Seminar Mathematical Physics, AAMP Spring 2005

- Assignment 1
- Sheets 24 March
- 1 page summary Variational Techniques
- Further reading: Applied Analytical Methods, part 2
The Korteweg-de Vries (KdV) equation describes the surface elevation \( \eta(x,t) \) of a layer of water above an even bottom. In normalised variables, and in a frame moving with the velocity of small, long waves, the equation is given by
\[
\partial_t \eta + \partial_x^3 \eta + \eta \partial_x \eta = 0.
\]

1. Write the equation in the form
\[
\partial_t \eta = -\partial_x \delta H(\eta)
\]
for a suitable functional \( H(\eta) \).

2. Specify a class of wave elevations so that for that class \( \partial_x \) is a skew-symmetric operator. Argue that on this class the KdV is a Hamiltonian system with Hamiltonian \( H \). In the following we will work on this class.

3. Show by direct verification that the functional \( I(\eta) = \int \eta^2 dx \) is an invariant integral, i.e. \( \partial_t I(\eta) = 0 \) for solutions, as a consequence of the fact that the integrand of \( H \) does not explicitly depend on \( x \).

4. Show that the 'flow' of \( I \), i.e. the solutions of
\[
\partial_t \eta = -\partial_x \delta I(\eta),
\]
is a spatial translation. Show in a direct way that the translation flow and the Hamiltonian flow (2) commute.

5. Look for uniformly translating solutions of (2): solutions of the form
\[
\eta(x,t) = S(x - \lambda t)
\]
for certain velocity \( \lambda \). Derive an equation for \( S \) (ode) and show it is a constrained critical point of \( H \) on a level set of \( I \):
\[
\text{Crit}\{ H(S) \mid I(S) = \gamma \}.
\]

6. Restrict the constraint variational principle to functions of the form \( S = a U(x/w) \) where \( a, w \) are (related to) the amplitude and width respectively. Use scaling arguments to derive relations between \( a, w, \lambda \) and \( \gamma \). Check how the multiplier is related to the value function.

The simplest linear, bidirectional wave equation for (long, small) surface waves above varying bottom is given by
\[
\partial_t^2 \eta = \partial_x \left[ c^2(x) \partial_x \eta \right]
\]
where \( c^2(x) = gh(x) \), with \( h(x) \) the depth.

1. Write the equation in Hamiltonian form.
2. Argue why the functional

\[ I = \int \partial_t \eta \cdot \partial_x \eta dx \]

is called the momentum functional. Under which conditions for \( c(x) \) is it an invariant integral?

3. In case \( c(x, t) \) depends explicitly on \( t \), verify that the Hamiltonian is not conserved.
Variational Modelling of Surface Gravity Waves

Assumptions

- inviscid, incompressible
- surface elevation $z = \eta(x, t)$
- fluid potential $\psi(x, z; t)$

Physical phenomena

- dispersion & nonlinearity
- steady states, ‘solitons’

Full Model, Hamiltonian formulation

- Hamiltonian $H(\phi, \eta) = \frac{1}{2} \rho g \eta^2 dx + \min_{\psi} \left\{ \frac{1}{2} \rho (\nabla \psi)^2 | \psi = \phi \text{ at free surface} \right\}$

  Potential Energy

  Kinetic Energy

- Hamilton Eqn’s $\partial_t \begin{pmatrix} \eta \\ \phi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_\phi H \end{pmatrix}$

- Motion Invariants
  
  Momentum, Mass

Variational Modelling

Approximating the total energy, retaining the Hamiltonian structure.

Examples: Boussinesq, KdV, Zakharov; NLS, phase-amplitude eqn’s, ...
**Relative Equilibria in Hamiltonian Systems**

**Ingredients**
- Hamiltonian system: \( \partial_t u = \Gamma \delta H(u) \)
  - Structure Map: \( \Gamma \) with \( \Gamma^* = -\Gamma \) (\& Jacobi)
  - Hamiltonian: \( H \)
  - Flow: \( u(t) = \Phi_t^H(u_0) \)
- Invariant Integrals: \( I \) with \( \partial_t I(u) = 0 \)

**Equilibria**
\( \Phi_t^H(U) = 0 \iff \delta H(U) = 0 \iff U \in \text{Crit}_u \{ H(u) \} \)

**Relative Equilibria == Coherent States**
- Constrained Energy Principle (CEP)
  - Constrained Critical Energy
    \( \delta H(U) = \lambda \delta I(U) \)
    \( \lambda = \partial h/\partial \gamma \)
  - Dynamic Evolution
    \( \Phi_t^H(U) = \Phi_{\lambda t}^I(U) \)
### Overview Basic Calculus of Variations (Chapter 1)

<table>
<thead>
<tr>
<th>Topic</th>
<th>Finite dimensional</th>
<th>Infinite dimensional</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconstrained minimization</td>
<td>( \min \left{ F(x) \mid x \in \mathbb{R}^n \right} )</td>
<td>( \min \left{ L(u) \mid u \in U \right} )</td>
<td>Sturm-Liouville</td>
</tr>
<tr>
<td>domain of definition</td>
<td>Linear space (or affine space) ( \mathbb{R}^n \ni x )</td>
<td>Function space + BoundaryCond’s : ( U \ni u )</td>
<td>( U = { u(x) \mid x \in (0,1), u(0) = 2 } )</td>
</tr>
<tr>
<td>function(0)</td>
<td>( F : \mathbb{R}^n \to \mathbb{R} )</td>
<td>( L : U \to \mathbb{R} )</td>
<td>( L(u) = \int \left( \frac{1}{2} \int \partial_x u \right)^2 + qu , dx )</td>
</tr>
<tr>
<td>Local Theory</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>differentiation</td>
<td>Partial derivatives ( \frac{\partial F}{\partial x_i} ) ( F : \mathbb{R}^n \to \mathbb{R} ), ( m = 1 \ldots n )</td>
<td>First variation ( \delta L(u) ) so that for all testfunctions ( \eta )</td>
<td>( \delta L(u) = \int \left( \frac{\partial F}{\partial x_i} \eta \right) , dx )</td>
</tr>
<tr>
<td>Directional derivative ( DF(x; \eta) = \left. \frac{d}{de} F(x + e \eta) \right</td>
<td>_{e=0} )</td>
<td></td>
<td>( \delta L(u) = \int \left( \frac{\partial F}{\partial x_i} \eta \right) , dx )</td>
</tr>
<tr>
<td>linear approximation</td>
<td>( F(x; \eta) = F(x) + \delta F(x; \eta) + O(\eta^2) )</td>
<td>( L(u; \delta u) = L(u) + \delta L(u; \eta) + O(\eta^2) )</td>
<td></td>
</tr>
<tr>
<td>Gradient</td>
<td>( \nabla F(x) = \left( \frac{\partial F}{\partial x_i} \right) ) so that ( DF(x; \eta) = \nabla F(x) \cdot \eta )</td>
<td>Variational derivative ( \delta L(u) ) so that for all testfunctions ( \eta )</td>
<td>( \delta L(u) = \int \left( \frac{\partial F}{\partial x_i} \eta \right) , dx )</td>
</tr>
<tr>
<td>Unconstrained Critical point</td>
<td>( D F(\dot{x}; \dot{\eta}) = \nabla F(\dot{x}) \cdot \eta = 0 )</td>
<td>( \delta L(u; \delta u) = \int \left( \frac{\partial F}{\partial x_i} \delta u \right) , dx )</td>
<td>( \delta L(u) = \int \left( \frac{\partial F}{\partial x_i} \delta u \right) , dx )</td>
</tr>
<tr>
<td>admissible variations</td>
<td>for all ( \eta \in \mathbb{R}^n )</td>
<td>for all ( \eta ) that vanish at prescribed boundary cond’s</td>
<td>( \nu ) arbitrary except satisfying ( \nu(0) = 0 )</td>
</tr>
<tr>
<td>Equation</td>
<td>Fermat’s algorithm ( \nabla F(\dot{x}) = 0 )</td>
<td>Euler-Lagrange equation ( \delta L(u) = 0 ) ++ Boundary Cond’s</td>
<td>( \nu(0) = 0 ) and ( \nu(1) = 1 )</td>
</tr>
<tr>
<td></td>
<td>for function(0)</td>
<td>++ Natural BC’s</td>
<td></td>
</tr>
</tbody>
</table>

