Title **Assignment Seminar Mathematical Physics**

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Organisation for finalising the course:
Each student has to finish three 'groups' of assignments:

1. The *general assignments about WKB* as given by van Groesen (described below, to be handed in electronically)
2. The *general assignments about existence travelling waves* by Van Gils (hand in withVan Gils)
3. A *special assignment* consisting of a more extensive exercise, to be chosen from a topic given by Van Groesen, or Van Gils or Molenaar (ask them).

Assignments 1 and 2 have to be finished before 16 October.
Assignment 3 has to be finished before December 1.

General Assignments WKB (4 exercises)
The standard equation to be considered is:

$$\partial_z^2 u + k^2(z)u = 0. \tag{1}$$

Physically u can be interpreted as the space-dependent part of the time-harmonic wave $u(z)\exp(-i\omega t)$. For general $k(z)$ the general solution cannot be written down in elementary functions. Based on the explicit solution for constant k, written in complex notation like

$$u(z) = Ae^{ikz} + Be^{-ikz}$$

(to the right and left-traveling wave) one can try to find approximations for the case that the function k is 'slowly varying'. We are interested in slowly varying function $k(z) = K(\sigma z)$, with σ small, and bounded derivative of K, but on 'long' interval of length $O(1/\sigma)$ over which k has change of order one.

1. Consider three possible approximations (for right traveling part):

   (a) naive quasi-homeogeneous approx:

   $$u_1(z) = A\exp(ik(z)\cdot z)$$

   (b) approx. with phase-function (and constant amplitude)

   $$u_2(z) = A\exp(i\theta(z)), \quad \theta(z) = \int^z k(\zeta)d\zeta$$

   (c) the WKB approximation u_{WKB}.

   Calculate the residue for each of these approximations.
   Choose your own function $k(z)$ that satisfies the assumptions and plot the three solutions on an interval of relevant length.

2. Since we have no exact solution, we use (standard) numerics to calculate a numerical solution:

   (a) Calculate numerically a solution u_{num}; use for instance MAPLE-diff-solve.

   (b) Make sure the calculated solution is 'accurate', and give full arguments why it is accurate enough to be considered as a 'nearly' exact solution.

   (c) Compare the four approximations, and describe your findings.

   (d) Change the parameter σ so that the condition of being 'slowly' varying is less and less well satisfied. Compare u_{WKB} with u_{num} and describe your conclusions.

3. Motivated by the form of the WKB-solution, introduce the following (WKB-) **transformation**:

$$\begin{aligned} z &\rightarrow \theta(z) = \int^z k(\zeta)d\zeta, \\ u(z) &\rightarrow B(\theta) = u(z) \cdot \sqrt{k(z)} \end{aligned}$$

   (a) Derive the ODE for $B(\theta)$ and find that it is of the form

$$\partial_\theta^2 B + [1 + \beta(\theta)]\, B = 0 \tag{2}$$

   where β depends on the function k. Observe that $\beta = O(\sigma^2)$.

   (b) How do you get the WKB-solution for (1) from this result?

   (c) Use this result to obtain a higher order approximation for the original equation by taking the WKB-solution of (2). What are the conditions to obtain reliable improvement?

   (d) Plot the improved solution and compare with u_{num} (do this in particular for cases of 'not so slowly varying' coefficient).

4. Consider the standard linear wave equation for surface waves above varying bottom:

$$\partial_t^2 \eta = \partial_x \left[c^2(x) \partial_x \eta \right]$$

   where $c^2(x) = gh(x)$, with $h(x)$ the depth. Give for time harmonic solutions the conditions for 'slowly varying' coefficient $c(x)$ (give physical interpretation) and derive a WKB-type of solution.