Advanced Modelling in Science
Mathematical Physics & Computational Mechanics
MSc 150910

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0.1 Preface

A mathematical model is a schematic representation of a part of reality. The modeller has to realize the purpose of the model when designing it and decisions have to be made about the different stages to be achieved. For instance, when the purpose is to understand a physical process, different requirements apply to the model then when an accurate prediction of numbers and sizes is needed. The first steps of modelling consist of a problem formulation and schematization. An abstraction has to be made of the “reality”, by making simplifying assumptions. This is followed by a model formulation, in which variables are defined and relations (e.g. physical laws) are given. The model equations and its solutions can then be studied analytically or numerically, and it may show necessary to make further simplifications or extensions to the model.

Making a mathematical model of a real life phenomenon from scratch is a difficult matter: slow progress in biological, neurological, societal modelling is an indication. However, when phenomena and processes from the physical and technical sciences are the topic of investigation, one can build on centuries of inventive work by scientific giants like Euler, Newton, Huygens, Fourier, Maxwell, and many others. They laid the basis for the description (models) of elastica, gases and fluids, of diffusion and light propagation, i.e. of the most common phenomena. Modern notation and education make it possible to acquire in a relatively easy way this accumulated knowledge. Therefore,

\[ \text{a first aim of this course is to give a brief} \]
\[ \text{description of the basic laws of nature,} \]
\[ \text{and the most important concepts used to} \]
\[ \text{describe characteristic phenomena,} \]

stress the mathematical structures with which these phenomena can be characterized.

But (un-)fortunately, the wide generality of these laws also implies that they are extremely difficult to attack directly. For instance, the Navier-Stokes equations, describing fluid motions, cannot be solved, except in very special cases. When solutions can be found, these are most times obtained by reducing the full set of equations into a simpler set that is specifically derived with the aim to study one specific aspect or phenomenon (for instance waves on the surface of a layer of fluid), thereby ignoring the many other phenomena also described by the full equations (such as swirling flows in outlets of industrial burners). The derivation of such simplified descriptions is actually the topic of modern day “modelling”. When a clear mathematical framework is exploited in this activity, it is justified to call this activity “mathematical modelling”, the area of activity of applied mathematicians.

\[ \text{Providing the mathematical tools to perform the mathematical modelling is the} \]
\[ \text{second aim of this course.} \]

More than to become a specialist in one of the areas covered, the aim will be to stress the mathematical structures that underlie various of the seemingly different areas and phenomena. It should be noted that with a choice of models in the physical sciences, this course also restricts to models that are deterministic of nature (in contrast to stochastic models), which are described by (systems of) differential equations.

Recognizing that any model is only an approximate description, two tasks turn up immediately. One is to know as precisely as possible the validity of the model. Accepting a certain description as a given model and then trying to relate it(s solutions) to (the solutions of) the full set of equations, is usually not possible since most times neither problem can be solved. Therefore it is more fruitful to keep track of the simplifying and asymptotic assumptions that are made during the modelling. This makes that the process of modelling and (partial) justification go hand in hand.

Since any model is an approximate description, a second task is to study the model for its robustness, its structural stability, its reliability. All these different names loosely refer to the same desired property, namely that the results of the model under consideration should not change “too much” when “small” perturbations are considered. However, since the possibility to perturb the model are abundant, constraints should be made on the class of perturbations that are considered; which perturbations are most interesting (i.e. most critical for the results of the model)
should also become clear from the modelling process itself. Besides that, these notions will in general critically depend on the precise description of “not too much” and “small”, i.e. on the norms that are used to measure certain deviations. From these remarks it will be clear that perturbation theory (in a broad sense) is an important ingredient in the modelling process, and various perturbation methods will be briefly described in this course.

0.2 Execution and organization of the course

0.2.1 Exercises and Maple Work Sheets

Many exercises can be found in the text and at the end of several chapters; besides that, some computer sessions will be arranged during which exercises with Maple Work Sheets can be made. Chapter 9 does not belong to the standard contents of the course. In Chapter 10 some examples of previous written tests and examinations are incorporated.

0.2.2 Examination

The final grade of the course will be based on the achievement in various activities: homework exercises (which will be assigned during the course), written tests, Maple assignments, and a final discussion.