### Overview Constrained Problems (Chapter 2)

<table>
<thead>
<tr>
<th>Topic</th>
<th>Finite dimensional</th>
<th>Infinite dimensional</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constrained minimization</td>
<td>( \min \left{ F(x) \mid x \in C \subset \mathbb{R}^n \right} )</td>
<td>( \min \left{ L(u) \mid u \in M \subset U \right} )</td>
<td>S-L Eigenvalue Problem</td>
</tr>
<tr>
<td>admissible constraints</td>
<td>Subset of linear space ( C \subset \mathbb{R}^n ) for instance: for given ( K : \mathbb{R}^n \to \mathbb{R} ) the levelset ( C = { x \mid K(x) = \gamma } \subset \mathbb{R}^n )</td>
<td>Subset of function space + BC’s : ( M \subset U ) for given ( K : U \to \mathbb{R} ) the levelset ( M = { u \mid K(u) = \gamma } \subset U )</td>
<td>( M \subset U = { u(x) \mid x \in (0,1), u(0) = 0 } )</td>
</tr>
<tr>
<td>for function(0)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>For function(0)</td>
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<td></td>
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</tr>
<tr>
<td>Constraint Critical point</td>
<td>( D F(\dot{x}; \dot{\nu}) = \nabla F(\dot{x}) \cdot \dot{\nu} = 0 )</td>
<td>( \delta L(u; \delta u) = 0 )</td>
<td>( \delta L(u) = \int \left( \frac{\partial F}{\partial x_i} \delta u \right) , dx )</td>
</tr>
<tr>
<td>admissible variations</td>
<td>for all ( \nu \in \mathbb{R}^n ) for which ( \dot{x} + \dot{\nu} ) satisfies the constraints up to ( O(\varepsilon^2) ) (the tangent space)</td>
<td>for all admissible ( \nu ) for which ( \dot{x} + \dot{\nu} ) satisfies the constraints up to ( O(\varepsilon^2) )</td>
<td>( \nu ) satisfying ( \nu(0) = 0 ) and ( \nu(1) = 1 )</td>
</tr>
<tr>
<td>for function(0)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equation: Lagrange’s Multiplier Rule</td>
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</tr>
<tr>
<td>There is a multiplier such that ( \nabla F(\dot{x}) = \lambda \nabla K(\dot{x}) ) &amp; ( K(\dot{x}) = \gamma )</td>
<td>There is a multiplier such that ( \delta L(u) - \lambda \delta K(u) = 0 ) &amp; ( K(u) = \gamma ) ++ Boundary Cond’s ++ Natural BC’s</td>
<td>( \lambda ) and ( \mu ) satisfying ( \lambda \mu = 0 ) and ( \mu(1) = 0 )</td>
<td></td>
</tr>
<tr>
<td>equivalently: is an unconstrained critical point of ( \dot{x} \to F(\dot{x}) - \lambda K(\dot{x}) ) for unconstrained critical point in ( U ) of ( U \ni u \to L(u) - \lambda K(u) )</td>
<td></td>
<td>Unconstrained critical point in ( U ) of</td>
<td></td>
</tr>
<tr>
<td>Multiplier &amp; Value function</td>
<td>Value function ( V(\gamma) = F(x) ) ( \min \left{ F(x) \mid K(x) = \gamma \right} ) ( \lambda = \frac{\partial V}{\partial \gamma} )</td>
<td>Value function ( V(\gamma) = L(u) ) ( \min \left{ L(u) \mid K(u) = \gamma \right} ) ( \lambda = \frac{\partial V}{\partial \gamma} )</td>
<td>Rayleigh quotient ( \lambda = \frac{\partial \left( \int \partial_x u \right)^2 , dx}{\int \partial_x u , dx} )</td>
</tr>
</tbody>
</table>

\( \lambda \) is a multiplier such that \( \delta L(u) - \lambda \delta K(u) = 0 \) & \( K(u) = \gamma \) ++ Boundary Cond’s ++ Natural BC’s equivalently: is an unconstrained critical point in \( U \) of \( \lambda \) Rayleigh quotient \( \lambda = \frac{\partial \left( \int \partial_x u \right)^2 \, dx}{\int \partial_x u \, dx} \)
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Assignment 2

The assignment consists of one of the following tasks

- EITHER: make and hand in all exercises of sections 1 and 2 specialised for the case of time-independent bottom, and make the exercise about travelling waves in section 3.1 for which you have to design an optimization code;
- OR be sure to understand everything from sections 1 and 2, and design the time-dependent code for the tsunami generation (exercise in Section 3.2); give an extensive report about the code and experiments.

The assignment should be finished completely in June. Please consult Gert Klopman (in period 15 April - 8 May) or me if serious problems arise.

