Seminar Mathematical Physics,
3 lectures on Optics

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Autumn 2000

Abstract

Lecture 1: Aspects of NLS, 28 Sept.
Lecture 2: Photonic Band Gaps
Lecture 3: Tunneling through Localised States

Lecture 1: 28 September

1 Lecture 1: Aspects of NLS

1.1 Appearance of NLS (Nonlinear Schrödinger equation)

in 1 space and time variable:

- in weakly-nonlinear, dispersive, conservative wave equation for physical quantity $\eta(x,t)$; dispersion given by $D(k,\omega)=0$, with solution $\omega=\Omega(k)$, or $k=K(\omega)$ for modes $e^{i[kx-\omega t]}$, Ω and K real and *odd* functions
- looking for slow, small deviations caused by nonlinearity of one monochromatic wave solution ('mode'), described by complex amplitude A

  $$A(x,t)e^{i[k_0x-\omega_0 t]}, \quad k_0=K(\omega_0)$$

  or by real amplitude-phase (A-Ph):

  $$A(x,t)=a(x,t)e^{i\phi(x,t)}$$

1.1.1 NLS for physical problems (quadratic or qubic nonlinearity)

Optics: pulses in 1D, initial value problem or signalling problem
Optics: beams in 2D
Surface waves (quadratic nonlinearity)

1.1.2 Derivation Linear (!) Schrödinger from first principles

General solution linear dispersive equation (signalling problem):

$$\eta(x,t)=\int \alpha(\omega)e^{i[K(\omega)x-\omega t]}d\omega$$

For spectral function α (satisfying $cc(\alpha(\omega)) = \alpha(-\omega)$) concentrated around $\pm\omega_0$, one writes

$$\eta = e^{i\theta_0} \int \alpha(\omega_0 + \nu) e^{i[K_0(\nu)x - \nu t]} d\nu + cc$$

where $\theta_0 = K(\omega_0)x - \omega_0 t$ is the phase of a mono-chromatic solution,and

$$\begin{aligned} K_0(\nu) &= K(\omega_0 + \nu) - K(\omega_0) \\ &= K'(\omega_0)\nu + \beta_2\nu^2 + \beta_3\nu^3 + ... = K'(\omega_0)\nu + \beta(\nu) \end{aligned}$$

From $\{\partial_t \hat{A}\} = -i\nu\hat{A}$ and $\{\partial_x \hat{A}\} = iK_0(\nu)$ follows what we will call the 'dispersive' LS eqn

$$\begin{aligned} \partial_x A - iK_0(i\partial_t)A &= 0, \\ [\partial_x + K'(\omega_0)\partial_t] A - i\beta(i\partial_t)A &= 0 \end{aligned} \tag{1}$$

or, by introducing $\tau = t - K'(\omega_0)x, \xi = x$:

$$\partial_\xi A - i\beta(i\partial_\tau)A = 0$$

Remark 1 *1. Usually the dispersion relation is approximated by a quadratic function, i.e.* $\beta_3 = ... = 0$*. Then we get the (standard) LS eqn:*

$$\partial_\xi A + i\beta_2\partial_\tau^2 A = 0$$

2. Including one higher order term, we get Dysthe modification (??)

$$\partial_\xi A + i\beta_2\partial_\tau^2 A - \beta_3\partial_t^3 A = 0$$

1.2 Summary (d)NLS equation

Adding a qubic nonlinearity, one gets corresponding versions of the NLS equation:

$$\text{dNLS: } \boxed{\partial_x A - iK_0(i\partial_t)A + i\gamma|A|^2 A = 0}, \tag{2}$$

$$\text{NLS: } \boxed{\partial_\xi A + i\beta_2\partial_\tau^2 A + i\gamma|A|^2 A = 0} \tag{3}$$

$$\partial_\xi A + i\beta_2\partial_\tau^2 A - \beta_3\partial_t^3 A + i\gamma|A|^2 A = 0 \tag{4}$$

1.2.1 Hamiltonian structure and VP (variational principle)

For dNLS Hamiltonian structure

$$\partial_x A = i\delta H(A), \text{with } H(A) = \int\int \left[K_0(i\partial_t)A.\bar{A} - \frac{1}{2}\gamma|A|^4\right] dt \tag{5}$$

from canonical action principle:

$$\int dx \left[\left[\int -i\bar{A}\partial_x A dt\right] - H(A)\right] \tag{6}$$

while for standard NLS from

$$\int d\xi \left[\left[\int -i\bar{A}\partial_\xi A d\tau\right] - H(A)\right]$$

$$\partial_\xi A = i\delta H(A), \text{ with } H(A) = \int \left[\beta_2|A_\tau|^2 - \frac{1}{2}|A|^4\right] d\tau \tag{7}$$

1.2.2 First integrals, symmetries and conservation laws

1. Quadratic functional, **wave energy (wave power)**, and related Gauge invariance:

$$\begin{aligned} N(A) &= \int |A|^2 dt, \\ \text{with flow} &: \quad \partial_x A = i\delta N(A) = iA, \ \text{ i.e. } \ A = c.e^{ix} \end{aligned}$$

   Corresponding conservation law:

$$\partial_x(|A|^2) + \partial_t(...) = 0$$

2. Quadratic functional, **Linear momentum**, and related translation symmetry:

$$\begin{aligned} L(A) &= \int i\bar{A}\partial_t A dt, \\ \text{with flow} &: \quad \partial_x A = i\delta L(A) = -\partial_t A, \ \text{ i.e. } \ A(x,t) = A(x-t,0) \end{aligned}$$

   Corresponding conservation law:

$$\partial_x(i\bar{A}\partial_t A) + \partial_t(...) = 0$$

1.3 Phase-Amplitude equations

1.3.1 Amplitude, phase and LOCAL wave number and frequency

Transformation (Madelung's transformation, polar coordinates for complex amplitude) and definitions:

$$\begin{aligned} A &= ae^{i\phi}, \text{ and so } \eta(x,t) = ae^{i\Phi} + cc + hot = 2a\cos(\Phi) + hot \\ a(x,t) &: \text{ real amplitude (envelope), } \geq 0; \ E := a^2 \\ \phi(x,t) &: \text{ relative phase, and } \Phi = k_0 x - \omega_0 t + \phi(x,t) \text{ total phase} \\ k(x,t) &: \equiv \partial_x \Phi = k_0 + \kappa, \quad \kappa = \partial_x \phi \text{ local wave number} \\ \nu(x,t) &: \equiv -\partial_t \Phi = \omega_0 + \nu, \quad \nu = -\partial_t \phi \text{ local frequency} \end{aligned}$$

Remark 2 *Be aware:*

1. *even when other coordinates are introduced* (ξ, τ)*, the wave number and frequency have to be defined as derivatives of phase with respect to* physical *variables* x, t *for a correct physical interpretation. And even then, these notions are not undisputable.*
2. *now* κ *and* ν *have nothing to do anymore with variables of Fourier transform!!*
3. *'Madelung' transformation has a different interpretation as complexification of real signal (using Hilbert transform).*

1.3.2 Transformation of VP

Use $\partial_t \left[f(t)e^{i\phi}\right] = [(\partial_t + i\phi_t) \circ f] \cdot e^{i\phi}$ for any $f(t)$ and correspondingly for ∂_x.
Then, transformation of action:

$$\int\int -i\bar{A}\partial_x A dt dx = \int\int \phi_x a^2 dt dx = \int\int \kappa E dt dx$$

and of Hamiltonian for dNLS

$$\begin{aligned}
H(a,\phi) &= \int \left[K_0(i\partial_t + \nu)a \cdot a - \frac{1}{2}\gamma a^4 \right] d\tau \\
&= \int \left[K'(\omega_0)\nu \cdot a^2 + \beta\,(i\partial_t + \nu)\,a \cdot a - \frac{1}{2}\gamma a^4 \right] d\tau \\
&= \int \left[K'(\omega_0)\nu \cdot a^2 + \beta_2 \left[(\partial_t a)^2 + \nu^2 a^2 \right] + .. - \frac{1}{2}\gamma a^4 \right] d\tau
\end{aligned}$$

and for NLS with action $\int\int \phi_\xi E d\tau d\xi$

$$\begin{aligned}
H(a,\phi) &= \int \left[\beta_2 \left[(\partial_\tau a)^2 + \nu^2 a^2 \right] - \frac{1}{2}\gamma a^4 \right] d\tau \\
&= \int \left[\beta_2 \left[\frac{(\partial_\tau E)^2}{4E} + \nu^2 E \right] - \frac{1}{2}\gamma E^2 \right] d\tau
\end{aligned}$$

1.3.3 Phase-amplitude eqn's for standard NLS

These eqn's follow immediately from canonical action functional A:

1. *energy equation* from variations with respect to phase $\delta\phi$:

   $$\partial_\xi \left[\frac{\partial A}{\partial \phi_\xi} \right] + \partial_\tau \left[\frac{\partial A}{\partial \phi_\tau} \right] = 0$$

   so

   $$\partial_\xi [E] - \partial_\tau [2\beta_2 \nu E] = 0$$

   which corresponds to 'energy' conservation in physical variables:

   $$\partial_x \left[\frac{\partial A}{\partial \kappa} \right] - \partial_t \left[\frac{\partial A}{\partial \nu} \right] = 0$$

   which can be written like

   $$\boxed{\partial_x E + \partial_t \left[K'(\omega_0 + \nu) \cdot E \right] = 0}, \tag{8}$$

   a variant of $\partial_t E + \partial_x(VE) = 0$, with V the groupvelocity.

2. *phase equation* from variations with respect to a (E):

   $$\phi_\xi - \beta_2 \nu^2 + \gamma E + \beta_2 \frac{a_{\tau\tau}}{a} = 0$$

   which can be written like a nonlinear dispersion relation (NDR)

   $$\text{NDR:} \quad \boxed{K(\omega) - k = \gamma E + \beta_2 \tfrac{a_{tt}}{a}} \tag{9}$$

   remembering the definitions of local wave number and frequency, i.e. the kinematic relation

   $$\boxed{\partial_x \omega + \partial_t k = 0} \tag{10}$$

Remark 3 *The* relevance of the phase-amplitude eqn's *is two-fold: a more direct physical interpretation (a describing the 'envelope' of the wave group, ω, k the properties of the carrier wave below the envelope), and an essential means in the analysis of (special) solutions, als we shall see in the next section. Therefore it would be interesting to study the more general dNLS in detail, in particular the NDR, and to study coherent structures and their stability for this generalised equation (could be that some of this is done for Dysthe-case).*

1.3.4 Integrals

1. From transforming the quadratic wave energy

$$\begin{aligned} N(a) &= \int a^2 d\tau, \\ \partial_\xi(a^2) - \partial_\tau(2\beta_2 \nu a^2) &= 0 \end{aligned}$$

   the energy equation.

2. The linear momenum related to translation symmetry:

$$\begin{aligned} L(a,\phi) &= \int \phi_\tau a^2 d\tau, \\ \partial_\xi(\nu a^2) - \partial_\tau(P) &= 0, P = 2\beta a_\tau^2 + 2\beta a^2 \phi_\tau^2 - \frac{3}{4}\gamma a^4 \end{aligned}$$

3. Observe the 'conserved flux property' for the density a^2, leading to the integral

$$\int \left[\tau a^2 + \xi 2\beta_2 P\right] d\tau$$

   This leads to the fact that the 'center of wave energy' (when finite) translates with constant speed, proportional to linear momentum functional:

$$\tau_C := \frac{\int \tau a^2 d\tau}{\int a^2 d\tau}, \partial_\xi \tau_C = 2\beta_2 \frac{L}{N} \text{ constant}$$

   For the variance

$$V = \int \tau^2 a^2 d\tau$$

   the 'virial theorem' holds:

$$\partial_\xi^2 V = 4\beta \int P = 8\beta H + 2\beta\gamma \int a^4 d\tau$$

   Then

$$\partial_\xi^2 \int (\tau - \tau_C)^2 a^2 d\tau = 8\beta H + 2\beta\gamma \int a^4 d\tau - 8\beta^2 \frac{L^2}{N}$$

1.4 Coherent structures and stability (for standard NLS)

NLS is a nonlinear dispersive equation for complex amplitude. It combines the diverging/converging effects as a consequence of dispersion and of nonlinearity. When signs are correct, these effects may balance eachother and a 'confined' solution may exist, the so-called 'focussing' NLS; in the other case one has the 'diverging' NLS. The converging NLS is most famous for its 1,2, .. N - soliton solutions, which can (accidentially) be written down relatively easy. Related to this, NLS is completely integrable, with infinitely many conserved densities (first integrals).
Converging or diverging depends on the sign of the coefficients

$$\text{NLS: } \boxed{\partial_\xi A + i\beta_2 \partial_\tau^2 A + i\gamma |A|^2 A = 0} \tag{11}$$

$$\begin{aligned} \text{converging if } sign(\beta_2\gamma) &> 0 \\ \text{diverging if } sign(\beta_2\gamma) &< 0 \end{aligned}$$

We will consider the CONVERGING NLS in the following.