0.2.3 Weekly overview

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Chapter 1

Dimensional Analysis and Scaling

Mathematical models are expressed by relations between various ‘quantities’ that are relevant for the problem under consideration. The physical quantities are usually measurable, which means that they have a certain specific dimension (like length, time, etc.). Combinations of measurable quantities have dimensions composed of the dimensions of the constituent quantities (such as velocity, which has dimension length divided by dimension time); investigating the dimensions of various expressions in the relations may lead to simplifying conclusions. This is the basis of ‘dimensional analysis’; in particular, dimensional analysis usually reduces the number of quantities necessary to describe the same relations, and hence to describe the same model. This is important for (experimental) modelling since the number of experiments can be reduced.

In the first section Buckingham’s pi-theorem is formulated and several examples are given. Most of the arguments in that section do not depend on the precise relation between the quantities. In the second section it is shown that if such a relation is known, a simple linear scaling may simplify the problem; scaling followed by dimensional analysis often lead to even more far reaching conclusions. Section 3 consists of two exercises, each of which covers the material of the preceding sections.

1.1 Dimensional analysis

In a research proposal “Biomarkers in wood dust associated respiratory effects” the following sentence appears about the costs for a company to give its 1000 employees a physical exam that takes one hour:

N.P., the company where all the research is being done, requires to be compensated for the loss of income of 1000 workers. The estimate would be

\[ 1000 \text{ (man.h)} \times 0.03 \frac{\text{m}^3}{\text{h/\text{man}}} \times 300 \text{ (USD/m}^3) \] = 9000 USD.

This illustrates the relevance of dimensions: the outcome in USD is a result of multiplying various quantities of different dimensions. In fact, the specification of the dimension of a term can define its meaning.

Exercise 1 Calculate the amount of work hour per USD in this example.

Important fundamental dimensions that appear in the natural and technical sciences are the so-called mechanical fundamental dimensions, the most important of which are listed below:

<table>
<thead>
<tr>
<th>quantity</th>
<th>symbol</th>
<th>SI-units</th>
<th>cgs-units</th>
</tr>
</thead>
<tbody>
<tr>
<td>length</td>
<td>( L )</td>
<td>m (meter)</td>
<td>cm (centimeter)</td>
</tr>
<tr>
<td>mass</td>
<td>( M )</td>
<td>kg (kilogram)</td>
<td>g (gram)</td>
</tr>
<tr>
<td>time</td>
<td>( T )</td>
<td>s (second)</td>
<td>s (second)</td>
</tr>
</tbody>
</table>

As indicated in the table, the ‘unit’ can be multiplied (scaled) by any number to make it suitable for the application under consideration. For instance ‘light-year’ may be a more convenient unit to measure distances in cosmography, while nm (nano-meter) and ps (pico-second) are more suitable in optics.

On the other hand, a length cannot be expressed in units that measure mass or time; the three quantities above are fundamentally different, and are examples of fundamental dimensions.

A quantity like volume in space has dimension \( \text{length}^3 \) and is not fundamental since it is the
product of quantities with fundamental dimension length.

\[ Q \text{ is a fundamental property that all known measurable quantities can be expressed in an algebraic way with a restricted number of fundamental dimensions.} \]

So if \( Q \) is a measurable quantity, a number of fundamental dimensions \( L_1 \ldots L_N \) can be found such that the dimension of \( Q \), written as \([Q]\), can be expressed like

\[ [Q] = L_1^{r_1} \cdots L_N^{r_N} \]

where \( r_j \) are rational numbers (positive, zero, or negative).

**Exercise 2** Give a simple argument why only rational powers of fundamental dimensions can appear in a measurable quantity. Argue why an expressions like \( \exp(x), \sin(x) \), etc. will not be measurable when \( x \) is measurable.

**Derived dimensions** are expressed in the fundamental dimensions; some examples:

<table>
<thead>
<tr>
<th>quantity</th>
<th>dimension</th>
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<tbody>
<tr>
<td>volume</td>
<td>( L^3 )</td>
</tr>
<tr>
<td>velocity</td>
<td>( L/T )</td>
</tr>
<tr>
<td>acceleration</td>
<td>( L/T^2 )</td>
</tr>
<tr>
<td>mass density</td>
<td>( M/L^3 )</td>
</tr>
<tr>
<td>mechanical energy</td>
<td>( M/L^2/T^2 )</td>
</tr>
</tbody>
</table>

Of particular importance, as we shall see, are dimensionless quantities, such as for instance the ratio of two quantities each of which has the dimension of length; e.g. the height of surface waves on a layer of fluid divided by the depth of the layer.

A quantity \( Q \) is called dimensionless if and only if \([Q] = 1\), i.e. if all exponents \( r_1 \ldots r_N \) are zero.