1 Variational formulation of full surface wave equations

The starting point is the next result.

Claim 1  Luke’s variational principle

Let \( \Phi(x,z,t) \) be the fluid potential for 2D irrotational fluid flow, i.e. \( \nabla \Phi \) is the Eulerian velocity; Let \( \eta(x,t) \) denote the surface elevation; Let \( h(x,t) \) be the given bathymetry (possibly time dependent). The full equations for incompressible, irrotational fluid flow follow from the critical point problem

\[
\text{Crit}_{\Phi, \eta} \left\{ \int P(\Phi, \eta) \, dt \right\} \quad \Phi = \phi \text{ at } z = \eta(x,t) \\
P(\Phi, \eta) = \int dx \left[ \int_{-h}^{\eta} \left\{ \partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + g \eta \right\} \, dz \right].
\]

Note that this is a ‘pressure principle’ since the integrand denotes the pressure in the fluid.

Exercise 2  1. Show that variations with respect to \( \Phi \) leads to

\[
\Delta \Phi = 0 \text{ for } -h < z < \eta, \\
\partial_t h = \nabla \Phi \cdot N_b \text{ at } z = -h, \text{ where } N_b = (-\partial_x h, -1) \\
\partial_t \eta = \nabla \Phi \cdot N_s \text{ at } z = \eta, \text{ where } N_b = (-\partial_x \eta, 1)
\]

Write out these equations in full, and interpret them as the continuity equation, the impermeability of the bottom and the kinematic surface equation respectively.

2. Show that variations with respect to \( \eta \) leads to

\[
\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + g \eta(x,t) = 0 \text{ at } z = \eta(x,t)
\]

Interpret this as the dynamic free surface equation.
The above problem is a formulation in the 2D fluid layer. Formally speaking we can reduce it to a problem in one space variable only. This will be an essential improvement, however at the cost that a functional (the kinetic energy) has to be introduced that is not easily expressed explicitly. We reason as follows to obtain a formulation in terms of the two basic quantities (canonical variables): \( \eta(x,t) \) and the free surface potential \( \phi(x,t) \).

Observe:

\[
\int_{-h}^{\eta} \partial_t \Phi dz = \partial_t \int_{-h}^{\eta} \Phi dz - \phi \partial_t \eta - \Phi \partial_t h
\]

which motivates to introduce the (modified?) kinetic energy which can be fully formulated as

\[
K(\phi, \eta) = \min_{\Phi} \left\{ \int dx \left( \int_{-h}^{\eta} \frac{1}{2} |\nabla \Phi|^2 dz - \Phi \partial_t h \right) \right\}, \quad \Phi = \phi \text{ at } z = \eta(x,t)
\]

Then

\[
P(\Phi, \eta) = \int dx \left[ \int_{-h}^{\eta} \left\{ \partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + gz \right\} dz \right].
\]

When \( \partial_t h \neq 0 \), energy will be inserted or subtracted from the fluid motion in the layer. Luke’s variational principle can then be rewritten as a canonical action principle:

\[
\text{Crit}_{\phi, \eta} \int dt \left\{ \int \phi \partial_t \eta dx - \mathcal{H}(\phi, \eta, t) \right\}
\]

with \( \mathcal{H}(\phi, \eta, t) = K(\phi, \eta) + \int \frac{1}{2} \eta \gamma^2 dx - \int \frac{1}{2} gh^2 dx \)

where we could leave out the unimportant term \( \int h^2 dx \) (unimportant for variations; important to measure energy change by bottom motions).

The functional \( \mathcal{H}(\phi, \eta) \) is called the Hamiltonian, and the governing equations are given by Hamiltons equations:

\[
\begin{align*}
\partial_t \eta &= \delta_{\phi} \mathcal{H}(\phi, \eta, t), \\
\partial_t \phi &= -\delta_{\eta} \mathcal{H}(\phi, \eta, t).
\end{align*}
\]

Note that, at least formally, the problem is now well formulated as a problem in basic variables that depend only on \( x \) but not on \( z \) anymore. The price that is paid is that we deal with the rather difficult functional \( K \). We will see the advantage in the next sections.

**Exercise 3**

1. Show that, even when \( h \) is time dependent, the Hamiltonian is actually invariant for adding a constant to \( \phi \). This motivates to introduce the quantity \( u(x, t) := \partial_x \phi \) and to consider \( u, \eta \) as the basic pair of variables, instead of \( \phi, \eta \). Writing the Hamiltonian then as \( \mathcal{H}(u, \eta, t) \), show that the equations become

\[
\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_z & 0 \end{pmatrix} \begin{pmatrix} \delta_{\eta} \mathcal{H}(u, \eta, t) \\ \delta_u \mathcal{H}(u, \eta, t) \end{pmatrix}
\]

Verify that this is still in the form of a Hamiltonian system.
2. We have derived these equations from simple manipulations in the variational formulation, which proves their validity. Yet it is useful to write these equations in full. This should coincide with the original equations. This is true, but be aware of the restriction to the surface and consequences for derivatives; for instance (nasty notation):

\[ \partial_t \phi = \frac{d}{dt} \Phi(x, \eta(x,t), t) = [\partial_t \Phi]_{z=\eta} + [\partial_z \Phi]_{z=\eta} \partial_t \eta \]

3. Suppose the bottom is time-independent. Then the Hamiltonian does not depend explicitly on time. Show that then the Hamiltonian is an invariant integral (constant of the motion):

\[ \frac{d}{dt} \mathcal{H}(\phi, \eta) = 0 \]

for solutions of the equations.

4. Suppose now that the bottom is flat: \( h(x, t) = \bar{h} \) constant. Show that then the horizontal momentum is an invariant integral:

\[ \frac{d}{dt} I(u, \eta) = 0, \text{ with } I(u, \eta) = \int u \eta dx. \]

Consider the \( I \)-flow, and explain from that why this functional is called the horizontal momentum.

## 2 Simplified models by variational restriction

We have observed that the kinetic energy is the crucial functional that makes the full surface wave problem so difficult. We will now consider various simplifications, simplified models all of which will keep the same variational (Hamiltonian) structure. This is achieved by simplifying the kinetic energy, that is to restrict the kinetic energy to specific type of wave fields (such as small amplitude waves, or waves above shallow layers) for which the kinetic energy gets a simpler form.