1.4.1 Idea of evolution, stability

NDR should be satisfied at each moment, position. Change in ω, k, a is coupled. However, special solutions arise for $\boxed{K(\omega) - k \equiv \mu}$=constant, which are basic solutions, actually 'coherent strutures' (steady states, standing waves), i.e. relative equilibria as constrained critical points of the Hamiltonian:

$$\text{crit } \{H(A) \,|N(A) = \text{constant}, \, L(A) = \text{const}\},$$

i.e. for some multipliers

$$\delta \left[(H(A) + \sigma_N N(A) + \sigma_L L(A) \right] = 0.$$

We first consider such solutions and investigate for some (in-)stability. Basic in the simple analysis is the Mechanical analogy, Newtons Law, of NDR:

$$\beta a_{\tau\tau} - \mu a + \gamma a^3 = 0 \tag{12}$$

as mass particle β, in potential field force

$$\boxed{\beta \partial_\tau^2 a = -\frac{\partial}{\partial a} P(a), \text{ potential } P(a) = -\tfrac{1}{2}\mu a^2 + \tfrac{1}{4}\gamma a^4} \tag{13}$$

Corresponding complex NLS-amplitude:

$$A(\tau, \xi) = a(\tau) e^{-i\mu\xi}$$

Remark 4 *The basic equation (**??**) can be scaled with a characteristic time T and amplitude q :*

$$a'_{\tau\tau} - \mu' a' + p^2 a'^3 = 0, \quad \textit{with } p^2 = \frac{\gamma q^2}{\beta T^2},$$

and where μ' could also be scaled to unity (up to sign) by scaling in ξ. The parameter p will play a special role in the following.

1.4.2 Dispersion plane,Steady soln's, evolution and instability

The dispersion plane (k, ω) is divided in two regions:

Ω_0	:	the linear dispersion curve (graph) $\{(k,\omega)\,\vert\omega = \Omega(k)\}$
Ω^+	:	'above' the linear dispersion curve (epi-graph), $\{(k,\omega)\,\vert\omega > \Omega(k)\}$
Ω^-	:	'below' the linear dispersion curve, $\{(k,\omega)\,\vert\omega < \Omega(k)\}$

Of particular interest are solutions for which k and ω are constant. Then the sign of μ will determine different possibilities. Observe,

$$\text{sign } (\mu) > 0 \text{ in } \Omega^+, \;\; \text{sign } (\mu) < 0 \text{ in } \Omega^-$$

Phase-plane analysis: trivial solution $a \equiv 0$ becomes unstable if μ crosses 0 ('pitch-fork bifurcation').
for positive μ : Lowest point for $a = \sqrt{\frac{\mu}{\gamma}}$, potential negative for $|a| < \sqrt{2}\sqrt{\frac{\mu}{\gamma}}$

1.4.3 Nonlinear harmonic

Standard form: Constant amplitude solution:

$$A = qe^{-i\gamma q^2 \xi}$$

corresponds to
$\mu > \mathbf{0}$: constant solution: $q = \sqrt{\frac{\mu}{\gamma}}$,

1.4.4 Nonlinear modulated harmonic

$\mu > \mathbf{0}$: periodic modulations

1.4.5 Benjamin-Feir modulational instability for $p > \frac{1}{\sqrt{2}}$

The constant mode **is linearly unstable (Benjamin Feir)**. To investigate this, consider

$$A = q(1+g)e^{-i\gamma q^2 \xi}$$

then the perturbation g has to satisfy the linearized eqn:

$$g_\xi + i\beta g_{\tau\tau} + i\gamma q^2 (g + g^*) = 0$$

(note: no real, ξ-dependent solutions). Substituting

$$g = \left[\alpha_+ e^{i\sigma\tau} + \alpha_- e^{-i\sigma\tau}\right] e^{\rho\xi}$$

the dispersion relation is found

$$\rho^2 + (\beta\sigma^2 - \gamma q^2)^2 = \left[\gamma q^2\right]^2$$

Defining the characteristic 'BF-parameter' p (see also Remark (**??**))

$$\boxed{p = \frac{q}{\sigma}\sqrt{\frac{\gamma}{\beta}}}$$

exponentially growing solution with real $\rho = \rho(\sigma, q)$ are found for $p > 1/\sqrt{2}$. The maximal growth factor is

$$\rho = \gamma q^2 \;\text{ for } p = 1, \text{ i.e. } \sigma = \pm\sqrt{\frac{\gamma}{\beta}} q$$

Introducing $\frac{\rho}{\gamma q^2} = \sin(2\psi)$, and writing $\alpha_+ = \frac{1}{2}\varepsilon e^{i\psi_0}$, (any ψ_0) the solution can be written like

$$g = \varepsilon e^{\rho\xi} \cos(\sigma\tau - \psi + \psi_0) e^{i\psi}$$

corresponding to unstable solutions of NLS of the form

$$A = q e^{-i\gamma q^2 \xi}\left[1 + \varepsilon e^{\rho\xi} e^{i\psi} \cos(\sigma\tau - \psi + \psi_0)\right]$$

Observe the phase shift connected to instability (for maximal growth a shift of $\pi/4$.).

Remark 5 ***Benjamin-Feir "instability"*** *of the constant solution with exponentially growing solutions, for $p > 1/\sqrt{2}$, shows LINEAR instability. Remarkably, the full nonlinear solution can be found, which is an $\tau-$periodic, $\xi-$soliton solution for which the linear expression is the limiting behaviour . See [[**?**]]!*

1.4.6 Soliton

$\mu >$ **0** : *Soliton*: $a = q\,\mathrm{sech}(q\sqrt{\frac{\gamma}{2\beta}}\tau)$ for $\mu = \gamma q^2/2$

1.4.7 Nonlinear bi-harmonic

μ< **0** : periodic, symmetric, zero-crossing (topological defects): 'nonlinear bi-harmonic'.

1.4.8 Bi-soliton: secondary modulation for small p

Exact formulae, observations/explanations.

1.4.9 Linear *Bi-harmonic* 'instability' for large p

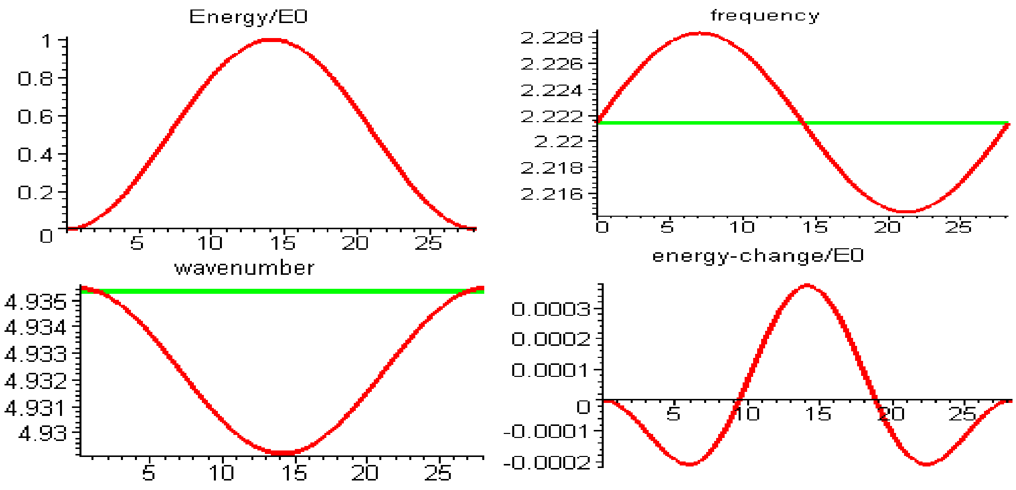
Different from the previous solutions, which are relative equilibria, we now consider the solution of a signalling problem for NLS which cannot be written down in closed form. It shows a characteristic behaviour of deformations that is present in more general solutions also. It is a challenge to see how much of the observed behaviour can be understood and described form the equations.
Consider the signal problem with a linear biharmonic signal:

$$\begin{aligned} \eta(\xi &= 0,t) = q\left[\cos(\bar{\omega}+\Delta\omega)t + \cos(\bar{\omega}\Delta\omega)t\right] \\ &= 2q\cos\Delta\omega t\cos\bar{\omega}t = q\cos\Delta\omega t\ \cdot e^{i\bar{\omega}t} + cc \end{aligned}$$

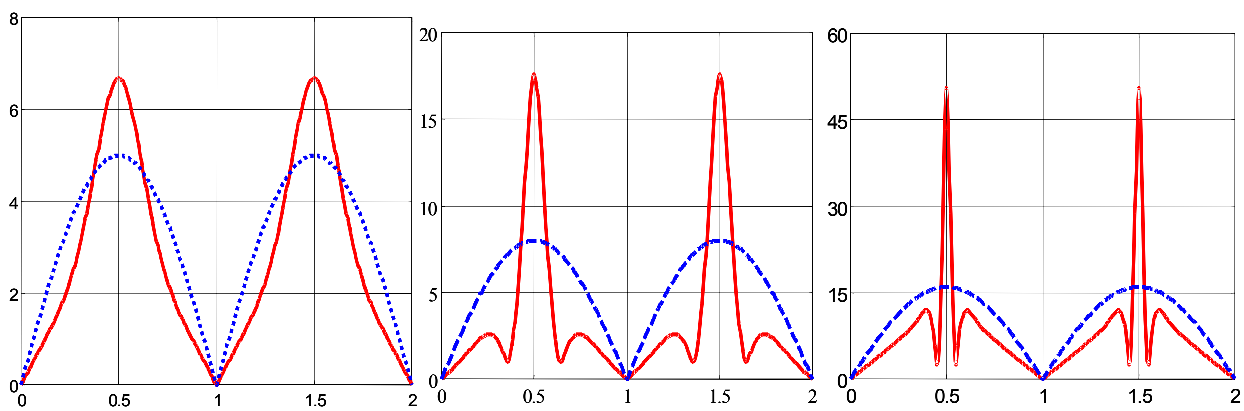
Numerical results[1] show that large deformations (up to twice initial amplitude) appear when the 'BF-coefficient' $p = q/\Delta\omega$ (see also Remark (**??**)) is sufficiently large. Yet, at first sight it is not clear how this parameter appears here: the dependence on the amplitude q is quite expected==nonlinearity, but the appearance of the dependence on the frequency difference =1/modulation time is not so obvious. Several theoretical explanations of the large deformations:

- initial evolution, deformation of wave packet from Phase-amplitude eqn's:
  Initial stages of Deformation

[1] Experiments executed at MARIN about surface waves in large wave tanks (200 m long, 5 m deep) show qualitatively the same behaviour; this also means that the NLS is a rather good description of this natural phenomena.



continued (numerical results for 3 different initial amplitudes):



- trajectory in dispersion plane: in Ω^- near vanishing envelope while in Ω^+ at maximal amplitude when p is sufficiently large;
- appearance of local minima in envelope (at which $a_{\tau\tau} > 0$) can only appear when μ is positive, (from (??)) i.e. at points on a trajectory that lie in Ω^+.
- a low dimensional can be derived and studied;
- soliton energy argument and induced modulation periodicity: Condition for soliton appearance
  - the generated signal has energy and period of the envelope

$$\text{energy } E_{sign} = \frac{q^2\pi}{2\Delta\omega}, \text{envelope period } T_{env.sign} \sim \frac{\pi}{\Delta\omega}$$

  - a soliton of amplitude C has energy and width given by

$$\text{energy } E_{sol} = 2C\sqrt{\beta/\gamma}, \text{width } W_{sol} \sim \frac{\sqrt{\beta/\gamma}}{C}$$

– in order that the signal carries enough energy to generate a soliton witha width inside the modulation periode leads to the condition for soliton appearance:

$$W_{sol} \precapprox T_{env.sign}, \; E_{sol} \precapprox E_{sign} :\Longrightarrow \frac{q}{\Delta\omega} > \text{const.}\sqrt{\frac{\beta}{\gamma}}$$