**Exercise 3** 1. Argue that any quantity expressed as a percentage is dimensionless.

2. Define a suitable dimensionless quantity for the length of a person in a country.

3. What is the meaning of the price of a stock at the stock exchange and of an economic index like the Dow-Jones Index?

4. Define a dimensionless quantity that could measure the 'economic prosperity' of a country and one that measures the 'economic prosperity' of the people in that country.

5. Indicate some factors that would determine a quantity measuring the 'quality of life.'

### 1.1.1 Buckingham

Consider a system that is described with a number of \( N \) quantities. In each application some of the quantities will have an interpretation as variable while others can be interpreted as parameters; irrespective their interpretation, let the number be denoted by \( N \). Suppose that these quantities can be expressed with a number \( d \) of basic dimensions.

By taking suitable products of the quantities a smaller number of dimensionless quantities can be defined, namely \( N - d \) dimensionless quantities.

The system will be characterized by the relation that is given between the quantities. A relation between the original quantities can be replaced with an equivalent relation between the dimensionless quantities.

These statements are formulated in the following lemma.

**Lemma 4** Buckingham Pi-Theorem

In a system with \( N \) quantities \( \{x_1, \ldots, x_N\} \) and \( d \) dimensions, a number of \( m = N - d \) dimensionless quantities \( \{q_1, \ldots, q_m\} \) can be defined as products of the original \( N \) quantities.

As a consequence: each scalar relation between the original variables

\[ f(x_1, \ldots, x_N) = 0 \]

can be replaced by a relation between the dimensionless variables:

\[ \tilde{f}(q_1, \ldots, q_m) = 0. \]

**Remark 5** 1. It should be remarked that this Lemma does not determine the precise choice of the dimensionless quantities; in fact these are far from unique: any algebraic (and even functional) combination of dimensionless quantities is dimensionless again.

2. Proof = exercise in linear algebra.
1.1.2 Examples

Pendulum

Consider the motion of a pendulum: a mass \( m \) (idealized to be a point) attached at the end of an inextensible, massless, rope of length \( \ell \) moving in a vertical plane. The position of the mass point at time \( t \) is completely specified by the angle \( \phi \) with the vertical. Neglecting friction and air-resistance, the angle at time \( t \) will depend on the initial position \( \phi_0 \), the initial velocity \( v_0 = \ell \frac{d\phi}{dt}(0) \), and the gravitational acceleration \( g \). Hence, it is argued that there is a certain relation between the various quantities; without knowing the precise relation, it can be written abstractly like

\[
F(\phi, t; \phi_0, v_0, m, \ell, g) = 0.
\]

In total 2 variables and 5 parameters can be recognized, so in total 7 quantities. The governing dimensions are the mechanical fundamental dimensions length, time and mass.

Hence according to Buckingham there are \((5+2)-3=4\) dimensionless quantities. Two dimensionless quantities are \( q_1 = \phi, q_2 = \dot{\phi}_0 \); to find the other two we form combinations of the dimensional quantities:

\[
q = t^{r_1} \ell^{r_2} g^{r_3} m^{r_4}.
\]

The dimension of this quantity is given by

\[
[q] = T^{r_1-r_2-2r_3} L^{2r_2+r_3+r_4} M^{r_4}
\]

and hence is dimensionless when

\[
r_1 - r_2 - 2r_3 = 0, \quad r_2 + r_3 + r_4 = 0, \quad r_5 = 0.
\]

These are three relations for five variables. A basis for the solutions is

\[
r_1 = 1, \ r_2 = 0, \ r_3 = -1/2, \ r_4 = 1/2, \ r_5 = 0
\]

and

\[
r_1 = 0, \ r_2 = 1, \ r_3 = -1/2, \ r_4 = -1/2, \ r_5 = 0
\]

corresponding to the dimensionless quantities

\[
q_3 \equiv t^* := \frac{t}{\sqrt{g/\ell}}
\]

and

\[
q_4 \equiv v_0^* := \frac{v_0}{\sqrt{g/\ell}}
\]

Hence the motion is governed by

\[
\ddot{\phi}(\phi, t^*; \phi_0, v_0^*) = 0.
\]

This shows that the mass of the particle is irrelevant for the motion.

Furthermore, if the motion is periodic, it is seen that the period scales with \( \sqrt{g/\ell} \).