### 2.1 Linear Theory

The kinetic energy is given by

\[ \mathcal{K}(\phi, \eta) = \frac{1}{2} \int \left\{ \left[ \frac{\partial \phi}{\partial N_S} \right]_{z=\eta} + \Phi_\omega \left[ \frac{\partial \Phi}{\partial N_B} \right]_{z=-\bar{h}} \right\} dx - \int \Phi_\omega \partial_t h dx \]

and contains the normal derivative to the free surface. If we would be able to rewrite this normal derivative \( \frac{\partial \phi}{\partial N_S} \) (and \( \Phi_\omega \)) and express it in \( \phi, \eta \) we would arrive at the kinetic energy as an explicit functional. This is possible in a certain approximation that describes small amplitude waves when the surface elevation and also the bottom variations are small. This will be done in detail in the next exercise. We will first give a more intuitive argument, for steady, time independent, bottom.

For small amplitude waves and small bottom variations, we replace the free surface and the bottom by a flat surface, i.e. we replace the fluid domain by a horizontal strip, \(-\bar{h} < z < 0\) where \( \bar{h} \) is the averaged depth. Then the normal derivative depends linearly on the surface potential, say through some linear operator \( D \):

\[ \frac{\partial \phi}{\partial N_S} = D \phi \]
This operator is called the *Dirichlet-to-Neumann operator*: it assigns to the Dirichlet value $\phi$ the normal derivative of the solution of the Laplace problem in the strip with bottom boundary condition. Hence it incorporates (in a nontrivial way) effects of the water motion in the interior of the layer. Then the kinetic energy becomes a quadratic form

$$K(\phi, \eta) = \frac{1}{2} \int \phi \cdot D\phi dx;$$

$D$ will be a pseudo-differential operator, which means that after Fourier transformation the operator becomes multiplication with the real function

$$\hat{D}(k) = \hat{D}(k) \hat{\phi}(k)$$

where a ‘hat’ denotes Fourier transformation. The function $\hat{D}(k)$ is called the symbol of the operator. In particular

$$K(\phi, \eta) = \frac{1}{2} \int \phi(x) \cdot D\phi(x) dx = \frac{1}{2} \int \hat{\phi}(k) \cdot \hat{D}(k) \hat{\phi}(k) dx$$

Clearly, the operator $D$ will be symmetric, and positive, corresponding to the fact that the symbol is real and positive. (Why??)

Then the governing linear Hamilton equations become

$$\partial_t \eta = D\phi, \quad \partial_t \phi = -g\eta,$$

i.e. the second order equation

$$\partial^2_t \eta = -gD\eta.$$  

This equation has dispersion relation\(^1\) given by $\omega = \Omega(k)$ with

$$\omega^2 = g\hat{D}(k) = : \Omega^2(k)$$

and the second order equation can then be written as

$$\partial^2_t \eta + \Omega^2 (-i\partial_x) \eta = 0.$$  

It may be known (to be derived in the next exercise) that $\Omega$ is explicitly given by

$$\Omega(k) = k\sqrt{g \tanh(kh) / k}.$$  

This means that surface waves have ‘dispersion’: the monochromatic mode $\exp[i(kx - \Omega(k)t)]$ travels with the phase velocity given by

$$c_p(k) = \Omega(k) / k = \sqrt{g \tanh(kh) / k},$$

which depends on $k$. Hence waves of different wavelength travel with different velocity, so that waves of different wavelength starting at the same position, travel at different speed and become separated: dispersed (NOT dissipated, they do not vanish after some time).

Denoting the corresponding operator by $C_p$, the kinetic energy can be written as

$$K_{\text{lin}}(\phi) = \frac{1}{2} \int \phi \cdot D\phi dx = \frac{1}{2y} \int u \cdot C_p^2 u dx =: K_{\text{lin}}(u)$$

\(^1\)The dispersion relation is obtained as the necessary condition such that $\hat{b}(x,t) = \hat{b} \exp[i(kx - \omega t)]$, where $k$ is the wave number and $\omega$ is the angular frequency, is a solution; this requires a certain relation between $\omega$ and $k$, say $\omega = \Omega(k)$. The solution $\exp[i(kx - \Omega(k)t)]$ is then called a mono-chromatic mode.
This gives the interpretation of the kinetic energy of the wave as the sum (integral) of the kinetic energy of each wave component, running with the phase velocity and mass given by $\hat{u}^2/g$:

$$K_{\text{kin}}(u) = \int \frac{1}{2} \hat{u}^2(k) g c_p^2(k) dk$$

Including small bottom motions, the strip-approximation leads to

$$\mathcal{K} = \frac{1}{2} \int \left[ \frac{\tanh(kh)}{k} |\hat{u}|^2 - \phi^* \frac{1}{\cosh(kh)} \partial_t \hat{h} \right]$$

$$- \frac{1}{2} \int \left[ \frac{\phi^*}{\cosh(kh)} \partial_t \hat{h} + \frac{\tanh(kh)}{k} |\partial_t \hat{h}|^2 \right]$$

$$= \frac{1}{2} \int \frac{\tanh(kh)}{k} |\hat{u}|^2 - \frac{1}{2} \int \frac{\phi^*}{\cosh(kh)} \partial_t \hat{h} - \frac{1}{2} \int \frac{\partial_t \hat{h}}{k}$$

The last term is irrelevant for obtaining the governing equations.

**Exercise 4** Find the solution of the Laplace problem $\Delta \Phi = 0$ on the strip $-\bar{h} < z < 0$ by determining the coefficients $A, B$ in the general solution at wavenumber $k$:

$$\Phi(k) = e^{ikx} \left[ Ae^{k(z+\bar{h})} + Be^{-k(z+\bar{h})} \right]$$

from the surface condition $\Phi = \phi$ at $z = 0$, and the bottom condition without, and then with, bed motion. Verify that it is given by

$$\Phi = \frac{\cosh(k(z+\bar{h}))}{\cosh(kh)} \Phi(k) - \frac{\sinh(kz)}{k \cosh(kh)} \partial_t \hat{h}(k).$$

Observe the dependence on the depth which will show itself in calculating the normal derivative (DtN operator) at the free surface. In fact, show that the DtN operator for horizontal fixed bottom is given by the operator with symbol

$$\hat{D}(k) = k \tanh(kh)$$

and derive from this the results given above.