1.4.10 Topological defects

In Madelung's transformation, the description in 'polar coordinates', the real amplitude is supposed to be non-negative (by definition). However, as seen in the examples above (e.g. 'nonlinear modulated harmonic') there are smooth solutions A that can be written as $A = ae^{i\phi}$, while then a, ϕ are smooth only if a is allowed to take negative values. If one would require a to be non-negative, the phase ϕ gets $\pi-$jumps at the zero's of a.
A zero of a complex-valued function of more variables is called a 'topological defect' (defined as the common zero of two real functions). Generically, for a function of two variables the defects are located on points. Consequently, zero's of $A(\xi, \tau)$ are generically isolated points. Hence, based on this argument, one would expect that the zero's of the initial signal of BH (at times at which $\cos(\Delta\omega\tau) = 0$) will be absent for $\xi > 0$. and the same for the 'corresponding' case of the nonlinear BH. However, the actual (numerical) solution shows that such zero's remain for each ξ : hence, non-generically, zero's appear on lines in the ξ, τ-plane. This should be the indication that the smooth solution has necessarily negative values for the amplitude, since zero's of a non-negative function (like amplitude) are isolated.
To consider this in some more detail, consider the linear biharmonic wave

$$BH = 2q * \cos(\Delta\omega t) * \cos\bar{\omega}t$$

and a variant

$$absBH = bh \cdot \Upsilon = 2q * |\cos(\Delta\omega t)| * \cos\bar{\omega}t$$

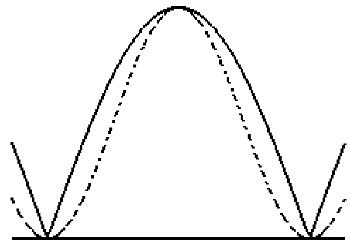
where $\Upsilon = [H(\cos(\Delta\omega t)) - H(-\cos(\Delta\omega t))]$ with H Heaviside's function. In both cases, the real amplitude is the same, $2q * |\cos(\Delta\omega t)|$, periodic with period $\frac{\pi}{\Delta\omega}$, but the phases are different:

$$\phi_{absBH} = \bar{\omega}t, \quad \phi_{BH} = \bar{\omega}t + \pi\Upsilon$$

Looking at the plots of the signals, at first sight the difference is hardly noticeable:

The difference near the 'topological defects' where the amplitude vanishes is small, but shows itself in successive in-phase and out-of phase periods. As a consequence, the spectra are quite different, the spectrum of $absBH$ obtained from convolution of spectrum of BH with that of Υ changes the sum of the two delta-functions at $\bar{\omega} \pm \Delta\omega$ for BH to contributions for $absBH$ at the central frequency $\bar{\omega}$ with 'side bands' at distances of multiples of $2\Delta\omega$. Taking into account only the main contributions one would have

$$2q|\cos(\Delta\omega t)| \approx q\,[1+\cos(2\Delta\omega t)]$$



which is a signal that leads to BF-instability for $\frac{\frac{1}{2}q}{2\Delta\omega} > \frac{1}{\sqrt{2}}$, i.e. $\frac{q}{\Delta\omega} > 2\sqrt{2}$. When calculated for the standard NLS equation, the evolution of BH shows that the topological defects are persistent (the phase jumps persist), while for $absBH$ the topological defects disappear and the amplitude becomes strictly positive; see Fig.??. Neglecting the nonlinear term in NDR near topological defects (vanishing amplitude), the Whitham Fornberg term relates the singularity in phase (frequency and wave number) with possible jumps in the derivative of the amplitude according to $a \cdot [K(\omega) - k] = a_{\tau\tau}$. This shows that in the intervals of vanishing amplitude, bounded variations of the phase lead to vanishing values of the second derivative, i.e. to a tendency to eliminate the defects and a strictly positive envelope.

1.5 Assignments

You can choose an assignment; if something is not clear, please don't waste time but come and see me.

Assignment 6 *Verify and specify the formula of NLS-equation, both for complex amplitude A as well as for the real amplitude phase equations. Derive for the more general dNLS (restrict to qubic approx. of dispersion relation) the govering equations from the canonical action principle. Observe that the energy and phase equation get additional terms. What is the effect of these terms for the special solutions (relative equilibria). For instance, (how) does the BF-instability change? how the argument of initial deformation of Bi-Harmonic?, etc.*

Assignment 7 *(1+2)-NLS*

1. *Consider a problem in time and 2 space dimensions; for instance 2-dimensional surface waves, or optical pulses in a plane, or 3-dimensional optical modes. Derive the governing NLS-equation for the signalling problem (prescribing the solution on a suitable plane in 3D space-time). First do it for linear wave groups, for a given linear dispersion relation, then by adding a qubic nonlinearity.*
2. *Consider for the 2+1 NLS a (nonlinear) plane wave solution. This will have Benjamin-Feir modulational instability for longitudinal perturbations. Investigate the instability for* transverse *perturbations (sec. 1.3.1. Sulem & Sulem [**?**]).*

Assignment 8 *Show that the soliton solution is **orbitally stable** for (1+1)-NLS. (This is not so for (1+2)-NLS which shows blowup in finite 'distance'). See Sulem & Sulem [**?**]. The proof uses beautiful arguments from nonlinear functional analysis to investigate a Lyapunov functional of the form $H(A) + \mu N(A)$ or from convexity arguments of a 'value function'. As a first step, linear stability should be investigated.*

References

[1] C. Sulem & P-L Sulem, *The Nonlinear Schrödinger Equation*, Springer Verlag, 1999.

[2] N. Akhmediev & A. Ankiewicz, Solitons, *Nonlinear pulses and beams*, Chapman & Hall, 1997.

Lecture 2: 26 October, revised and extended 28 October

2 Lecture 2: Variational characterization of Photonic Band Gaps (Improved and extended version 28/10/00)

2.1 State variables and Maxwell's equations

E *electric field*	**H** *magnetic field*
D *electric displacement*	**B** *magnetic induction*
P *polarization*	**M** *magnetic moment*

Maxwell's equations for zero current:

$$\frac{1}{c}\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \qquad \nabla \cdot \mathbf{B} = 0,$$
$$\frac{1}{c}\partial_t \mathbf{D} = \nabla \times \mathbf{H}, \qquad \nabla \cdot \mathbf{D} = 0$$

complemented with *constitutive equations*:

$$\begin{aligned} \mathbf{D} &= \mathbf{E} + \mathbf{P} \\ \mathbf{B} &= \mathbf{H} + \mathbf{M} \end{aligned}$$

Restriction to non-magnetic material: $\mathbf{M} = 0$ gives $\mathbf{B} = \mathbf{H}$ and upon eliminating $\mathbf{H}$ there results a single vector wave eqn:

$$\frac{1}{c^2}\partial_t^2 \mathbf{D} = \nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E})$$

Materials exhibit responses to an applied optical field, in general nonlinear, and one writes

$$\mathbf{P} = (nonlin).\ response\ operator\ \mathbf{E}$$

Several notations are in use:

$$\begin{aligned} \mathbf{D} &= \mathbf{E} + \mathbf{P} = \mathbf{E} + \mathbf{P}^{(L)} + \mathbf{P}^{(NL)}, \\ &= \hat{\varepsilon}\mathbf{E} + \chi^{(NL)}\mathbf{E} \end{aligned}$$

with $\hat{\varepsilon} = 1 + \hat{\chi}^{(1)}$ where:

$\hat{\varepsilon}$	$permittivity = dielectric\ constant = refractive\ index^2$
$\hat{\chi}$	*susceptibility*

In general, the permittivity (and susceptibility) are operations working on (the products of) the components, possibly as a convolution operator to account for time delay; this leads to *dispersion.*

First order dispersion for local media
This takes into account the time delay for the first order polarization to respond to changes in the electric field. This is necessary when pulses are short in comparison with the response time of the medium.

$$\begin{aligned} \mathbf{D}^{(L)}(\mathbf{x}, t) = &\quad \hat{\varepsilon} * \mathbf{E} \\ \equiv &\quad \int \varepsilon(t-\tau)\mathbf{E}(\mathbf{x}, \tau)\, d\tau \end{aligned}$$

The kernel function defines integral operator; after Fourier transform with respect to time t the response would be a multiplication with function $\bar{\varepsilon}$:

$$\varepsilon(t) = \int \bar{\varepsilon}(\omega) e^{-i\omega\, t}\, d\omega \quad \text{with} \quad \bar{\varepsilon}(\omega) = 1 + 4\pi \bar{\chi}^{(1)}(\omega).$$

Second and third order nonlinearity

The nonlinear dependence of the polarization on the electric field. When taken instantaneously and for homogeneous materials in components:

$$P_i^{(2)} = \chi_{ijk}^{(2)}\, E_j\, E_k$$

$$P_i^{(3)} = \chi_{ijk\ell}^{(3)}\, E_j\, E_k\, E_\ell$$

Higher order effects may be taken into account in the same way.

By letting χ depend on position (for inhomogeneous media) and/or delay properties (for short pulses), successively more (difficult effects) are incorporated.

2D materials

Consider 2D linear inhomogeneous material (uniform in $y-$ direction) without time delay (dispersion), and denote the index by n:

$$\mathbf{D} = n^2(x,z)\mathbf{E}$$

Then the TE-mode, the solution with $\mathbf{E} = (0, E_y, 0)$, satisfies $\nabla \cdot \mathbf{E} = 0$, and hence the equation

$$\frac{1}{c^2}\partial_t^2 D_y = \Delta E_y$$

Looking for a mode with frequency ω :

$$E_y = u(x,z)e^{i\omega t}$$

the equation becomes:

$$\left[\Delta + \frac{\omega^2 n^2(x,z)}{c^2}\right] u = 0$$

For nonlinear, third order material, $D_y = n^2 E_y + \chi |E_y|^2 E_y$ the equation modifies to

$$\left[\Delta + \frac{\omega^2}{c^2}\{n^2(x,z) + \chi u^2\}\right] u = 0$$

Both problems can be formally seen (without specifying boundary conditions) as an eigenvalue problem for the Laplace operator

$$-\Delta u = \frac{\omega^2}{c^2}\left[n^2(x,z) + \chi u^2\right] u$$

which is the equation for constrained critical points of the variational problem

$$\mathrm{Crit}_u \ \left\{ \int (\nabla u)^2 \middle| \int (n^2 u^2 + \frac{\chi}{3} u^4) \right\} = \text{constant}$$

with the multiplier interpreted as the squared frequency: $\lambda = \frac{\omega^2}{c^2}$.

Photonic crystal: A material with periodic index distribution. In such a material PBG's, *Photonic Band Gaps*, may exist: intervals of frequencies for which no propagation of light is possible. Such band gap devices may be used as narrow-band filters, rejecting all the frequencies inside the gap.

We will now study in detail 1D gratings as an example of a simple 'photonic 1D crystal' to demonstrate the appearance of PBG's. First for a special case that is explicitly solvable, then more general with variational characterization of the PBG.

Standard literature is the book of Joannopoulos et al., [**?**]; for the variational methods we will use, see the Lecture Notes [**?**].

References

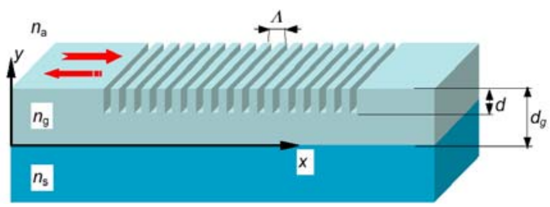
[1] E. van Groesen, *Applied Analytical Methods*, Lecture Notes UTwente.

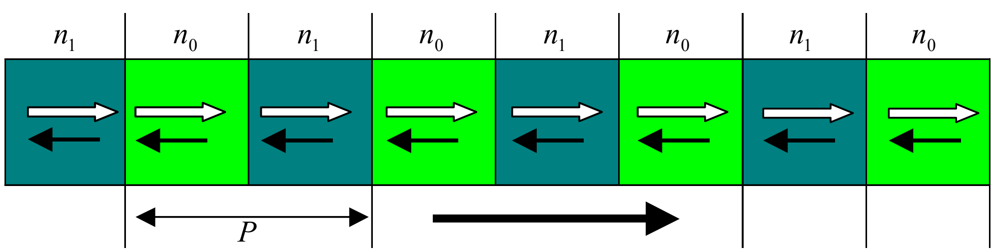
[2] J.D. Joannopoulos, R.D. Meade & J.N. Winn, *Photonic Crystals, Molding the flow of light*; Princeton University Press, 1995.

2.2 1D Linear grating: Transfer matrix approach

The standard problem is a material with piecewise constant indices, periodically repeated. For instance,

$$\begin{aligned} \left[\partial_z^2 + \frac{\omega^2 n^2(z)}{c^2}\right] u &= 0 \qquad (14)\\ n(z) &= \begin{cases} n_0 & \text{for} \quad 0 < z < p/2 \\ n_1 & \text{for} \quad p/2 < z < p \end{cases} \end{aligned}$$





This is a nice example since the solution can be found explicitly.