Exercise 6 1. Verify the results obtained above by taking as model for the pendulum motion:

(a) the linear equation (harmonic oscillator)

\[
m \ell \ddot{\phi} + m g \phi = 0
\]

(b) the ‘exact’ pendulum equation

\[
m \ell \ddot{\phi} + m g \sin \phi = 0
\]

2. If the motion is found to be periodic with period \( \tau \), what is the period if (with the same initial conditions) the length is twice as long, and what if the pendulum is placed on the moon?

Froude number

Consider a ship sailing at sea (deep water); the influence of wind is neglected. The motion of the ship induces the appearance of waves that disturb the surrounding water; of the energy needed to move through the water, an essential part is used to generate these waves. The precise amount of energy in the generated waves is difficult to calculate (strongly determined by the shape of the ship), but when the waves are free from the ship, the gravity is the only force that determines the further behaviour of these waves. (The roughness of the hull also causes small whirling flows close to the hull and is another way how energy is absorbed; we neglect this effect here, but see below for the Reynolds number.)

It can be argued (expected, anticipated) that the waves generated by the ship depend on the length (and width, depth) \( L \) and the velocity \( V \) of the ship. (Convince yourself from every-day experience.)

Hence, the generation of the waves by the ship is governed by three quantities: acceleration of gravity \( g \), length \( L \) and velocity \( V \) of the ship. These
quantities depend on two dimensions (time and length), and so one dimensionless quantity can be defined
\[ Fr := \frac{V}{\sqrt{Lg}} \]
This is the Froude number, called after a ship builder.

Exercise 7 To do experiments on a ship of 100 m length that sails with maximum speed of 35 km/hr, one uses in a laboratory a model of the ship. The towing in the laboratory is restricted to velocities of at most 7 km/hr. What is the smallest scale of the model that can be used?

Reynolds number

Every fluid has a certain viscosity: an internal friction that causes every (unforced) motion of the fluid to be damped out after some time. For instance, for syrup the viscosity is larger than for water as is known from everyday experience. The Reynolds number is the dimensionless quantity that measures the ratio of the convective force (of the motion of the water) and the viscous force, and is given by
\[ Re := \frac{\rho L V}{\mu} \]
where \( \rho \) is the mass density, \( L \) some characteristic length, \( V \) some characteristic velocity, and \( \mu \) the viscosity coefficient.

Exercise 8 1. Determine the dimension of \( \mu \).

2. Will a model in the laboratory make whirling flows of the same Reynolds number as the ship in real sea? Can test facilities be designed in such a way that both the Froude and the Reynolds number are the same? Can you give an argument why the scaling of a ship is usually determined by the Froude number and not the Reynolds number? Will this be the same when testing an airplane in a (low speed) wind tunnel?

3. The motion of water (or air) is called turbulent if, roughly speaking, whirling motions of various scales (from small to large) are present. How do the velocities of whirling motions of different length scales compare?

Atomic explosion of 1945

Details of the strength of the first atomic bomb in 1945 were classified until the sixties. However, the British physicist G.I. Taylor (see Bluman & Kumei [2] for a description) was able to give a very accurate estimate of the strength using a movie picture that was available of the expansion of the ‘mushroom’-shape of the explosion, and using dimensional analysis, and a small scale experiment to estimate the value of a parameter. His arguments proceed as follows.

The basic appearance of the explosion is an expanding spherical fireball whose edge corresponds to a powerful shock wave. Let \( R \) be the radius of the shock wave, and assume
\[ R = f(E, t, \rho, P) \]
where
\[ E \] is the energy released by the explosion;
\[ t \] the time elapsed since the explosion;
\[ \rho \] the initial and ambient air density;
\[ P \] the initial and ambient air pressure.

Together with \( R \) we recognize 5 quantities. There are 3 mechanical dimensions (length \( L \), time \( T \) and mass \( M \)) since
\[ [E] = L^2 M T^{-2}, \quad [\rho] = L^{-3} M; \]
\[ [P] = L^{-1} M T^{-2}, \quad [R] = L. \]

Hence \( 5 - 3 = 2 \) dimensionless quantities can be found, for instance
\[ \tau_1 = P \left( \frac{\mu^6}{E^2 \rho^4} \right)^{1/5}, \quad \tau_2 = \frac{P^6 \rho}{E T^2} \]
As a consequence, any relation between the five original quantities can be transferred to a relation between \( \tau_1 \) and \( \tau_2 \). Written differently, this leads to the conclusion that \( R \) can be expressed like
\[ R = \left( \frac{E t^2}{\rho} \right)^{1/5} g(\tau_1) \]
where the function \( g \) is still unknown.
Assuming \( g \) to be continuous, Taylor approximated (!!!) the function \( g \) by a constant, for instance the same value as for \( \tau_1 = 0 : g(\tau) \approx g(0) \). (Of course there is no a priori justification for this,
1.2. SCALING

but with nothing else available, this is a very inventive and useful assumption.)