**Remark 5** Dispersion, Long wave approximations

A Taylor expansion of the dispersion relation $\Omega(k)$ near $k = 0$ leads to

$$\frac{\Omega^2 h}{g} = (kh)^2 - \frac{1}{3} (kh)^4 + \frac{2}{15} (kh)^6 - \frac{17}{315} (kh)^8 + \mathcal{O} ( (kh)^{10} )$$

In lowest order of $(kh)$ it gives

$$\Omega^2 = ghk^2 = c_\infty^2 k^2$$

where $c_\infty = \sqrt{gh}$ is the (maximal) phase speed of infinitely long waves. Observe that the next higher order expansion is non-definite, leading to unstable approximation; this can be ‘stabilised’ at the cost of changing the differential operator by the inverse of a definite differential operator:

$$\frac{\Omega^2 h}{g} \approx (kh)^2 - \frac{1}{3} (kh)^4 \sim -h^2 \partial_x \left( 1 + \frac{h^2}{3} \partial_x^2 \right)^{-1} \left( \text{unstable} \right)$$

$$\approx \frac{(kh)^2}{1 + (kh)^4/3} \sim -h^2 \partial_x \left( 1 - \frac{h^2}{3} \partial_x^2 \right)^{-1} \left( \text{stable} \right).$$
2.2 SWE: Shallow water equations

The linear theory in the limit for \((kh) \to 0\) (infinitely long waves \(k \to 0\), or shallow layer \(h \to 0\)) leads to \(K = \int h u^2/2dx + \ldots\). This effectively means that the fluid potential is approximated by \(\Phi(x,z) \approx \phi(x)\) independent of \(z\). Taking this approximation in the kinetic energy integral, but taking into account the integration till the free surface, leads to the shallow water approximation

\[
K = \frac{1}{2} \int (\eta + h) u^2 dx - \int \phi \partial_t h dx
\]

The non-quadratic term \(\eta u^2\) in the integrand is the only term leading to nonlinearity of the equations:

\[
\begin{align*}
\partial_t \phi &= -g \eta - u^2/2 \\
\partial_t \eta &= -\partial_x [(h + \eta) \partial_x \phi] - \partial_t h
\end{align*}
\]

The last equation can be rewritten like \(\partial_t (h + \eta) = -\partial_x [(h + \eta) u]\), and the first one is Bernoulli’s equation at the surface with approximation of the squared velocity. Both equations rewritten leads to the SWE in ‘conservation’ form

\[
\begin{align*}
\partial_t u &= -\partial_x \left[ g \eta + u^2/2 \right] \\
\partial_t (h + \eta) &= -\partial_x [(h + \eta) u]
\end{align*}
\]

2.3 New Boussinesq equation

Typically a Boussinesq equation is an approximation for which dispersive and nonlinear effects are taken into account in a simplified and balanced way. Hence, the dispersion operator is approximated, and the nonlinearity is taken as in the SWE. A simple addition of the corresponding terms in the kinetic energy leads to

A non-definite (unstable) B-eqn is obtained for \(c_{p, \text{approx}}^2 = gh \left( 1 - (kh)^2/3 \right)\), a positive definite one is obtained for \(c_{p, \text{approx}}^2 = gh/ \left( 1 + (kh)^2/3 \right)\).

We will here treat another B-eqn, that is derived directly from the exact formulation by taking a simple Ansatz for the potential in the fluid and then deriving the B-eqn by restriction of the original variational principle.

The details are given now\(^2\).

We make the following Ansatz for the potential \(\phi(x, z, t)\), corresponding with a parabolic behaviour over depth with \(\partial_z \phi = 0\) at the bed and \(\Phi = \phi\) at the free surface:

\[
\begin{align*}
\Phi(x, z, t) &= \phi(x, t) + f(z; h, \eta) \psi(x, t), \\
\text{with } f(z; h, \eta) &= \frac{1}{2} (z - \eta) \frac{2h + z + \eta}{h + \eta}.
\end{align*}
\]

We take this choice, because we only want time derivatives of \(\eta(x, t)\) and \(\phi(x, t)\) to appear in the Euler-Lagrange equations, and because we know that for a horizontal bottom we have \(\partial_z \Phi = 0\) at \(z = -h\) and therefore expect parabolic behaviour at leading order.

\(^2\)This new formulation is based on a recent paper: G.Klopman, M. Dingemans & E. van Groesen, A variational model for fully non-linear water waves of Boussinesq type, accepted for Proceedings of 20th International Workshop on WWFB2005 (Water Waves and Floating Bodies), Spitsbergen, Norway, 29 May - 1 June 2005.
Under the assumption of a mildly sloping bottom, i.e. neglecting spatial derivatives of \( h(x,t) \), the velocity components become:

\[
\partial_x \Phi = \partial_x \phi - \frac{1}{2} \left[ 1 + \left( \frac{h + z}{h + \eta} \right)^2 \right] \psi \partial_x \eta + f(z; h, \eta) \partial_x \psi
\]

(1)

and \( \partial_z \Phi = \frac{h + z}{h + \eta} \psi \).

(2)

Note that \( \psi(x,t) \) is the vertical velocity \( \partial_z \Phi \) at \( z = \eta(x,t) \).

At the bottom the potential is \( \Phi = \phi - \frac{1}{2} (h + \eta) \psi \). From these, we find for the energy density \( H \), writing as before \( u = \partial_x \phi \)

\[
H = \frac{1}{2} (h + \eta) \left[ u - \frac{2}{3} \psi \partial_x \eta - \frac{1}{3} (h + \eta) \partial_x \psi \right]^2
\]

\[
+ \frac{1}{90} (h + \eta) \left[ \psi \partial_x \eta - (h + \eta) \partial_x \psi \right]^2
\]

\[
+ \frac{1}{6} (h + \eta) \psi^2 + \frac{1}{2} g \eta^2.
\]

\[
- \left[ \phi - \frac{1}{2} (h + \eta) \psi \right] \partial_t h
\]

The best results are obtained using this expression. However, to simplify matters we will only consider a further approximation, namely a weakly nonlinear model.

### 2.3.1 Weakly nonlinear model

We assume that the free surface slope \( \partial_x \eta \) is small, and this term will therefore be neglected in the Hamiltonian. This then leads to the (still positive definite) Hamiltonian

\[
H = \frac{1}{2} (h + \eta) \left[ u - \frac{1}{3} (h + \eta) \partial_x \psi \right]^2
\]

\[
+ \frac{1}{90} (h + \eta) \left[ (h + \eta) \partial_x \psi \right]^2
\]

\[
+ \frac{1}{6} (h + \eta) \psi^2 + \frac{1}{2} g \eta^2.
\]

\[
- \left[ \phi - \frac{1}{2} (h + \eta) \psi \right] \partial_t h
\]

and find the set of equations

\[
\partial_t (h + \eta) + \partial_x \left\{ (h + \eta) \left[ u - \frac{1}{3} (h + \eta) \partial_x \psi \right] \right\} = 0,
\]

\[
\partial_t u + \partial_x \left\{ \frac{g \eta + \frac{1}{2} (h + \eta) \partial_x \psi}{(h + \eta)^2} \left[ u - \frac{2}{3} (h + \eta) \partial_x \psi \right]^2 \right\} = 0,
\]

\[
(h + \eta) \psi + \partial_x \left\{ (h + \eta)^2 u - \frac{2}{5} (h + \eta)^3 \partial_x \psi \right\} + \frac{3}{2} (h + \eta) \partial_t h = 0.
\]
2.3.2 Linearised equations