2.2.1 Period-p Poincaré map

The solution can be written using either real or complex notation. Write, $k_0 = \frac{\omega n_0}{c}, k_1 = \frac{\omega n_1}{c}$ and then in successive regions:

$$\begin{aligned} u(z) &= \alpha_0 e^{ik_0 z} + \beta_0 e^{-ik_0 z} = a_0 \cos(k_0 z) + b_0 \sin(k_0 z), \text{ in layer of index } n_0 \\ &= \alpha_1 e^{ik_1 z} + \beta_1 e^{-ik_1 z} \text{ in successive layer of index } n_1 \\ &= \alpha_2 e^{ik_0 z} + \beta_2 e^{-ik_0 z} \text{ in sucessive layer of index } n_0 \end{aligned}$$

Then, conditions of continuity of field & flux through boundaries leads to transfer matrix. (This is a simple example of Coupled Mode Theory.)

Introduce

$$\begin{aligned}\Omega &= \omega p\sqrt{n_0 n_1}/c \\ s &= \frac{k_0+k_1}{2\sqrt{k_0k_1}} = \frac{n_0+n_1}{2\sqrt{n_0n_1}}, v = \frac{n_1-n_0}{2\sqrt{n_0n_1}};\ \text{then } s^2-v^2=1\end{aligned}$$

Observe that s, v are independent of ω, just material properties. The transformation over a full period leads to

$$T = \left[\begin{array}{cc} a & b \\ \bar{b} & \bar{a}\end{array}\right] = \left[\begin{array}{cc} a(\Omega) & b(\Omega) \\ \bar{b} = b(-\Omega) & \bar{a} = a(-\Omega)\end{array}\right]$$

where

$$\begin{aligned} a &= s^2 e^{i(k_1+k_0)p/2} - v^2 e^{-i(k_0-k_1)p/2} \\ &= \left[s^2\cos s\Omega - v^2\cos v\Omega\right] + i\left[s^2\sin s\Omega + v^2 \sin v\Omega\right] \\ b &= sv\left[e^{i(k_1-k_0)p/2} + e^{-i(k_0+k_1)p/2}\right] \\ &= sv\left[\cos v\Omega + \cos s\Omega\right] + isv\left[\sin v\Omega - \sin s\Omega\right]\end{aligned}$$

Observing $\bar{a}(\Omega) = a(-\Omega), \bar{b}(\Omega) = b(-\Omega)$, the structure of T can be understood:

$$\begin{aligned}\alpha_2 &= a\alpha_0 + b\beta_0,\ \text{right travelling wave} \\ \beta_2 &= \bar{b}\alpha_0 + \bar{a}\beta_0,\ \text{left travelling wave}\end{aligned}$$

For real notation: Poincaré map over one period Φ

$$\Phi = RTR^{-1}\ \text{ with }\ R = \left(\begin{array}{cc} 1 & 1 \\ i & -i\end{array}\right), \Phi = \left(\begin{array}{cc} \text{re}(a+b) & \text{im}(a-b) \\ \text{-im}(a+b) & \text{re}(a-b)\end{array}\right)$$

2.2.2 Main property: Volume preserving map

The determinant of the map is the product of its eigenvalues; denoting the eigenvalues by $\lambda_{1,2}$, the determinant is the product of the eigenvalues. In this case the value of the determinant is equal to one:

$$Det = |a|^2 - |b|^2 = \left[s^2 - v^2\right]^2 = 1 = \lambda_1 \cdot \lambda_2$$

This could be foreseen since the basic equation (**??**) is a Hamiltonian system, and hence its flow is area conserving.
Explicitly, the eigenvalues are given by

$$\begin{aligned}\lambda_{1,2} &= r \pm \sqrt{r^2-1} = \cos(\phi) \pm i\sin(\phi),\ \text{with }\ r = \text{Re}(a) = \cos(\phi) \\ r &= s^2\cos s\Omega - v^2 \cos v\Omega\end{aligned}$$

Two possibilities:

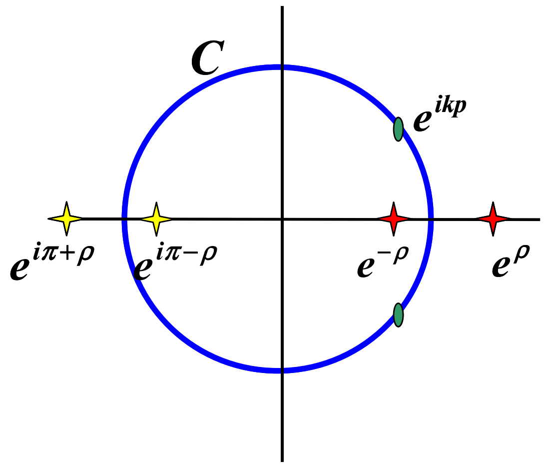
- either both eigenvalues are not real, and then complex conjugate on the unit circle in the complex plane (periodic). We then define the real 'wave number' k by writing

  $$\lambda_1 = e^{ikp}$$

  This is motivated by writing the eigenvalues like $\lambda_{1,2} = e^{\pm i\phi}$ and interpret ϕ (the so-called Floquet-exponent) as the change in phase over one grating period, so that

  $$\phi = k * p,\ \text{with } k \text{ the real 'wave number'}$$

- or both eigenvalues are real (exponential solutions),
  - either both positive, and then they an be written like e^{ρ} and $e^{-\rho}$ for some real ρ,
  - or both negative, and then can be written like $e^{i(\pi-i\rho)}$ and $e^{i(\pi+i\rho)}$ for some real ρ



2.2.3 Photonic Band Gap

We observed that there are no periodic solutions for those values for which eigenvalues are real (not on unit circle). The values for which this is the case define the *Photonic Band Gap*. This is for

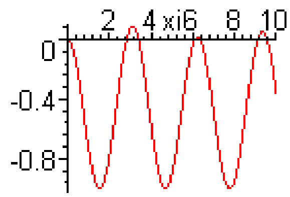
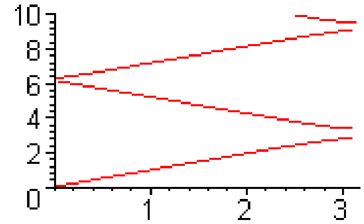
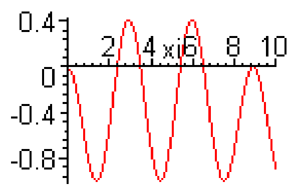
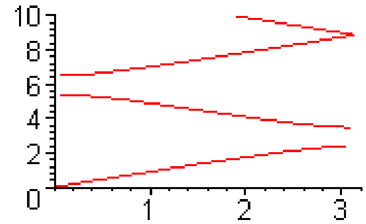
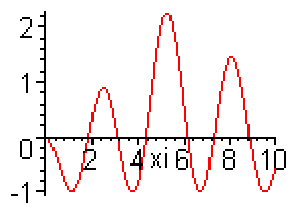
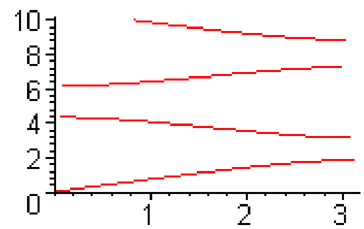
$$PBG: \ r^2 > 1$$

Below we present some pictures, for different cases (different index-contrasts); plotted is vertically $r^2 - 1$, horizontally Ω. Outside the PBG the phase ϕ is real, and

$$\cos(\phi) = \mathrm{Re}(\lambda) = \mathrm{Re}(a) = r, \text{ so } \phi = \arccos(r)$$

Remembering that Ω is a scaled version of the frequency and ϕ of the wave number, the 'dispersion plot' of 'ω vs k' corresponds to Ω vs ϕ, shown also in the pictures below.

Dispersion and PBG plots for given index difference Periodicity makes it possible to restrict the horizontal axis to the interval $[-\pi, \pi]$ (and evenness to $[0, \pi]$).

indices:	PBG:$r^2 - 1 > 0$	dispersion plot
$n_0 = 1.5, n_1 = 2$		
$n_0 = 1, n_1 = 2$		
$n_0 = 1.5, n_1 = 5$		

Allowing complex-valued 'wave numbers', we can interpret the finding above as characterizing the dispersion relation

$$k = K(\omega)$$

with different branches as follows:

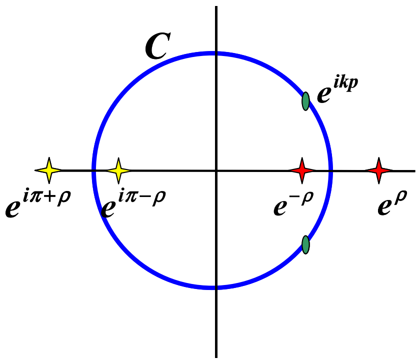
- 'branches' where $k = K(\omega)$ is real for ω outside the PBG; the solutions can be interpreted as travelling waves, $e^{i(kz-\omega t)}$ with phase velocity ω/k;
- the Photonic Band Gaps, for which the complex wave number can be defined as

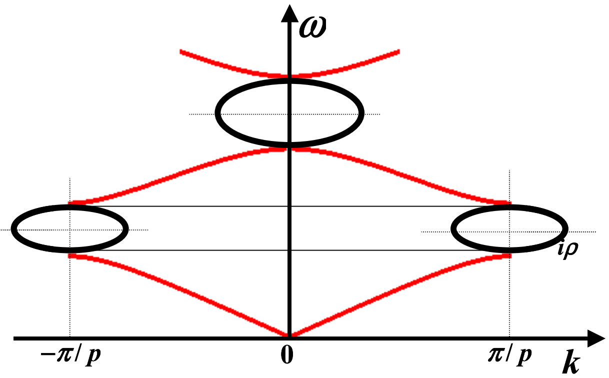
$$k = K(\omega) = k_g(\omega) \pm i\rho(\omega) \text{ for } \omega \in PBG$$

  of two different types:

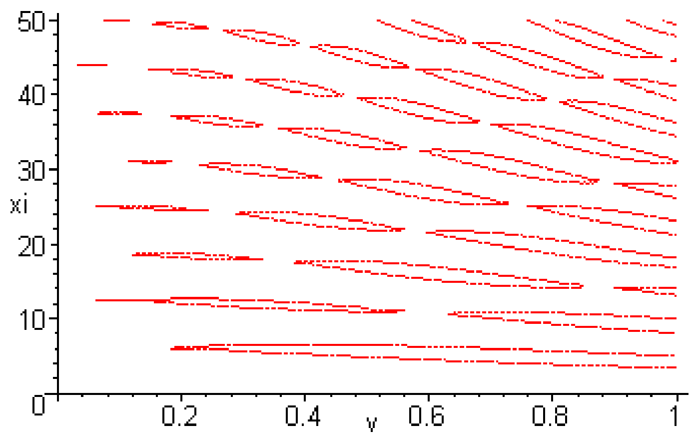
  - the branches with $k_g = 0$ for which the solutions are harmonic in time and exponentially decreasing/increasing: $e^{\pm\rho z - i\omega t}$ (the long wave length band-gap, exponential modulation of standing waves with period p)
  - the branch with $k_g = \pi/p$ for the which the solution looks like $e^{i(z\pi/p-\omega t)}e^{\pm\rho z}$, (the short wave length band-gap, exponential modulation of standing waves with period $2p$).

In one picture, introducing locally imaginary axis in the PBG's, the situation can be sketched as follows:





Plots of Band gap as function of index-difference We can also take the index difference v as variable (horizontally) and Ω vertically, and then plot level sets where $r = 1$ that enclose the areas for PBG. The result is as follows:



2.3 Floquet-approach

We will now generalize the previous results to periodic index distributions that are not necessarily piece-wise constant and for which explicit solutions cannot be written down.
For non-autonomous ODE's with periodic coefficients,the solutions can be represented in a special way, known as Floquet's Theorem (or Bloch's theorem). We formulate the result for the special form of the equation we are interested in.

Theorem 9 *(**Floquet**) Consider the first order system of ODE's*

$$\partial_z u + A(z)u = 0$$

where A is a p-periodic matrix: $A(z + p) = A(z)$. Then each solution is of the following form:

$$u(z) = v(z)e^{\rho z}, \quad \text{with } \rho \in C, \text{ and } v(z) \text{ is } P\text{ -periodic}$$

Note that $u(z+P) = u(z)e^{\rho p}$; the value ρ is called the Floquet exponent, and $e^{\rho p}$ the Floquet multiplier.

