_m-2)^+$. It has no optimal stopping time. It is easy to see that $V_0 = \mathrm{e}^{-rm}\mathbb{E}(S_m-1)^+$. Indeed, both sides equal $\sup_{0\leq s<m}\mathrm{e}^{-rs}\mathbb{E}(S_s-1)^+$, using the fact that the European option price $\mathrm{e}^{-rs}\mathbb{E}(S_s-1)^+$ is increasing in expiration s. But, if we successfully sell this contract at a pitched "fair" price of V_0, we will make an arbitrage profit, no matter how small, with the hedge that goes long one European call of expiration m and strike 1. Indeed, the buyer will eventually have to exercise early before m if the payoff is in the money, for his payoff will drop at m if he procrastinates.

But, even if he wisely exercises at some time $t < m$, we pay him only $S_t - 1$, which is *less* than what our European option hedge is worth then.

At least in theory therefore, an investor is strictly better off at the same cost with the European option than this contract. This suggests that such contracts are not proper financial options. As in [20], one can view an option as a pair (T, O), with random expiry $T \in \mathcal{T}$ and random payoff $O \in \mathcal{F}_T$. For a reward process Z that does have an optimal stopping time T^*, the American option can then properly be defined as the pair (T^*, Z_{T^*}).

Acknowledgements

I am grateful to the anonymous referees for valuable comments. I would also like to thank many of the participants at the Jan-06 Symposium on Optimal Stopping, especially A. Shiryaev and R. Lerche, for interesting discussions, helpful comments and constructive criticism.

References

[1] Andersen, L. and Broadie, M., 2004, A primal-dual simulation algorithm for pricing multidimensional American options, *Management Science*, **50**, 1222–1234.

[2] Beibel, M. and Lerche, H.R., 1997, A new look at optimal stopping problems related to mathematical finance, *Statistica Sinica*, **7**, 93–108.

[3] Bensoussan, A., 1984, On the theory of option pricing, *Acta Applicandae Mathematicae*, **2**, 139–158.

[4] Bolia, N., Glasserman, P. and Juneja, S., 2004, Function-approximation-based importance sampling for pricing American options. *Proceedings of the 2004 Winter Simulation Conference*, pp. 604–611.

[5] Carr, P., Jarrow, R. and Myneni, R., 1992, Alternative characterizations of American put options, *Mathematical Finance*, 87–105.

[6] Chen, N. and Glasserman, P., 2005, Additive and multiplicative duals for American option pricing, Columbia Univversity Working paper.

[7] Chiarella, C. and Ziogas, A., 2006, American call options on jump diffusion processes: a Fourier transform approach, Sidney University of Technology working paper.

[8] Davis, M.H.A. and Karatzas, I., 1994, A deterministic approach to optimal stopping. In: F.P. Kelly (Ed.) *Probability, Statistics and Optimisation* (NewYork, Chichester: John Wiley and Sons), pp. 455–466.

[9] Delbaen, F. and Yor, M., 2002, Passport options, *Mathematical Finance*, **12**(4), 299–328.

[10] Dellacherie, C. and Meyer, P.A., 1978, *Probabilities and Potential* (North-Holland: Elsevier).

[11] Detemple, J., Feng, S. and Tian, W., 2003, The valuation of American call options on the minimum of two dividend-paying assets, *The Annals of Applied Probability*, **13**(3), 953–983.

[12] Elliott, R.J., 1982, *Stochastic Calculus and Applications* (Berlin, New York: Springer).

[13] El Karoui, N. and Quenez, M.C., 1995, Dynamic programming and pricing of contingent claims in an incomplete market, *SIAM Journal on Control and Optimization*, **33**(1), 29–66.

[14] Glasserman, P., 2003, *Monte Carlo Methods in Financial Engineering* (Berlin-Heidelberg-New York: Springer).

[15] Haugh, M. and Kogan, L., 2004, Pricing American options: a dual approach, *Operations Research*, **52**, 258–270.

[16] Jacka, S.D., 1991, Optimal stopping and the American put, *Mathematical Finance*, **2**, 1–14.

[17] Jacod, J. and Shiryaev, A.N., 2002, *Limit Theorems for Stochastic Processes*, Volume 288 of Grundlehren der mathematischen Wissenschaften, 2nd ed. (Berlin-Heidelberg-New York: Springer).

[18] Jaillet, P., Lamberton, D. and Lapeyre, B., 1990, Variational inequalities and the pricing of American options, *Acta Applicandae Mathematicae*, **21**, 263–289.

[19] Jamshidian, F., 1992, An analysis of American options, *Review of Future Studies*, 73–80 Formulas for American options. Working paper, (1989).

[20] Jamshidian, F., 2003, Linear invariant option pricing and application to American and Bermudan options, A collection of three working papers and two presentations substantially consolidated in this paper.

[21] Joshi, M.S., 2006, A simple derivation of and improvements to Jamshidian's and Roger's upper bound method for Bermudan options, Working paper.

[22] Joshi, M.S. and Theis, J., 2002, Bounding Bermudan swaptions in a swap-rate market model, *Quantitative Finance*, **2**(5), 370–377.

[23] Karatzas, I., 1988, On the pricing of American options, *Applied Mathematics and Optimization*, **17**, 37–60.

[24] Kim, I.J., 1990, The analytic valuation of American options, *Review of Financial Studies*, **3**, 545–572.

[25] Kolodko, A. and Schoenmakers, J., 2004, Upper bounds for Bermudan style derivatives, *Monte Carlo Methods and Applications*, **10**, 331–343.

[26] Kolodko, A. and Schoenmakers, J., 2006, Iterative construction of the optimal Bermudan stopping time, *Finance and Stochastics*, **10**, 27–49.

[27] Kramkov, D.O., 1996, Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets, *Probability Theory and Related Fields*, **105**, 459–479.

[28] Lepeltier, J.P., 1984, Mainguenau: le jeu de Dynkin en theorie generale sans L'hypothese de Mokobodski, *Stochastics*, **13**, 25–44.

[29] McKean, H.P., 1965, Appendix: a free boundary problem for the heat equation arising from a problem of mathematical economics, *Industrial Management Review*, **6**(32–39).

[30] Protter, P., 2001, A partial introduction to financial asset pricing theory, *Stochastic Processes and Their Applications*, **91**, 169–203.

[31] Rogers, L.C.G., 2002, Monte Carlo valuation of American options, *Mathematical Finance*, **12**, 271–286.

[32] Shepp, L. and Shiryaev, A.N., 1993, The Russian option: reduced regret, *Annals of Applied Probability*, **3**, 631–640.

[33] Shiryaev, A.N., 1963, On optimum methods in quickest detection problems, *Theory of Probability and its Applications*, **8**, 22–46.