To get some experience with the equations, we first consider the linear limit for horizontal bed, i.e. the still-water depth \( h \) is constant, we get:

\[
\begin{align*}
\partial_t \eta + h \partial_x u - \frac{1}{3} h^2 \partial_x^2 \psi &= 0, \\
\partial_t u + g \partial_x \eta &= 0, \\
\left[1 - \frac{2}{9} h^2 \partial_x^2\right] \psi + h \partial_x u &= 0.
\end{align*}
\]

Formally inversion of the last equation, makes it possible to eliminate the variable \( \psi \) and leads to the equation

\[
\begin{align*}
\partial_t \eta + h \partial_x u + \frac{1}{3} h^3 \left[1 - \frac{2}{5} h^2 \partial_x^2\right]^{-1} \partial_x^3 u &= 0, \\
\partial_t u + g \partial_x \eta &= 0.
\end{align*}
\]

We find for the above linearized Boussinesq-type equations the following dispersion relation:

\[
\frac{\omega^2 h}{g} = (kh)^2 \left(1 + \frac{1}{15} (kh)^2 + \frac{2}{75} (kh)^4 + \frac{4}{75} (kh)^6 + \mathcal{O}((kh)^8)\right).
\]

The first terms of a Taylor-series expansion around \( kh = 0 \) are:

\[
\frac{\omega^2 h}{g} = (kh)^2 - \frac{1}{3} (kh)^4 + \frac{2}{15} (kh)^6 - \frac{4}{75} (kh)^8 + \mathcal{O}((kh)^{10}).
\]

which start differing from the expansion of the exact dispersion relation with the term proportional to \((kh)^8\). The dispersion curves differ less than 1% for \( kh < 2.3 \) and less than 2.8% for \( kh < \pi \).

3 Travelling waves and tsunami generation

In the next sections we use a numerical Finite Element Method to simulate two specific topics. (See Annex for the FEM approach as a variational restriction method.)

3.1 Periodic travelling waves as relative equilibria

From general Hamiltonian theory we have seen that travelling waves can be expected when besides the Hamiltonian also the horizontal momentum is an invariant functional. For this to be true we have to consider the case of a flat constant bottom. Then the principle of constrained minimal energy will lead to translating waves:

\[
\min \left\{ \mathcal{K}(u, \eta) \mid I(u, \eta) = \gamma \right\}.
\]

We will be interested to investigate the waves for the newly derived B-eqn. This cannot be done analytically, therefore we will design a numerical optimization scheme.

Exercise 6 Linear and Nonlinear normal modes; Soliton

Consider first the periodic case. Design and implement a FEM.

1. First implement the linear scheme, and check numerical results with explicitly known modes at specific wave numbers. Verify/comparing the translation speed with phase velocity of linear theory. Investigate robustness of the scheme.
2. Now add the nonlinear term to the formulation. (Introduce an artificial parameter that multiplies the nonlinear term.) Design an iteration scheme to solve the nonlinear problem; investigate robustness. Calculate the nonlinear mono-chromatics; investigate the shape and the propagation speed.

3. Realise that a soliton-type of solution can only be found in a nonlinear system. Anticipate symmetry in the choice of your boundary condition at one end of the interval. Find a boundary condition at the other side to be able to calculate a soliton. Investigate shape and propagation speed.

**Exercise 7** Can you get the qualitative results above in a more analytical way using Maple(-type of) computer algebra package.

### 3.2 Symmetric tsunami generation

We now want to simulate the generation of waves caused by bottom motions, as a simple model to simulate the generation of tsunamis caused by earthquakes or bed motion from moving tectonic plates. For numerical efficiency, consider symmetric or skew-symmetric situations in the following.

**Exercise 8**

1. Design and implement the time-accurate spatial FEM-code; for time-discretization use RK45 for instance.

2. Test your code for constant bottom profiles in a sufficiently convincing way.

3. Study bottom motions on a relatively small part of the interval to simulate bed motions from volcanic or tectonic activities. Investigate the resulting surface waves and give arguments for the accuracy of the results.
4 Annex: Scientific Computing

This is material from the book AMS, Advanced Modelling in Science (Van Groesen & Molenaar, 2005).

As was indicated already in the section about variational approximation and restriction, the difference between (deriving in a variational consistent way) low-dimensional models and high-dimensional models for numerical calculations is not seen as a principle difference. Different is that for low-dimensional models the choice of the model manifold is essential and determines directly the quality and validity of the model; usually it will be a nonlinear manifold. When looking for numerical algorithms, however, the model manifold is taken to be a linear subspace, of finite but high dimension, say dimension $N$. Characteristic is then to consider $N$ as a parameter that can be varied according to desire of accuracy: not a single manifold but a monotonically increasing sequence of linear manifolds which will span the whole space in the limit $N \to \infty$. Hence, roughly speaking, the solution will always be captured (with arbitrary accuracy) by taking $N$ sufficiently large. Of course, one specific choice of the $N$-dimensional manifolds (choice of base functions) may be better (or more efficient, or more easy) for one problem than for another, but for large enough $N$ the differences with which the solution can be approximated will be small.

For variational problems, the variational restriction method guarantees that the variational property of the basic equation is reflected in the high-dimensional problem since the so called discretized system will be obtained as the equation for critical points of the restricted functional. Hence the discretization procedure is variationally consistent. This consistency in itself is not directly related to accuracy or efficiency. These are conditions that will depend on the specific application, and that in general can be accommodated, or at least not violated, by the requirement of consistency. In the basic description of the ideas to follow, we will not dwell upon efficiency conditions, while accuracy should be improved by consistency.

The basic ingredients of any model are the state variables and the relations between them. For numerical discretizations of complicated sets of pde’s, the approximation of the state variables belongs to ‘approximation theory’, and is formally not related to the way of approximating the equations. However, in the variational restriction method, the discretization of the equations follows directly from the choice of the model manifold, i.e. from the approximation of the state variables. So the discretization of the state variables, and the discretization of the equations are closely connected in the process of variational restriction. In fact, we have seen in Section 2.2?? that in a general Ritz-Galerkin way, the restriction of the equation is the discretized system that is obtained by projecting the original equation in directions of the base functions with which the functions is approximated. For restricted manifolds that are nonlinearly depending on the parameters, the projection of the original equation is along the the tangent directions of the manifold.