Proof. Denote the fundamental matrix solution by $\Phi(z)$. Because of periodicity of $A(z)$, if $u(z)$ is a solution, then so is $u(z+P)$. Hence, there exists a non-singular matrix T such that $\Phi(z+p) = T\Phi(z)$. Since T is non-singular, a matrix B exists such that $T = \exp(Bp)$. Now define $\Psi(z) = \Phi(z)\exp(-Bz)$; then Ψ is p-periodic and Φ is given by

$$\Phi(z) = \Psi(z)\exp(Bz)$$

from which the theorem follows[2]. ■

Theorem 10 *Consider the second order Hamiltonian system*

$$\partial_z^2 u + q(z)u = 0,$$

where $q(z)$ is a function periodic with period p. Then each solution is of the following form:

$$u(z) = v(z)e^{\pm iKz}, \text{with } v(z) \text{ is } p \text{ -periodic}$$

and with three possibilities (depending on the function q) for the Floquet exponent ρ

- *either $K = k$ is real and bounded solutions result;*
- *or $K = \pm i\rho$ with ρ real, giving rise to exponential multiplying a p-periodic function,*
- *or $K = \pm\pi/p \pm i\rho$ with ρ real, giving rise to exponential solutions multiplying the $2p$-periodic function $v(z)e^{i(z\pi/p)}$.*

Proof. (sketch, without using the result of the general Floquet theorem, but using the basic idea) Motivated by the specific case of the previous section:
Let $u_0(z)$ and $u_1(z)$ be two independent solutions; the linearity makes it possible to express any solution as a linear combination of these two. From the periodicity of q, also $u_0(z+p)$ and $u_1(z+p)$ are solutions, which can therefore be expressed in terms of u_0, u_1, thus defining a transfer matrix T :

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(z+p) = T\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(z)$$

The determinant of T is one, as follows from the volume-conservation: $\partial_z\left(u_0\partial_z u_1 - u_1\partial_z u_1\right)(z) = 0$, and hence in particular

$$\left(u_0\partial_z u_1 - u_1\partial_z u_1\right)(z+p) = (u_0\partial_z u_1 - u_1\partial_z u_1)(z).$$

Then the arguments are as in the previous section. ■

2.3.1 Characterization of PBG-standing waves

Specified for the equation of interest,

$$\left[\partial_z^2 + \omega^2 n^2(z)\right]u = 0$$

we come to the following conclusions:

- Including the time dependence for the optical field

  $$E_y = v(z)e^{iKz}e^{i\omega t},$$

  solutions for which $K(\omega)$ is real (for real ω) are wave-like solutions (travelling or standing waves);

[2] That is to say, provided the matrix B is (complex) diagonalizable; if B is nil-potent, quasi-exponentials of the form $z^k e^{\rho z}$ arise, with $k = 0, 1, ...$

- when for given ω there does not exist a solution with a real value of k, the solution will exponentially increase/decrease in the propagation -direction, and light cannot propagate: the characteristic property of a PBG;
- Of special interest are the values that define the boundary of the PBG. Distinguish the 'long wave length' PBG for which $k = 0$ (Floquet multiplier = 1), and the short wave length PBG for which $k = \pi/p$ (Floquet multiplier = −1).

Our aim is now to characterize the PBG's, by characterizing the PBG-boundary-solutions.
First some simple, yet important, observations.

Lemma 11 *1. Let $k = 2\pi/p + \kappa$, then*

$$u(z) = v(z)e^{ikz} = v(z)e^{i2\pi z/p}e^{i\kappa z}$$

Since v is p-periodic $\Longleftrightarrow$ $ve^{i2\pi z/P}$ is p-periodic, it is sufficient to restrict the interval of real k values to $(-\pi/p, \pi/p]$, and actually to $k \in [0, \pi/p]$.

2. Observe that u is p-periodic, if and only if

$$u = ve^{ikz} \text{ with } k = 0, v \text{ is p-periodic}$$

3. Define u to be "p-skew-periodic" if

$$u(z + p) = -u(z);$$

then certainly u is $2p$-periodic. Furthermore,

$$u \text{ is } p - \text{skew-periodic } \Leftrightarrow u = ve^{ikz} \text{ for } kp = \pi \text{ and } v \text{ is (real) p-periodic}$$

4. Both the real p-periodic and the real p-skew-periodic solutions give rise to standing waves for the optical field.

Summarizing we get:

- the PBG at $k = 0$ contains the p-periodic solutions as boundary solutions;
- the PBG at $k = \pi/p$ contains the p-skew-periodic solutions as boundary solutions;
- all these solutions correspond to *standing-waves*: $E = u(z)e^{i\omega t}$

2.3.2 Effective index method: Envelope equation

In many optical problems one wants to define a kind of 'effective' index $\bar{n}$ that would simplify the actual system under consideration, with real index n, in such a way that the most important properties of the real system are also described by the effective index system. From the fact that the grating allows exponential solutions for ω in the PBG it should be clear that it is not possible to define a real effective index $\bar{n}$ such that

$$\partial_z^2 u + \omega^2 n^2 u = 0$$

has for all ω the same behaviour. It is clear that an effective index should be complex valued. However, another complication is that in the band gap at $k = \pm\pi/p$ there are actually 4 solutions. Nevertheless, an 'effective' model can be defined by looking only at a model with the correct dispersion relation found above without paying attention to the p-periodic functions $v(z)$. To 'neglect' the p-periodic carrier wave $v(z)$, define the 'envelope' $W(z) = e^{iK(\omega)z}$, exploiting the notation $K(\omega) = k(\omega) + i\rho(\omega)$ as introduced before,

$$u(z) = v(z)W(z).$$

The original equation can be written like

$$(\partial_z + i\omega n)(\partial_z - i\omega n)u = 0,$$

or, making it a fourth order equation

$$\left[(\partial_z + i\omega n)(\partial_z - i\omega n)\right]^2 u = 0.$$

An effective index method, motivated by the dispersion branches, would consider the envelope equation of fourth order;

$$[\partial_z + iK(\omega] \left[\partial_z + i\overline{K(\omega)}\right] [\partial_z - iK(\omega] \left[\partial_z - i\overline{K(\omega)}\right] W = 0;$$

upon restricting to non-negative values of the real part of K the second order 'envelope equation' can be written like

$$[\partial_z - iK(\omega] \left[\partial_z - i\overline{K(\omega)}\right] W = 0;$$

- outside the PBG this reduces to the first order ODE

  $$[\partial_z - ik(\omega)]\, W = 0, \text{ for } \omega \notin PBG$$

  with the bounded harmonic solution $e^{ik(\omega)z}$;

- Inside a PBG the equation becomes

  $$[\partial_z - ik_g - \rho(\omega)]\,[\partial_z - ik_g + \rho(\omega)]\, W = 0, \text{ for } \omega \in PBG$$

  - For the long-wave length PBG, $k_g = 0$, there results

    $$\left[\partial_z^2 - \rho^2\right] W = 0$$

    with the exponential solutions.

  - For the short wave length PBG there results

    $$\left[\partial_z^2 - 2ik_g\partial_z - (k_g^2 + \rho^2)\right] W = 0$$

    By introducing

    $$W = e^{ik_g z} W_2,$$

    the equation for W_2 becomes

    $$\left[\partial_z^2 - \rho^2\right] W_2 = 0,$$

    where now W_2 is the envelope multiplying the $2p$-periodic carrier wave $v(z)e^{ik_g z}$.

2.4 Nonlinear effects, Band gap solitons

Now consider propagation in a material with third order nonlinearity

$$\partial_z^2 u + \omega^2 \left[n^2 + \chi|u|^2\right] u = 0.$$

In a material with constant index n, periodic nonlinear modes exist as generalizations of the linear modes $e^{i\omega n z}$.

For constant index, 'soliton'-type of solutions (decaying at infinity) do not exist since for near vanishing values the solutions are periodic. For a grating in this material the situation may be different, since in the absence of nonlinearity the trivial solution $u = 0$ is unstable for $\omega \in PBG$; in fact, inside the PBG the non-smooth function

$$S_{dis} = v(\omega, z) e^{ik_g z} e^{-\rho|z|}$$

satisfies the correct equation except at $z = 0$. Inclusion of the nonlinearity may allow the concave corner of S_{dis} to be removed and smoothly connected to form a soliton-type of solution.

To investigate this, consider once again a carrier-envelope Ansatz:

$$u = v(z)W(z).$$

A simple argument (that is not completely correct) would be to take the nonlinearity directly into the equation for the envelope W or W_2:

$$\left[\partial_z^2 - \rho^2\right] W + \omega^2 \chi W^3 = 0 \tag{15}$$

which is indeed the equation for soliton solutions when χ is positive.

However, the equation (**??**) does not follow rigorously as envelope equation. To that end find the full equation to be satisfied as

$$W\partial_z^2 v + 2\partial_z W \partial_z v + v\partial_z^2 W + \omega^2 (n^2 v W + \chi v^3 W^3) = 0$$

(For $\chi = 0$, W_{dis} satisfies this equation outside $z = 0$ for $v = v(\omega, z), W = e^{ik_g z} e^{-\rho|z|}$; for $\chi > 0$ the nonlinearity could possibly compensate a suitable continuisation of the discontinuity).

Averaging technique would start with the assumption that W varies on a much longer length scale than the function v that varies on the scale of the grating period. Therefore multiply the equation by v and integrate over a period, assuming W and its derivatives to be constant. The result is

$$\overline{v^2}\partial_z^2 W + \left[\omega^2 \overline{n^2 v^2} - \overline{|\partial_z v|^2}\right] W + \omega^2 \chi \overline{v^4} W^3 = 0$$

Now, for $v = v(\omega, z)$ it holds (verify) that

$$\omega^2 \overline{n^2 v^2} - \overline{|\partial_z v|^2} = -\rho^2 \overline{v^2}$$

and hence there results

$$\partial_z^2 W - \rho^2 W + \omega^2 \chi \frac{\overline{v^4}}{\overline{v^2}} W^3 = 0 \tag{16}$$

This averaged envelope equation admits, for $\chi > 0$, standard NLS-type soliton solutions. Such solutions are called *Band Gap Solitons.*

Remark 12 *Observe that the analysis above is not yet a complete proof of the existence of such solitons; technical considerations have to be added for a full proof that the conclusion for this averaged equation carries over for the complete equation. Also, it cannot be excluded that there are also other solutions, such as multi-hump soliton solutions.*

2.5 Variational description

Remark 13 ***WARNING.*** *The material in this section seems not to be in the literature and may be new; but be aware that the statements are not yet extensively checked!! It seems possible to generalise the given results to 2D (and 3D??) photonic crystals, and to nonlinear materials.*

Original equation

$$\left[\partial_z^2 + \omega^2 n^2(z)\right] u = 0$$

results from the variational principle

$$\delta \int \left[(\partial_z u)^2 - \omega^2 n^2(z)\, u^2\right] dz = 0$$

or from constrained problem

$$\text{“crit”} \quad \delta \left\{ \int (\partial_z u)^2 dz \middle| \int \left[n^2(z)\, u^2\right] dz = 1 \right\}$$

withcritical value the multiplier ω^2

or equivalently (different normalization)

$$\text{“crit”} \quad \delta \left\{ \int \left[n^2(z)\, u^2\right] dz \middle| \int \left[(\partial_z u)^2\right] dz = 1 \right\}$$

with critical value the multiplier $\frac{1}{\omega^2}$

The interval of integration is not specified here, and is the main problem to characterize arbitrary solutions.
However, for the boundary-PBG-solutions, we have more information. We will use this information to construct the variational characterization.

2.5.1 PBG at $k = \pi/p$

All boundary solutions are p-skew-periodic; hence each solution is an odd continuation of a solution in one period with Dirichlet boundary conditions. However, in general the precise position of these points of vanishing field is unknown. For the ‘first’ (lowest) PBG there are two extremal solutions that determine the gap:

$$PBG^1(k = \pi/p) = [\omega_-^1, \omega_+^1]$$

The lowest value ω_-^1 and the corresponding PBG-boundary solution are found from the value (=multiplier) and as the solution of the constrained minimization problem:

$$\left(\omega_-^1\right)^2 = \mathrm{Min}_{\xi,u} \left\{ \int_\xi^{\xi+p} \left[(\partial_z u)^2\right] dz \middle| \int_\xi^{\xi+p} \left[n^2(z)\, u^2\right] dz = 1, u(\xi) = u(\xi+p) = 0 \right\}$$

or equivalently:

$$\left(\frac{1}{\omega_-^1}\right)^2 = \mathrm{Max}_{\xi,u} \left\{ \int_\xi^{\xi+p} \left[n^2(z)\, u^2\right] dz \middle| \int_\xi^{\xi+p} \left[(\partial_z u)^2\right] dz = 1, u(\xi) = u(\xi+p) = 0 \right\}$$

Observe that this boundary-solution will have its maximal value at the high-index region.