This then led Taylor to the approximate formula

\[ R \approx A t^{2/5} \text{ where } A = \left( \frac{E}{\rho} \right)^{1/5} g(0). \]

The next step was to estimate \( g(0) \); from experiments with light explosions (\( T_1 \) small) it was found that \( g(0) \approx 1 \). Using the motion picture, \( R \) was plotted vs. \( t \), which confirmed approximately a linear relation between \( R \) and \( t^{2/5} \); the proportionality constant (steepness of the linear relation) then led to an estimate for the value of \( \frac{E}{\rho} \) and hence \( E \).

1.2 Scaling

In the dimensional analysis above we didn’t exploit the precise relation between the various quantities. If such a relation is known, which may be either algebraic or differential, scaling followed by dimensional analysis can be exploited to lead to more detailed results, sometimes even to (an indication of) the actual solution.

A relation between independent and dependent variables, and parameters, can often be simplified by performing a suitable scaling of the variables. For simplicity we will consider only linear scaling of the variables in the following. A special choice of the scaling may eliminate various parameters and reduce the relation to a fundamental relation between less (scaled) quantities. It should be observed that the scaling does not have to make the quantities dimensionless; successive dimensional analysis to the fundamental relation may yield further reductions or conclusions. We will give several examples in the following.

As a simple example, consider for given nonzero parameters \( a, b \) the relation

\[ y^2 + ax + b = 0 \]

Apply a linear scaling in \( x, y \)

\[ x = \alpha \eta, \quad y = \beta \xi \]

the equation becomes

\[ \beta^2 \eta^2 + a\alpha \beta \xi \eta + b = 0. \]

Then, choosing \( \alpha, \beta \) to satisfy

\[ \frac{a\alpha}{\beta} = 1, \quad \frac{b}{\beta^2} = \text{sign}(b) \]

the resulting relation for \( \xi, \eta \) is free of parameters:

\[ \eta^2 + \xi \eta + \text{sign}(b) = 0. \]

Observe that in this case both \( \xi \) and \( \eta \) are dimensionless; in general that doesn’t have to be the case (for instance when \( b = 0 \)).

1.2.1 Example: Spring systems

When the position of a particle of mass \( m \) is described by its displacement \( u \) measured from some equilibrium position, its motion in time \( t \) is specified by a function \( u(t) \). When the motion is caused by a force \( f \), Newton’s law relates the acceleration and the force:

\[ m \frac{d^2 u}{dt^2} = f. \]

The expression \( m \frac{d^2 u}{dt^2} \) is called the inertia force.

Here we consider the case that the force \( f \) is caused by two effects:

- a spring attached to the particle; the corresponding force will be denoted by \( f_{spring} \) and two models will be specified below;

- a damper which dissipates the energy of the motion; this damper alone would drive the particle to its equilibrium position. As a simple model we assume that this damping force is proportional to the velocity, with coefficient of proportionality \( \mu \geq 0 \).

Then the system is described by Newton’s law:

\[ m \frac{d^2 u}{dt^2} + \mu \frac{du}{dt} + f_{spring} = 0. \]

The evolution is determined by specifying initial data, i.e. the displacement and velocity at some initial time. For the initial time we take \( t = 0 \) and for simplicity we take zero initial velocity:

\[ u(0) = u_0, \quad \frac{du}{dt}(0) = 0. \]
**Linear spring**

In particular for a *linear spring* with spring-coefficient $k > 0$, the spring force is given by $f_{spring} = ku$ and the equation is linear and reads

$$m \frac{d^2 u}{dt^2} + \mu \frac{du}{dt} + ku = 0.$$

The dimensions of the coefficients follow from the equation:

$$[\mu] = \frac{M}{T}, \quad [k] = \frac{M}{T^2}.$$

The dimensional quantities $u, t$ can be made dimensionless in various ways. For instance, for the displacement it is natural to scale with the initial displacement; for the time we specify a scaling factor $T$, with dimension of time, later. Denoting scaled quantities by a superscript asterisk, we have

$$u^* := \frac{u}{u_0}, \quad t^* := \frac{t}{T}.$$

In these variables the initial values are

$$u^*(0) = 1, \quad \frac{du^*}{dt^*}(0) = 0$$

and the equation becomes

$$\frac{d^2 u^*}{dt^*} + \mu^* \frac{du^*}{dt^*} + k^* u^* = 0$$

where the dimensionless coefficients are both non-negative and given by

$$\mu^* = \frac{\mu T}{m}, \quad k^* = \frac{k T^2}{m}.$$

Referring to the original variables, the ratio of damping force and inertia force is given by

$$\frac{\text{damping force}}{\text{inertia force}} = \frac{\mu \frac{du}{dt}}{m \frac{d^2 u}{dt^2}} = \frac{\mu \frac{du}{dt} / \frac{d^2 u}{dt^2}}{m}.$$