In the following we will consider briefly some simple ways to approximate state variables, and then show the resulting discretizations for characteristic problems. We will use the Helmholtz equation from optics as a simple case to illustrate the numerical approach using the Finite Element Method (FEM).

4.1 Approximation of the state variables

Let the state of a system be described by an element from a state space which will be infinite dimensional for our main interest. A simple, but illustrative example that we will use in the rest of this section to illuminate the ideas, is to take real or complex valued functions defined on an interval, say

$$ U = u : [0, L] \to \mathbb{R} \text{ (or } \mathbb{C}) $$

We will not specify smoothness of the functions higher than continuity, but when required, we will assume sufficient regularity to perform certain operations later on. In the first instance we will also disregard possible boundary conditions.
Approximation theory deals with approximating functions with a finite number of variables: approximating the infinite dimensional space by a finite dimensional one, say of $N$ dimensions. We list and comment on a few of the most well-known methods.

**Collocation** The interval is divided into $N$ intervals (of equal or different mesh-size) by specifying grid points $x_0 = 0 < x_1 < x_2 < \ldots < x_N = L$. Then function values at these points are collected in a vector $(u_0, u_1, \ldots, u_N)$. For a given continuous function $u(x)$ the ‘discretization’ is defined by taken for $u_k$ the value at the point $x_k : u_k = u(x_k)$. Note that in this procedure, the function $u(x)$ is represented by an $N$-dimensional vector. The reverse way, interpreting the vector as an approximation of the function, is in fact not defined, since in the intervals between the gridpoints there is no information. Intuitively one often thinks of linear or spline interpolation in between the points. In fact, many modern-day plotting programmes on computers will easily transform such a bunch of points by a more or less smooth line, often without the user knowing what is actually plotted. It is clear that this is no basis for any serious mathematics. Stated differently, without additional information, the collocation method does not define a model manifold that can be considered as a subspace of the original state space. Related to this is the fact that the map $u(x) \to (u_0, u_1, \ldots, u_N)$ can be described with functionals like

$$u_k = \int u(x) \delta(x-x_k)dx,$$

but since function values at a specific point are not continuously defined in $L_2$-sense (represented by the ‘delta-function’ that is not square integrable), this is not a continuous functional in the usual $L_2$-sense. This collocation method of a function has also its consequence when looking for approximations of the derivatives of such a function. As said above, no information in between the grid points is available, so one has to define the derivatives without this information. The usual approach is to replace the differential quotients by difference quotients, i.e. taking ‘finite differences’, which is also the common name for such methods. Euler forward, backward, central differences etc. are then some of the (many, non-unique) choices.

**Finite Elements** By specifying the function in the intervals between the grid points, different from the collocation method, we get subsets of the original space. In Finite Element Methods this is done by using ‘local’ functions, i.e. by specifying a basis of local functions, each of which vanishes outside a small interval; usually these are nonzero only on a few adjacent ‘elements’, which are the intervals between grid points. The smoother (more differentiable) the functions are taken for approximating a given function and its derivatives up to certain order accurately, the longer the support of the base functions will have to be. We will consider only the simplest example where the base functions are ‘hat’-functions illustrated before already. These are the piecewise linear functions with support on two adjacent ‘elements’, the intervals left and right of a mesh point $x_k$ at which the function takes the value one. For simplicity we will also assume a uniform grid of mesh size $h = L/N$. In formula the base functions are then given by

$$T_k(x) = \left(1 - \frac{|x-x_k|}{h}\right)H(x-x_{k-1}) * H(x_{k+1} - x)$$

where $H$ denotes the Heaviside function. Then any continuous function will be approximated by the piecewise linear function through the function values at the grid points:

$$u(x) \approx \sum_{k=0}^{N} u_k T_k(x)$$

(5)

where in a pointwise way: $u_k = u(x_k)$. By using instead of these hat-functions a basis of quadratic or higher order splines, a function is approximated by parabola or higher order polynomials.

A slightly different way to define the coefficients $u_k$ in (5) is to use a continuous functional to determine the coefficients so that they depend continuously on the function. This can be done for instance by taking

$$u_k = \int u(x) T_k^*(x)dx$$
with a set of ‘dual’ base functions $T_k^*$ satisfying for all $k, j$

$$\int T_k T_j^* \, dx = \delta_{kj}.$$ 

**Fourier-truncation** If the functions to be considered are periodic, with period $L$ say, standard Fourier representation may be used. This means that a function that is represented by its Fourier series will be approximated by a finite sum: Fourier truncation. A truncation to $n$ modes, then leads to a $N = 2n + 1$-dimensional manifold and is described, using complex notation, by

$$u(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx/2\pi/L} \rightarrow \sum_{k=-n}^{n} c_k e^{ikx/2\pi/L}$$

Note that in this way the approximations form a subspace of the original space. The Fourier coefficients giving the mapping to the finite dimensional space, are the values of $N$ linear functionals $C_k$:

$$u \rightarrow C_k(u) := \frac{1}{L} \int u(x) e^{-i2\pi kx/L} \, dx$$

The exponentials, or the corresponding real functions $\cos(2\pi kx/L), \sin(2\pi kx/L)$, form a set of **global** base functions. Global since the support covers the whole interval, and, as a consequence, local changes of a function effect all Fourier coefficients, as is seen from the action of the functionals $C_k$.

### 4.2 Variational discretization by restriction

We will now show a few simple examples of variational discretization.