The upper value ω_+^1 and the corresponding boundary solution are found by looking for solutions of the following Mini-Max problem:

$$\left(\omega_+^1\right)^2 = \text{Max}_\eta \ \text{Min}_u \ \left\{ \int_\eta^{\eta+p} \left[(\partial_z u)^2\right] dz \middle| \int_\eta^{\eta+p} \left[n^2\left(z\right)u^2\right] dz = 1, u(\eta) = u(\eta+p) = 0 \right\}$$

or equivalently

$$\left(\frac{1}{\omega_+^1}\right)^2 = \text{Min}_\eta \ \ \text{Max}_u \ \left\{ \int_\eta^{\eta+p} \left[n^2\left(z\right)u^2\right] dz \middle| \int_\eta^{\eta+p} \left[(\partial_z u)^2\right] dz = 1, u(\eta) = u(\eta+p) = 0 \right\}$$

Higher PBG's at $k = \pi/p$ can be found using successive characterizations by restricting the competing functions to satisfy reciprocity conditions with the eigenfunctions of the previous gaps. It may be possible that a few of such 'gaps' are actually 'closed' (see plot in first subsection for $n_0 = 1, n_1 = 2$).

2.5.2 PBG at $k = 0$.

All PBG-boundary-solutions are p-periodic. To be specific, take the origin $z = 0$ in a region of high index. To avoid constant functions u as 'solution' in some of the following variational characterizations, observe that by integrating the equation over one period there results

$$\int_{period} n^2(z)u(z)dz = 0.$$

For the 'first' (lowest) PBG (with $\omega > 0$) there are two extremal solutions that determine the gap:

$$PBG^1(k=0) = [\omega_-^1, \omega_+^1]$$

The lowest value ω_-^1 and the corresponding PBG-boundary solution are found from the value (=multiplier) and as (a continuation of) the solution of the constrained minimization problem (with the additional natural constraint):

$$\left(\omega_-^1\right)^2 = \text{Min}_{\xi,u} \ \left\{ \int_{\xi-p/2}^{\xi+p/2} \left[(\partial_z u)^2\right] dz \middle| \int_{\xi-p/2}^{\xi+p/2} \left[n^2\left(z\right)u^2\right] dz = 1, \int_{\xi-p/2}^{\xi+p/2} n^2(z)u(z)dz = 0, u(\xi-p/2) = u\left(\xi+p/2\right) \right\}$$

or equivalently:

$$\left(\frac{1}{\omega_-^1}\right)^2 = \text{Max}_{\xi,u} \ \left\{ \int_{\xi-p/2}^{\xi+p/2} \left[n^2\left(z\right)u^2\right] dz \middle| \int_{\xi-p/2}^{\xi+p/2} \left[(\partial_z u)^2\right] dz = 1, \int_{\xi-p/2}^{\xi+p/2} n^2(z)u(z)dz = 0, u(\xi-p/2) = u\left(\xi+p/2\right) \right\}.$$

The natural boundary conditions obtained for the extremal solutions, $u_z(\xi - p/2) = u_z\left(\xi + p/2\right)$, make it possible in this case to continue the solution in a p-periodic way. Observe also that this boundary-solution will have its maximal value at the high-index region.
The upper value ω_+^1 is found analogously as above by Mini-Max-formulation.
Higher PBG's at $k = 0$ can be found using successive characterizations.

2.6 Remarks

2.6.1 Wave groups through gratings

To consider the evolution of pulses through a linear grating, consider the superposition of grating modes around a certain frequency ω_0 (outside gap). Then

$$\begin{aligned} u(z,t) &= \int \alpha(\omega) v(\omega,z) e^{iK(\omega)z-\omega t} d\omega \\ &= e^{ik_0 z-\omega_0 t} \int \alpha(\omega_0+\nu)\left[v_0(z)+\nu v_0' + \frac{1}{2}\nu^2 v''\right] e^{i\left[\nu^2 z-\nu\tau\right]} d\nu + cc \\ &= e^{ik_0 z-\omega_0 t}\left[v_0(z)A(z,\tau)+iv_0'(z)\partial_\tau A - \frac{1}{2}v''(z)\partial_\tau^2 A\right] + cc \end{aligned}$$

where $v_0(z) = v(\omega_0, z)$, primes denote derivatives with respect to ω and A satisfies the standard LS-equation. (The expansion in ω of $v(\omega, z)$ is to the relevant order compared to the order of terms in the LS-equation.)

2.6.2 Phase-amplitude equation

Remark 14 *Motivated by Floquet's theorem (Bloch), substitute in the governing equation the transformation*[3]

$$\begin{aligned} u(z) &= e^{ikz} v(z), \text{ with } v \text{ periodic with grating period } p; \\ & \quad k \text{ is interpreted as wave number.} \end{aligned}$$

there remains

$$\left[\partial_z^2 + 2ik\partial_z + \left(\frac{\omega^2 n^2(z)}{c^2} - k^2\right)\right] v = 0, \quad v \text{ is } p\text{-periodic}$$

This equation may be viewed as an eigenvalue problem for ω, looking at given k for a periodic solution. It is not clear how this can be used in a constructive way.

Remark 15 *Write the solution to be found in a phase-amplitude-type of way*

$$u(z) = a(z)\cos(\theta(z))$$

and find

$$\left[2a_z\theta_z + a\theta_{zz}\right]\sin(\theta) + \left[a_{zz} + (\omega^2 n^2 - \theta_z^2)a\right]\cos(\theta) = 0$$

This is satisfied (for instance!!) when

$$2a_z\theta_z + a\theta_{zz} = 0, \quad \text{with solution } \theta_z = \frac{const}{a^2(z)}$$

(hence: θ and a vary on same scale) and

$$a_{zz} + (\omega^2 n^2 - \theta_z^2)a = 0,$$

leading to nonlinear equation for amplitude:

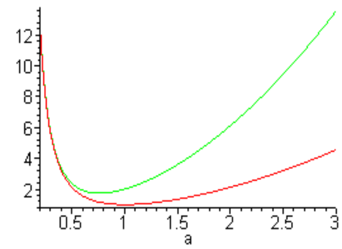
$$a_{zz} + \omega^2 n^2 a - \frac{const^2}{a^3} = 0,$$

[3] At first sight this resembles the transfromation to PHASE AMPLITUDE variables, as was done in NLS!!, However, be aware that now v to be interpreted as carrier-wave, modulated with long wave e^{ikx}, very different from the NLS-situation!

a Newtonian system with potential energy

$$a_{zz} = -\partial_a V(a), V(a) = \frac{1}{2}(\omega^2 n^2 a^2 + \frac{const^2}{a^2})$$

Graph of potential for $const = 1$ *for two different values of* n *:*



Minimum attained for

$$a^4 = \frac{const^2}{\omega^2 n^2}$$

and then $V_{\min} = const \cdot n$.
Interpretation: index change brings particle in another potential, leading to 'motion' away from minima, and either bounded or unbounded motion may result (outside and inside PBG respectively).

2.6.3 Analogy with Crystal Band Gaps

Mono-atomic crystals In solid state physics, considering lattice dynamics, investigate a ring of atoms (periodic bdy conditions, Born-van Karman), mass M, at equal equilibrium distances a, connected by linear springs, spring constant α.
Equations for displacement ψ_ℓ

$$M\partial_t^2 \psi_\ell = \alpha\left[\psi_{\ell+1} - 2\psi_\ell + \psi_{\ell-1}\right] =: \alpha D^* D\psi_\ell$$

Remark 16 *1. Continuisation for* $a \to 0, \frac{\alpha a^2}{M} = c^2 > 0$ *constant, leading to*

$$\partial_t^2 \psi = c^2 \partial_z^2 \psi$$

2. This is Potential-Hamiltonian system; easy to generalize by generalizing potential

3. Continuization process using Hamiltonian equally possible.

Look for solutions of form

$$\psi(z,t) = A_0 \exp i(kz - \omega t)$$

find

$$\omega = 2\sqrt{\alpha/M}\sin(ka/2) = \omega_0 \sin(ka/2)$$

For N atoms, length L, it holds: $L = Na, k = \frac{2\pi}{L}n, n \in Z$.

$$\text{Phase-velocity:} \quad v_p = \frac{\omega}{k} = \begin{cases} \text{for } ka \to 0 : a\omega_0/2 \\ \text{for } ka \to \pi : a\omega_0/\pi \end{cases}$$

$$\text{Group-velocity:} \quad v_g = \frac{\partial\omega}{\partial k} = \begin{cases} \text{for } ka \to 0 : a\omega_0/2 \\ \text{for } ka \to \pi : 0 \quad \text{(standing wave)} \end{cases}$$

First *Brioullin zone I*: the restriction of wavenumbers to $-\pi/a \le k \le \pi/a$, -(Lower) branch is known as Acoustic branch
Observe: this is 'discrete-homogeneous' case. Discretisation causes

- inhomogeneity through mass-distribution;
- deviation from linear dispersion relation to sin;
- group-velocity to vanish at π/a

2-atomic crystals The same situation, 'discrete inhomogeneous'; successive masses m and M, at distance (NOTE: half-length!!) $a/2$; equations

$$\begin{aligned} M\partial_t^2\psi_{2\ell} &= -\alpha D^* D\psi_{2\ell} \\ m\partial_t^2\psi_{2\ell+1} &= -\alpha D^* D\psi_{2\ell+1} \end{aligned}$$

Dispersion relation from

$$\begin{aligned} \psi_{2\ell} &= A_2 \exp i(k(2\ell)a/2 - \omega t) \\ \psi_{2\ell+1} &= A_1 \exp i(k(2\ell+1)a/2 - \omega t) \end{aligned}$$

$$\det \begin{bmatrix} 2\alpha - \omega^2 m & -2\alpha\cos(ka/2) \\ -2\alpha\cos(ka/2) & 2\alpha - \omega^2 M \end{bmatrix} = 0$$

solutions

$$\omega_\pm^2 = \alpha\left(\frac{1}{m} + \frac{1}{M}\right) \pm \alpha\left[\left(\frac{1}{m} + \frac{1}{M}\right)^2 - \frac{4\sin^2(ka/2)}{mM}\right]^{1/2}$$

Remark 17 *Note:*

$$\left(\frac{1}{m} + \frac{1}{M}\right)^2 - \frac{4\sin^2(ka/2)}{mM} > \left(\frac{1}{m} + \frac{1}{M}\right)^2 - \frac{4}{mM} = \left(\frac{1}{m} - \frac{1}{M}\right)^2 > 0$$

and so, ω^2 always real and positive, hence: All frequencies ω are real

Properties (suppose $m < M$)

$$\begin{aligned} \text{for } \omega_+ &: \begin{cases} \text{for } ka \to 0 : \omega^2 = 2\alpha\left(\frac{1}{m} + \frac{1}{M}\right) \\ \text{for } ka \to \pi : \omega^2 = 2\alpha\left(\frac{1}{m}\right) \end{cases} \\ \text{for } \omega_- &: \begin{cases} \text{for } ka \to 0 : \omega^2 = 0 \\ \text{for } ka \to \pi : \omega^2 = 2\alpha\left(\frac{1}{M}\right) \end{cases} \end{aligned}$$

Crystal Band GAP:

$$CBG: \quad \sqrt{\frac{2\alpha}{M}}a < \omega a < \sqrt{\frac{2\alpha}{m}}a$$

Observe the vanishing of groupvelocity for lower region just as in mono-atomic crystals.

Comparison CBG and PBG Compare CBG with PBG for discontinuous index, as in first subsection. Both cases concern

$$\partial_t^2 u = Lu$$

with operator

$$\begin{aligned} L &= L_C = 1/Mass \;\; D^* D \text{ is discrete-discrete for crystal} \\ L &= L_P = 1/index \;\; \partial_z^2 \text{ is discrete-continuous} \end{aligned}$$

Then, considering $u = \exp(i\omega t)$ is looking for the spectrum of L :

$$-\omega^2 u = Lu$$

Remark 18 *1. Observe,*

$$L_c \to L_P \text{ by continuization, } a \to 0, \alpha a^2/\bar{m} = constant$$

Since

$$CBG = (\sqrt{2\alpha/M}, \sqrt{2\alpha/m})$$

this gap in frequency will go to 'infinity' under continuisation, but remains fixed for the scaled frequency $\omega \cdot a$. *In 'continuisation', the two atoms should be smeared out to* n_0, n_1 *values on intervals.*

2. *For crystal there does not exist a steady solution for* $\omega \in CBG$; *for* $\omega \in PBG$ *there are exponential solutions. Hence: if the crystal is externally driven at frequency* $\omega \in CBG$ *the solution will desintegrate in many frequency components, while if light of frequency* $\omega \in PBG$ *is sent into a grating, the light will not be desintegrated but just decays exponentially.*

Dispersion Plots Crystal motions

2.7 Exercises/assignments

Exercise 19 ***/Assignment***

1. *Calculate explicitly the boundary PBG-solutions for the grating and verify the values of the PBG with those from the variational characterization. (Use the most natural intervals on which the variational problems should be posed: centered inside the layer of high index, and the layer of low index.)*

2. *Investigate a finite length grating of N periods. Determine the evolution (attenuation factor) of incoming waves in the PBG. In particular, calculate the transmission coefficient as function of ω, and note the appearance of band gaps.*

3. *Investigate the presence of a defect in a grating, for instance a layer of different index or a wider high-index layer. Show that there may exist 'localised states': solutions that decay exponentially outside the defect in the PBG of the undisturbed grating.*

4. *Investigate the PBG for a grating consisting of 2 half-infinite gratings with periods p_1 and p_2.*

5. *Consider a periodic index structure given by*

$$n^2(z) = 2 + \varepsilon \cos(z)$$

   Estimate the first PBG using the variational formulation. For $\omega \in PBG$, calculate a solution using MAPLE. Determine the bandgaps from perturbation theory, using the variational characterizations.