The left hand side is dimensionless, and the right hand side depends in the same way on $\mu$ and $m$ as $\mu^*$, which is why $\mu^*$ is referred to as the ratio of damping and inertia forces. In view of the expression for $k^*$, a convenient choice for $T$ would be

$$T := \sqrt{m/k}$$

and then

$$k^* = 1 \quad \text{and} \quad \mu^* = \frac{\mu}{\sqrt{mk}}.$$

This value for $T$ is the so-called *natural period*, since it is the period of the periodic motion that results when the friction is neglected (i.e. if $\mu = 0$). This shows that by choosing a specific scaling, the number of essential parameters in the given equation is reduced to be one.

Compare this with the result from Buckingham’s Lemma: there are two variables and four parameters $u, u_0, t, m, \mu, k$, and three dimensions mass, time, length; the number of resulting dimensionless quantities should be $6 - 3 = 3$, which are given above by $u^*, t^*, \mu^*$.

The reduction of the number of parameters is important for a practical investigation of the original problem with three parameters: understanding the scaled equation for all possible values of $\mu^*$ is sufficient.

**Exercise 9**

1. Sketch/Plot the solution $u^*(t^*)$ as function of $t^*$ and also the phase plane portraits of the motion; first for $\mu^* = 0$, then for various values of $\mu^* > 0$.

2. Determine the critical value of $\mu^*$ for which the damping is super-critical.

3. The equation is a simple model for the stiffness of an air wing when $u$ is interpreted as the normal deflection of the wing at a certain position on the wing. During flight, the mass decreases due to the fact that fuel in the wing is used. Investigate the difference of the damping properties at two different times during flight.

**Nonlinear spring**

For a nonlinear spring, the spring force depends nonlinearly on $u$. As a specific example we take a quadratic relation: $f_{spring} = k u + k_2 u^2$. Then the equation reads

$$m \frac{d^2 u}{dt^2} + \mu \frac{du}{dt} + ku + k_2 u^2 = 0.$$

With the same scaling as above, the following scaled equation is found

$$\frac{d^2 u^*}{dt^*} + \mu^* \frac{du^*}{dt^*} + u^* + \varepsilon u^* u^* = 0$$

where

$$\varepsilon = \frac{k_2}{k} u_0.$$
1.2. SCALING

Hence this scaled problem depends on two dimensionless parameters: $\mu^*$ (as before) and $\varepsilon$. Compare this with the result from Buckingham’s Lemma: there are two variables and five parameters: $u, u_0, t, m, \mu, k, k_2$, and three dimensions: mass, time, length. The number of resulting dimensionless quantities should be $7-3=4$, and 4 of such dimensionless quantities are given here by $u^*, t^*, \mu^*, \varepsilon$.

**Remark 10** The (scaled) spring force, given by $u^* + \varepsilon u^{*2}$, contains the parameter $\varepsilon$. Since $u^*(0) = 1$, at the initial time the linear restoring force is equal to 1, while the nonlinear restoring force is $\varepsilon$. When $\varepsilon$ is small, the nonlinearity contributes only a little at the initial time, and this remains true for later time as long as the solution remains of ‘order’ unity (which is the case in this example for all positive time since the deflection decreases with increasing time; in other equation’s this may not be true for all time). In that case, $\varepsilon$ can be used as a perturbation parameter. In this and similar cases, one speaks of a *weakly nonlinear problem* when $\varepsilon$ is small, $\varepsilon << 1$; else the problem is called *strongly nonlinear*. Observe that whether $\varepsilon$ is small or not depends on the ratio $k_2 / k$ and also on the initial position $u_0$. Investigating the unscaled system for given ratio $k_2 / k$, it depends on the initial value $u_0$ whether the system is weakly or strongly nonlinear. The precise notion of ‘smallness’ is not specified; actually, the distinction between weakly and strongly is determined by the actual effect the term $\varepsilon u^{*2}$ on the solution, and can be decided only a posteriori. Nevertheless, the words are often used to indicate when a perturbation method is – expected to be – successful (for weakly nonlinear) or not successful (for strongly nonlinear).