#### 4.2.1 Sturm-Liouville boundary value problem

Consider a variational problem with functional $L(u)$ on a space $U = \{u : [0, L] \rightarrow \mathbb{R}\}$. As specific example we will first take the Sturm-Liouville type of functional

$$L(u) = \int_0^L \left[ \frac{1}{2} p(x) (\partial_x u)^2 + \frac{1}{2} q(x) u^2 - f(x) u \right] \, dx$$

where $p, q$ and $f$ are given functions. We consider the functional for functions that satisfy homogeneous Dirichlet conditions: $u(0) = u(L) = 0$. The governing Euler-Lagrange equation leads to the standard Sturm-Liouville boundary value problem

$$-\partial_x (p\partial_x u) + qu = f \text{ on } (0, L)$$

$$u(0) = u(L) = 0. \quad (6)$$

We will use FEM with the piecewise linear hat-functions introduced above on a uniform grid with mesh size $L/(N + 1)$ ($N + 1$ for ease of notation further on). Hence each function is approximated by an element from the model manifold

$$u^{(N)}(x) = \Sigma_{k=0}^{N+1} \mu_k T_k(x).$$

To satisfy the boundary conditions we take $\mu_0 = \mu_{N+1} = 0$, and there results

$$u^{(N)}(x) = \Sigma_{k=1}^{N} \mu_k T_k(x).$$
Since \( u^{(N)} \) is affine on each element, and continuous everywhere, its derivative is piecewise continuous, and the functional can be evaluated for such functions. Inserting the superposition into the functional \( \mathcal{L} \) there results a function of the \( N \)-vector \( \mu := (\mu_1, \ldots, \mu_N) \) defined by

\[
\mathcal{L}(\mu) := \mathcal{L}\left(u^{(N)}\right).
\]

The FE approximation of the S-L problem is then given by the variational problem for the critical points of \( \mathcal{L} \), i.e. by the \( N \)-algebraic equations

\[
\nabla_{\mu} \mathcal{L}(\mu) = 0, \text{ i.e. } \partial_{\mu_1} \mathcal{L}(\mu) = \ldots = \partial_{\mu_N} \mathcal{L}(\mu) = 0.
\]

An explicit expression for the function \( \mathcal{L}(\mu) \) is easily obtained:

\[
\mathcal{L}(\mu) = \int_0^L \left[ \frac{1}{2} p(x) \left( \Sigma_{k=1}^N \mu_k \partial_x T_k(x) \right)^2 + \frac{1}{2} q(x) \left( \Sigma_{k=1}^N \mu_k T_k(x) \right)^2 - f(x) \left( \Sigma_{k=1}^N \mu_k T_k(x) \right) \right] dx
\]

This can be recognised to be a quadratic expression which can be written with matrix notation like

\[
\mathcal{L}(\mu) = \frac{1}{2} \mu \cdot P \mu + \frac{1}{2} \mu \cdot Q \mu - F \cdot \mu
\]

where the elements of the matrices \( P, Q \) are given by

\[
P_{kj} = \int_0^L \left[ p(x) \left( \partial_x T_k(x) \right) \left( \partial_x T_j(x) \right) \right] dx \tag{7}
\]

\[
Q_{kj} = \int_0^L \left[ q(x) T_k(x) T_j(x) \right] dx \tag{8}
\]

and the vector \( F \) by

\[
F_k = \int_0^L f(x) T_k(x) dx.
\]

The governing algebraic equation is then

\[
\nabla_{\mu} \mathcal{L}(\mu) \equiv P \mu + Q \mu - F = 0,
\]

a system of \( N \) linear equations for the \( N \) components of the vector \( \mu \). With usual conditions for the function \( p(x) \) that guarantee that the S-L problem has a solution (for instance, positivity of \( p(x) \) and \( q(x) \) suffices), this algebraic system has a solution that can be found numerically. Comparing this with the original S-L equation, it is seen that the procedure has discretized the differential operator \(-\partial_x p(x) \partial_x\) into the matrix \( P \), and the multiplication operator \( q(x) \) into the matrix \( Q \), and the function \( f \) into the vector \( F \).

It may be observed that the matrices \( P \) and \( Q \) are symmetric; this is an immediate consequence of the consistent variational approach: the restriction of the quadratic parts of the functional \( \mathcal{L} \) of \( u \) have become symmetric bilinear functions of \( \mu \), which, as any symmetric bilinear function (quadratic form) can be written with a symmetric matrix.

Actually, in the formulas above we have not yet used the specific definition of the basis functions \( T_k \); the same formulas apply when the function \( u \) is represented by a Fourier-sine series (the restriction to sines in order to satisfy the boundary conditions) which are truncated to \( N \) modes:

\[
u(x) \rightarrow \Sigma_{k=1}^N \mu_k \sin(\pi k x / L)
\]
when \( T_k(x) \) is defined as \( \sin(\pi k x/L) \). The difference becomes apparent when the matrices are explicitly calculated. Then also the practical advantage of the FEM may be appreciated: different from the matrices obtained with the global base functions of the Fourier method, the FEM matrices obtained with the local hat functions are very sparse, in this case tri-diagonal (the product of any two hat functions is restricted to at most three elements, see below). On the other hand, the matrices obtained with the global sine-functions will usually be full matrices when the functions \( p, q, f \) are not trivially constant. From a computational point of view, the FEM-case is preferred to solve the linear algebra problem.

To show the sparcity of the FEM-matrices, observe that, although the integrals in (2) are over the entire interval \((0, L)\), the confinement of the tent functions has as consequence that the product \( T_k T_j \), and likewise the product of their derivatives, vanishes except for \( k = j \), or \( k = j \pm 1 \). Hence, both \( P \) and \( Q \) are tri-diagonal matrices. There structure and character can best be interpreted for the special case that the functions \( p, q \) are constant. Then:

\[
\begin{align*}
\text{for } p(x) &= 1 : & P &= \frac{1}{h} \text{tri-diag}(1, -2, 1), \\
\text{for } q(x) &= 1 : & Q &= \frac{1}{h} \text{tri-diag}\left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right).
\end{align*}
\]

and we find more specifically that the above procedure has discretized the differential operator \( -\partial_x^2 \) and the multiplication operator \( q(x) \) in the following way at the point \( x_k \):

\[
\begin{align*}
\partial_x^2 u(x_k) &\rightarrow \frac{\mu_{k+1} - 2\mu_k + \mu_{k-1}}{h^2} \\
u(x_k) &\rightarrow \frac{1}{6} \mu_{k+1} + \frac{2}{3} \mu_k + \frac{1}{6} \mu_{k-1}.
\end{align*}
\]

The expression for the differential operator may be recognized as being the same expression that would be obtained when the central difference method of the second derivative is used. However, the discretization of the multiplication operator, here the identity for \( q = 1 \), shows that the present discretization is not simply the value at the point of consideration as would be the case with Taylor-type of arguments. Instead, it is a weighted value, weighted with the value at two adjacent grid points.

**Exercise 9** Write a numerical implementation of the discretization scheme above, for instance using Matlab. Check your implementation by taking special cases for which the solution is known (at least qualitatively), for instance for the uniform case \( p = 1, q = 0, f = 1 \). Determine the 'order' of the numerical scheme: the error of the solution as function of the grid size. Take for \( p = 1 = -q \) different loading functions \( f \) and convince yourself by plotting the resulting deflections that the shapes are what you expect from experience and the interpretation of the equation as a loaded string.