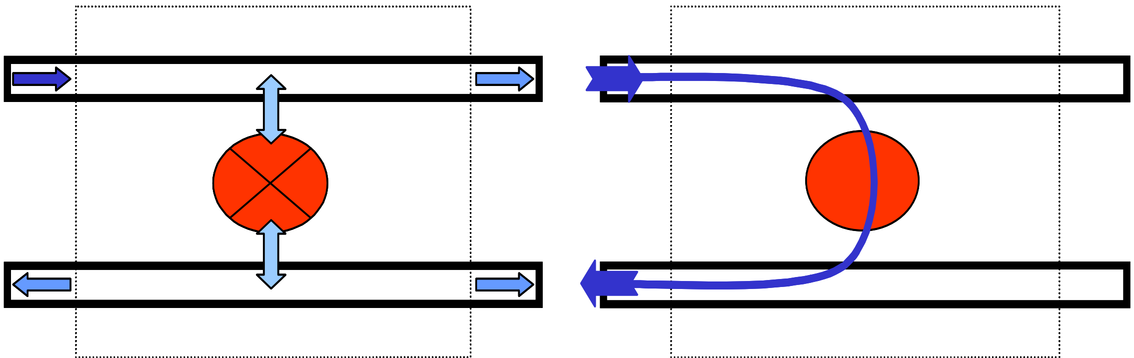
6. *Consider a grating in material with third order nonlinearity. Write down the equations and variational formulations for the PBG-boundary solutions. Try to find bounds for this PBG.*

Lecture 3: November 9

3 Lecture 3: Tunneling through Localised States (9 Nov. 2000)

3.1 Introduction

Many optical devices (for switching, etc.) consist of input and output waveguides (which will carry the 'global states' which extent to 'infinity') and another localised component that has the function to route a signal from one of the incoming guides into one of the outgoing guides. To achieve this, the incoming light needs to get a specific influence inside/through local components of the device (which will determine the 'local state').
A typical 2D example is a 'micro-resonator' that can act as wavelength filter, consisting of two parallel waveguides with a resonator (ring,or disc) in between that transfers light of one specific wavelength from one waveguide to the other one, and transmits other wavelengths (almost undisturbed) through the same waveguide.
That only specific wavelengths are transferred has as another interpretation/requirement that the local device determines the discrete spectrum of the total device: modes of the local component couple only with selective modes of the waveguides that carry a continuum of states.
The general geometric structure of the device (without specifying the local component) is sketched below.



Transfer and transmittance of light from upper left port; 'generically' output through all ports, 'exceptionally' full transmittance (at right).

The general problem is to look for solutions of the Helmholtz equation

$$\Delta u + \omega^2 n^2 u = 0, \tag{17}$$

where the device is characterised by the refractive index structure $n = n(x,z)$. To be well-posed, specific boundary conditions should be formulated. This is the real problem in optics, which is partly resolved (or obscured) with the idea of light fluxing in and out through certain continua (waveguides). To make this vague statement more explicit, consider a single waveguide along the z-axis; then for each ω (continuous spectrum), looking for waveguide modes (satisfying translation symmetry in the z-direction) reduces to the eigenvalue problem for the x-dependent modes ϕ with eigenvalues (the propagation constants) β:

$$u = \phi(x)e^{i\beta z}, \quad \partial_x^2\phi + \omega^2 n^2(x)\phi = \beta^2\phi$$

This leads to a discrete spectrum for β when looking for (x-) decaying modes.
From this example it is clear that boundary conditions are essential to determine whether a continuous or discrete spectrum can be expected.
A simpler but comparable case is the following example.

Example 20 *Role of boundary conditions for the simplest Sturm-Liouville problem. The problem*

$$u_{zz} + \lambda u = 0, \quad u(0) = 0, u(\pi) = 0$$

is the 'standard EVP' with discrete spectrum

$$\psi_k = \sin(kz), \quad \lambda_k = k^2$$

However, for each $\alpha \neq 0$ (the generic case), the problem

$$u_{zz} + \lambda u = 0, \quad u(0) = 0, u(\pi) = \alpha$$

has continuous spectrum,

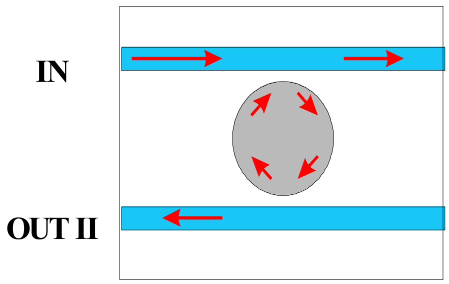
$$\lambda \in IR \backslash \{k^2 | k \in IN\}$$

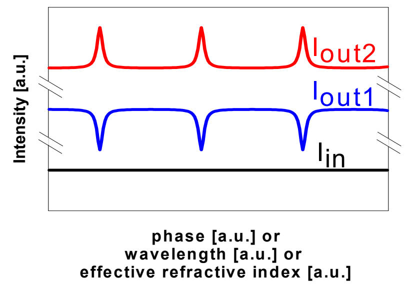
the complement of the above discrete spectrum for the non-generic value $\alpha = 0$.

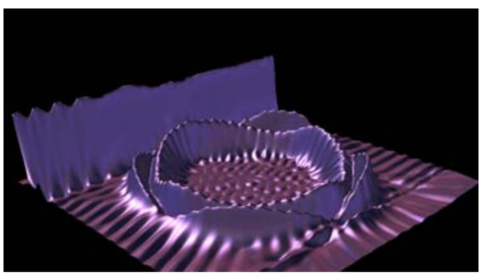
The same is true, but more difficult to formulate, for the micro-resonator: in-phase states correspond to discrete spectrum for ω, but out-of-phase to the complement (continuous). This may be 'explained' by first looking at the upper-waveguide: without 'perturbation' of the presence of the resonator, the spectrum is continuous, which (in the generic case) remains to hold when the resonator is present and acts as a perturbation; only exceptionally (non-generic), the spectrum changes. It would be nice to be able to give an explicit characterization for this discrete spectrum; one (theoretical) possibility would be to require the solutions of **??** to vanish for $z \to \infty$ (??). However, the global variational characterization

$$\omega^2 = \text{Crit} \left\{ \int |\nabla u|^2 \middle| \int n^2 u^2 = 1, \quad u \to 0 \text{ for } z \to \infty \right\}$$

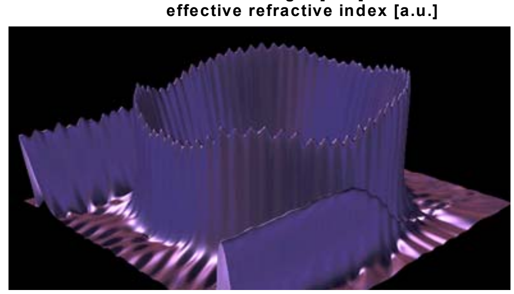
should be 'renormalised' in an appropriate way before it could be used in a constructive way.







Micro-resonator out-of-phase.



Micro-resonator in-phase, for discrete wave lengths only.

In the following we consider the much simpler analogy of (two) strings coupled to oscillators. This example has been motivated to understand the general formula in [**?**, **?**], and to prepare for treatment of the more difficult optical systems. One main lesson to be learned is the importance of symmetry arguments. Different from optical systems, the interaction between the strings is modeled in a simple ad hoc way, while this is one of the most difficult parts for optical devices.

References

[1] S.Fan, P.R. Villeneuve, J.D. Joanopoulos & H.A. Haus, Channel drop tunneling through localized states, *Phys.Rev. Letters*, 80(98)960-963

[2] S.Fan, P.R. Villeneuve, J.D. Joanopoulos & H.A. Haus, Channel drop filters in photonic crystals, *Optics Express*, 3(98)4-11

3.2 String coupled to oscillators

Consider an infinite, 1D, string described by a conservative wave equation that is coupled to oscillators.

3.2.1 First order (uni-directional) equation

We first consider the simplest example of a uni-directional eqn.

$$\partial_t u - c\partial_z u = \alpha r(t)\delta(z).$$

The forced solution is given by

$$R = \frac{\alpha}{c} r(t - z/c)H(z)$$

When the motion is coupled to an oscillator:

$$\dot{r} - i\omega_0 r = \alpha u(z = 0, t),$$

then, in the presence of an initial free wave $f(t - z/c) = Ae^{i\omega^*(t-z/c)}$ in the string, the oscillator eqn becomes

$$\begin{aligned} \dot{r} - i\omega_0 r &= \alpha\left[f(t) + \alpha r(t)/c\right], \\ \text{i.e.}\quad \dot{r} - (i\omega_0 + \alpha^2/c)r &= \alpha A e^{i\omega^* t} \end{aligned}$$

with solution

$$r(t) = \frac{\alpha A}{i\omega^* - i\omega_0 - \alpha^2/c} e^{i\omega^* t}.$$

This then leads to the solution in the string of the coupled system

$$\begin{aligned} u(x,t) &= f(t - z/c) + R \\ &= Ae^{i\omega^*(t-z/c)}\left[1 - \frac{i\alpha^2/c}{\omega^* - \omega_0 + i\alpha^2/c}\right] H(z) + Ae^{i\omega^*(t-z/c)} H(-z) \end{aligned}$$

Interpretation:

- Coupling with the resonator (oscillator) leads to addition of the original signal with the resonator signal; the resulting *interference* may lead to constructive or destructive interference for this unidirectional case:
- when the incoming mode has the same frequency as the oscillator $\omega_0 = \omega^*$, the incoming monochromatic wave $Ae^{i\omega^*(t-z/c)}H(-z)$ is completely damped by destructive interference with the resonator . This frequency is called the *'resonant' frequency*.
- for modes of other, non-resonant, frequencies, partial destruction takes place; then the incoming wave is 'mildly' attenuated in a Lorentzian-way, i.e. with (tanh-profile) attenuation coefficient:

$$\left|1 - \frac{i\alpha^2/c}{\omega^* - \omega_0 + i\alpha^2/c}\right| = \left|\frac{\omega^* - \omega_0}{\omega^* - \omega_0 + i\alpha^2}\right| = \frac{\varepsilon}{\sqrt{\varepsilon^2 + 1}} \quad \text{with} \quad \varepsilon = \frac{|\omega - \omega_0|}{\alpha^2}$$

Plot of Lorentzian $\frac{|\varepsilon|}{\sqrt{\varepsilon^2+1}}$

Remark 21 *Suppose a delta-pulse is injected i.e. an initial wave* $\delta(t - z/c)$*. This then leads to an excitement of the resonator* $\dot{r} - i\omega_0 r = \alpha\delta(t)$*, with solution* $r(t) = \alpha e^{i\omega_0 t} H(t)$*. For the string the result is an excitement given by*

$$R = \frac{\alpha^2}{c} e^{i\omega_0 (t - z/c)} H(t - z/c) H(z).$$

By measuring the signal at $z > 0$*, the properties of the resonator can be found: the coupling constant* α *and the resonant frequency* ω_0*.*

3.2.2 Bi-directional wave equation

Now consider bi-directional (non-dispersive) wave eqn with two types of forcing:

$$\partial_t^2 u - c^2 \partial_z^2 u = \alpha \dot{r}_e(t)\delta(z) + \beta c r_o(t)\delta'(z)$$

or as system

$$\begin{aligned} \partial_t u - c\partial_z v &= \alpha\delta(z) r_e(t) \\ \partial_t v - c\partial_z u &= \beta\delta(z) r_o(t) \end{aligned}$$

Coupling to oscillator/resonator is found from variational formulation (Lagrangian):

$$\begin{aligned} &\int\int \left[u_t^2 - c^2 u_z^2\right] dz dt \\ &+ \int \left[\dot{r}_e^2 - \omega_e^2 r^2 + \dot{r}_o^2 - \omega_o^2 r_o^2 + 2\alpha \dot{r}_e u\,(z = 0, t) - 2\beta c r_o \partial_z u(z = 0, t)\right] dt \end{aligned}$$

leading to equations:

$$\begin{aligned} u_{tt} - c^2 u_{zz} &= \alpha \dot{r}_e(t)\delta(z) + \beta c r_o(t)\delta'(z) \\ \ddot{r}_e + \omega_e^2 r &= -\alpha\partial_t u(z = 0, t) \\ \ddot{r}_o + \omega_o^2 r &= -\beta c \partial_z u(z, t)|_{z=0} \end{aligned}$$

If u is interpreted as vertical deflection, the forcing effect $r_e(t)$ represents forced vertical motion of one point of the string; the forcing $r_o(t)$ can be viewed as twisting (rotating) the string at $z = 0$.