**Exercise 11**

1. Sketch/Plot the phase portrait of the nonlinear system, for $\mu^* = 0$ and $\mu^* \neq 0$; for instance, for $\mu^* = 0$ as in figs. 1.1, 1.2, using the potential energy function.

2. Determine the upper bound of the energy $E_0$ for which all solutions with energy less than $E_0$ are bounded.

3. For $\mu^* = 0$ explain from the phase portraits the meaning of the notion ‘weakly nonlinear’.

![Figure 1.1: Potential energy function](image1)

$V(u) = \frac{1}{2} u^2 + \frac{1}{3} u^3$ for the nonlinear spring equation $\ddot{u} = -\frac{dV(u)}{du}$.

![Figure 1.2: Phase portrait of the nonlinear spring equation with potential as given above.](image2)

1.2.2 Example: Surface wave models

Consider a layer of fluid (such as water) above a horizontal bottom. When the fluid is set in motion, by whatever cause, the fluid particles will start to move, interacting with each other and influenced by the force of gravity. When the upper surface of the fluid is *free*, this surface will also be
deformed by the particle motions. Looking only at the surface, the deformation is often of a characteristic type, like waves that are caused by throwing a stone in a pond. In principle, the particles below the surface determine the surface elevation. However, when the fluid is incompressible, and when so-called irrotational flow is considered, it turns out to be possible to describe approximately the elevation of the surface without reference to the internal motion. Below we will describe some aspects of this physical problem. From a point of view of modelling in the physical/technical sciences this is a good example how starting with a certain 'given' equation information about the problem can be extracted. (Hence we will not derive the governing equations from first principles, but investigate a postulated description of the phenomenon.)

In order to restrict to essentials, we make the following simplifying assumptions.

- The fluid is assumed to be incompressible with constant mass density; then without restriction the mass density will be taken to be unity.

- When undisturbed, the layer has a horizontal free surface. Taking the $z$ axis in the vertical direction, opposite to the direction of gravity, the free (undisturbed) surface will correspond to the plane $z = 0$. The depth of the layer is taken to be $H$; $g$ will denote the acceleration of gravity.

- It is assumed that no forces act from above on the free surface, a so called pressure free atmosphere; for instance, effects of wind are neglected.

- In the two horizontal directions, we assume that the motion is uniform in one direction; that is, we consider plane waves. Taking the $X$ axis as the direction of propagation, this means that the motion does not depend on the transverse horizontal direction.

Then, the elevation of the surface is described by specifying its $z$ value at each horizontal position $X$ and at each time $T$. Hence, the free surface will be described by a function $(X, T) \rightarrow \eta(X, T)$.

**Exercise 12**

1. Leaving the (constant) mass density outside consideration, recognize that the problem above has 3 variables $(\eta, X, T)$, 2 parameters $(g, H)$ and 2 dimensions (length, time). Find (Buckingham) $5 - 2 = 3$ dimensionless quantities.

2. A rough idea about wave propagation can be obtained in the following way. Although the surface deformation can be rather difficult, it is often reasoned as if the surface takes a harmonic profile, say of the form

$$a \cos \left( \frac{2\pi X}{\lambda} \right)$$

where $a$ is the wave amplitude and $\lambda$ is the wave length. This surface profile will not be stationary, but propagate with a certain speed, the wave velocity, say $V$. Only taking these three quantities into account, together with the parameters $g, H$ show that the problem is described by a relation between normalized amplitude and wave length and dimensionless velocity:

$$f\left( \frac{a}{H}, \frac{\lambda}{H}, \frac{V}{\sqrt{gH}} \right) = 0.$$  

Another interesting dimensionless quantity is the wave steepness $\frac{a}{\lambda}$.

3. From the above result, try to explain the observation (to be made at any coast when looking at the waves running into the shore) that waves approach the coast perpendicular, even when the coastal boundary is rather irregular.

In 1895 Korteweg and De Vries published an equation for $\eta(X, T)$ that in a certain order of approximation describes the surface elevation. In the derivation of the equation it was assumed that the waves were ‘rather low’ (small amplitude) and ‘rather long’. As said, we will not follow the derivation, but simply state the partial differential equation for $\eta(X, T)$ that they derived; this equation is known as the *Korteweg-de Vries equation* (KdV-equation)\(^1\).