Exercise 22 *Show that the 'twisting' can be viewed as limiting case of opposite vertical forcing of the string at two different points, say at* $x = -a$*, and* $x = a$*.*

Even mode The forcing $r_e(t)$ leads to waves propagating outward (away from the origin), in equal amounts to the left and the right: the forced solution is an even function in z explicitly given by

$$R_e(z, t) = \frac{\alpha}{2c}\left[r_e(t - z/c)H(z) + r_e(t + z/c)H(-z)\right]$$

With free solution $f(t - z/c) + g(t + z/c)$ the eqn for resonator:

$$\begin{aligned} \ddot{r} + \omega_e^2 r &= -\alpha\left[\dot{f} + \dot{g} + \alpha\dot{r}/c\right], \\ \ddot{r} + \left(\alpha^2/c\right)\dot{r} + \omega_e^2 r &= -\alpha\left[\dot{f} + \dot{g}\right] \end{aligned}$$

and hence:

$$r_e(t) = \int \frac{i\alpha\omega\left[\hat{f}(\omega) + \hat{g}(\omega)\right]}{-\omega^2 - i(\alpha^2/c)\omega + \omega_e^2} e^{-i\omega t} d\omega$$

and total solution:

$$\begin{aligned} u(z,t) &= f(t-z/c) + H(z)\int \left[\frac{1}{2}\frac{i(\alpha^2/c)\omega}{-\omega^2 - i(\alpha^2/c)\omega + \omega_e^2}\right]\left[\hat{f}(\omega)+\hat{g}(\omega)\right] e^{-i\omega(t-z/c)}d\omega \\ &+ g(t+z/c) + H(-z)\int \left[\frac{1}{2}\frac{i(\alpha^2/c)\omega}{-\omega^2 - i(\alpha^2/c)\omega + \omega_e^2}\right]\left[\hat{f}(\omega)+\hat{g}(\omega)\right] e^{-i\omega(t+z/c)}d\omega \end{aligned}$$

Odd mode The odd mode can be found by differentiating the even mode solution with respect to z. Briefly:
The forcing $r_o(t)$ leads to waves propagating outward (away from the origin), in equal but opposite-signed amounts to the left and the right: the forced solution is an odd function in z explicitly given by

$$R_o(z,t) = \frac{\beta}{2c}\left[-r_o(t-z/c)H(z) + r_o(t+z/c)H(-z)\right]$$

With free solution $f(t-z/c)+g(t+z/c)$ the eqn for resonator:

$$\begin{aligned} \ddot{r}_o + \omega_o^2 r &= -\beta\left[-\dot{f} + \dot{g} + \beta/c\dot{r}\right], \\ \ddot{r} + \beta^2/c\dot{r} + \omega_o^2 r &= \beta\left[\dot{f} - \dot{g}\right] \end{aligned}$$

and hence:

$$r_o(t) = \int \frac{i\beta\omega\left[-\hat{f}(\omega)+\hat{g}(\omega)\right]}{-\omega^2 - i(\beta^2/c)\omega + \omega_o^2} e^{-i\omega t} d\omega$$

and total solution:

$$\begin{aligned} u(z,t) &= f(t-z/c) + H(z)\int \left[-\frac{1}{2}\frac{i(\beta^2/c)\omega}{-\omega^2 - i(\beta^2/c)\omega + \omega_o^2}\right]\left[-\hat{f}(\omega)+\hat{g}(\omega)\right] e^{-i\omega(t-z/c)}d\omega \\ &+ g(t+z/c) + H(-z)\int \left[\frac{1}{2}\frac{i(\beta^2/c)\omega}{-\omega^2 - i(\beta^2/c)\omega + \omega_o^2}\right]\left[-\hat{f}(\omega)+\hat{g}(\omega)\right] e^{-i\omega(t+z/c)}d\omega \end{aligned}$$

3.2.3 Combined effect of even and odd mode

Combination of the free solution and the forced motions leads to

$$\boxed{u(z,t) = f(t-z/c) + g(t+z/c) + R_e + R_o}$$

in detail:

$$\begin{aligned} u(z,t) &= H(z) * \left[f(t-z/c) + \frac{\alpha}{2c}r_e(t-z/c) - \frac{\beta}{2c}r_o(t-z/c) + g(t+z/c)\right] \\ &+ H(-z) * \left[f(t-z/c) + g(t+z/c) + \frac{\alpha}{2c}r_e(t+z/c) + \frac{\beta}{2c}r_o(t+z/c)\right] \end{aligned}$$

Conclusion: If the free solution consist of an incoming wave at $z=-\infty$ only, i.e. $g=0$, then

$$\begin{aligned} u(z,t) &= H(z) * \left[f(t-z/c) + \frac{\alpha}{2c}r_e(t-z/c) - \frac{\beta}{2c}r_o(t-z/c)\right] \\ &+ H(-z) * \left[f(t-z/c) + \frac{\alpha}{2c}r_e(t+z/c) + \frac{\beta}{2c}r_o(t+z/c)\right] \end{aligned}$$

and the solution vanishes identically for $z > 0$ if we could take for all arguments

$$r_o(s) = -\frac{\alpha}{\beta} r_e(s), \quad r_e(s) = -\frac{c}{\alpha} f(s),$$

which is the case if

$$\alpha = \beta, \omega_e = \omega_o$$

This special situation is an *'accidental degeneracy'*. When satisfied, (only) for a single mode signal $f(t) = e^{-i\omega^* t}$ with resonant frequency $\omega^* = \omega_e = \omega_o$, full cancellation for $z > 0$ is possible.

3.3 Transfer in coupled strings

It is simple to have the change in the free wave (the effect of R_e and R_o) 'transferred' to another string by taking the same coupling. Denote the deflections of the second string by $v(z,t)$ and assume for simplicity that it has identical properties as the first string. Then from the Lagrangian

$$\int\int \left[u_t^2 - c^2 u_z^2 + v_t^2 - c^2 v_z^2\right] dzdt + \int \left[\dot{r}_e^2 - \omega_e^2 r^2 + \dot{r}_o^2 - \omega_o^2 r_o^2\right] dt +$$
$$\int \left[2\alpha \dot{r}_e \left[u\left(z=0,t\right) - v\left(z=0,t\right)\right] - 2\beta c r_o \left[\partial_z u(z,t) - \partial_z v(z,t)\right]_{z=0}\right] dt$$

one finds the coupled system

$$\begin{aligned}
u_{tt} - c^2 u_{zz} &= \alpha \dot{r}_e(t)\delta(z) + \beta c r_o(t)\delta'(z) \\
\ddot{r}_e + \omega_e^2 r &= -\alpha \partial_t \left[u(z=0,t) - v(z=0,t)\right] \\
\ddot{r}_o + \omega_o^2 r &= -\beta c \partial_z \left[u(z,t) - v(z,t)\right]_{z=0} \\
v_{tt} - c^2 v_{zz} &= -\alpha \dot{r}_e(t)\delta(z) - \beta c r_o(t)\delta'(z)
\end{aligned}$$

With accidental degeneracy, this system is able to completely transfer the mode with resonant frequency.

Exercise 23 *Show that by changing the sign of the interactions in the v-string, the transferred mode propagates in the opposite (backward) direction, see plot.*

3.3.1 Generalization in mode formulation

The above example can be formulated in terms of modes only (replacing the pde by an infinite number of ode's) by using Fourier-expansion in space. If we write and use standard notation

$$u(z,t) = \int u_k(t) e^{ikz} dk, \quad \delta(z) = \int e^{ikz} dk, \delta'(z) = \int ik e^{ikz} dk$$

the string equations become

$$\begin{aligned}
\partial_t^2 u_k + c^2 k^2 u_k &= \alpha \dot{r}_e(t) + ik\beta c r_o(t) \\
\partial_t^2 v_k + c^2 k^2 v_k &= -\alpha \dot{r}_e(t) - ik\beta c r_o(t)
\end{aligned}$$

while the oscillator equations become

$$\begin{aligned}
\ddot{r}_e + \omega_e^2 r &= -\alpha \partial_t \int \left[u_k(t) - v_k(t)\right] dk \\
\ddot{r}_o + \omega_o^2 r &= -\beta c \partial_t \int ik \left[u_k(t) - v_k(t)\right] dk
\end{aligned}$$

More general systems with Hamiltonian structure for 'mode-oscillator' coupling can be written down. For instance, when the spatial dependence is from waveguide modes, $\psi_k(x,z)$, and the signal in waveguide is given by

$$u(z,t) = \int u_k(t)\psi_k$$

the generalised Fourier-coefficients play the same role as above.
More generally, concentrating on the Hamiltonian structure, the set-up is as follows. In order to guarantee that 'even' and 'odd' modes are possible trough coupling, we consider two local oscillators coupled to 'continua':

$$\begin{aligned} \partial_t u_k &= i\delta_{u_k} H, \;\; \partial_t v_k = i\delta_{v_k} H \\ \partial_t r_e &= i\delta_{r_e} H, \;\; \partial_t r_o = i\delta_{r_o} H \end{aligned}$$

where H is the Hamiltonian. With the Hamiltonian given in general by

$$\begin{aligned} H(u,v,r) &= \int \left[\frac{1}{2}\omega_1(k)u_k^2 + \frac{1}{2}\omega_2(k)v_k^2 + \frac{1}{2}\omega_e r_e^2 + \frac{1}{2}\omega_o r_o^2\right] dk \\ &\quad + \int \left[\alpha_e(k)u_k r_e + \alpha_o(k)u_k r_o + \beta_e(k)v_k r_e + \beta_o(k)v_k r_o\right] dk \end{aligned}$$

where interaction coefficients are denoted by α, and $\omega_{1,2}$ describe the properties of the continua, the full equations become

$$\begin{aligned} \partial_t u_k - i\omega_1(k)u_k &= \alpha_e(k)r_e + \alpha_o(k)r_o \\ \partial_t r_e - i\omega_e r_e &= \int \left[\alpha_e(k)u_k + \beta_e(k)v_k\right] \\ \partial_t r_o - i\omega_o r_o &= \int \left[\alpha_o(k)u_k + \beta_o(k)v_k\right] \\ \partial_t v_k - i\omega_2(k)v_k &= \beta_e(k)r_e + \beta_o(k)r_o \end{aligned}$$

Observe that we now have described the complete system as a set of coupled oscillator equations. This formulation allows for easy generalizations, to take into account nonlinearity, additional coupling between the oscillators, etc.

3.3.2 String model for micro-resonator

The micro-disc resonator, a circular disk of high index in between two straight waveguides, shows the characteristic coupling of light through the upper waveguide, that is coupled to 'whispering gallery' modes in the resonator which will transfer some light in the lower waveguide; at the upper part, the gallery mode will either experience constructive or destructive interference with the wave guide, resulting into amplification in the resonator (leading to continued transfer to the lower waveguide) or attenuation in the resonator (and hence in the lower waveguide). This principle can be given a simpler setting of the string model as used above, now by modelling the resonator by a resonator-string (instead of an oscillator). Geometrically, the resonator string can be thought to be bent in a circle, coupling two straight strings. Full transfer (in backward direction) of an incoming wave is possible for frequencies that lead to waves in the resonator string that fit precisely the natural period of the resonator string. (This model is severely simplified; for one reason, different from real optical problem, incoming waves do not change properties of the resonator string.

Exercise 24 *Take a (uni-directional) guiding string with vertical deflection $u(z,t)$ and assume it is non-dispersive with propagation speed c_1. Take a resonator string (without dispersion) of length*

$L = 2\pi R$ and propagation speed c_0; denote the (radial) deflection by $\ell(\zeta, t)$. Design a 'coupling' at $z = 0$ that connects to $\ell(0, t)$ and $\ell(L, t)$; for instance from interaction functional

$$\alpha u(z = 0, t)\ell(\zeta = 0, t) - \beta u(z = 0, t)\ell(\zeta = L, t).$$

Find the response to an incoming wave $F(t - z/c_1)$.

Exercise 25 *Make a more complete model: two guiding strings to make transfer possible, bi-directional, with dispersion in the guiding strings given by $k = \beta(\omega)$ (to model the propagation constant of uni-modal waveguides).*