\(^1\)The KdV equation became the most famous partial differential equation when in the sixties it was found that the
1.2. SCALING

The equation for $\eta(X, T)$ in the physical variables reads as follows:

$$\frac{\partial \eta}{\partial T} = -c \frac{\partial \eta}{\partial X} - \frac{cH^2}{6} \frac{\partial^3 \eta}{\partial X^3} + \frac{3c}{2H} \eta \frac{\partial \eta}{\partial X} \tag{1.1}$$

where the parameter $c = \sqrt{gH}$ has been introduced since it plays an important role in the physical phenomenon, as we shall see later.

This equation shows the time evolution of the free surface: at a fixed position, the time derivative of the elevation depends (in a complicated way) on the spatial derivatives of the elevation: *temporal variations and spatial variations are coupled*, which is characteristic for a *partial* differential equation. Understanding the coupling would mean that the meaning of each of the three terms in the right hand side should be clear, which at this moment is not possible. Also the equation is rather difficult in the sense that it is not easy to find solutions explicitly.

In the following we will try to interpret the various terms in the right hand side, and particularly, try to understand how the underlying modelling assumptions of 'long' and 'low' waves show itself in this equation.

The starting point is rather characteristic: we perform a scaling of the variables without specifying at this moment the scaling factors; these will be determined, or chosen, at a later instant.

Specifically, consider a scaling of $X, T, \eta$ to $x, t, u$ according to

$$X = Lx; \ T = \tau t; \ \eta(X, T) = a u(x, t).$$

It should be remarked that at this moment we do not aim to make the new variables $x, t, u$ dimensionless. Inserting the scaling in the equation leads to the following form of the equation$^2$

$$\partial_\tau u + \alpha u_x + \beta u_{xx} + \gamma u u_x = 0, \tag{1.2}$$

where

$$\alpha = \frac{\tau c}{L}, \ \beta = \frac{\tau cH^2}{6L^3}, \ \gamma = \frac{3\tau c a}{2HL}.$$ 

Observe that by the scaling we introduced 3 additional parameters ($L, \tau$ and $a$), which together with $H$ and $g$ (or equivalently $H$ and $c$) brings the total number of parameters to 5. However, the equation shows that only specific combinations of these parameters (combinations given by $\alpha, \beta$ and $\gamma$) play a role. We show how scaling can be used for various purposes.

In order to study the original equation (1.1) it is sufficient to study equation (1.2). Therefore we would like to reduce the numbers of parameters in this equation as much as possible. This can be done by choosing the scaling coefficients in an appropriate way:

**Exercise 13** Determine scaling coefficients such that the KdV equation becomes

$$\partial_\tau u + \frac{1}{6} u_{xxx} + \frac{1}{6} u u_x = 0; \tag{1.3}$$

observe that no parameters appear in this equation anymore. Compare this with the result obtained above (Buckingham).

From a mathematical point of view, this result is very satisfactory: only studying the equation (1.3), in which no parameters appear anymore, this knowledge can be transferred to various different physical situations.

Scaling can also be used for another purpose as we shall show now. In particular we show how scaling arguments can give an interpretation of the meaning, and importance, of the three terms in the right hand side of (1.1) or, equivalently, of (1.2). Talking about relative importance, it is necessary to make explicit that when dealing with the scaled variables, we assume these to be of unit order; so in the following $x, t$ and $u$ and *all of the derivatives of $u$* are considered to be of order unity; all notions of smallness are then compared to unity.

First we consider some limiting cases.
Exercise 14  1. Realize that the value of \( a \) is a measure of the amplitude of the surface elevation: for large \( a \) the physical wave heights are large (remember that we think of it to be of order one), while the limit \( a \to 0 \) means that waves of infinitesimal amplitude are considered. Since \( a \) only appears in the coefficient \( \gamma \), we can simply take the limit \( a \to 0 \) in the scaled equation, keeping all other parameters fixed. Show that \( \gamma \to 0 \), and that the resulting equation becomes
\[
\partial_t u + \alpha u_x + \beta u_{xxx} = 0. \tag{1.4}
\]
Observe that this equation is linear\(^3\), while (1.2) is nonlinear. Hence, explain the statement: "The nonlinear term in the KdV equation describes effects that are due to the finiteness of the wave heights".

2. The value of the parameter \( L \) determines the length of the physical spatial interval in which changes take place: for small \( L \) this interval is small while for large \( L \) changes are over a large physical interval. Small, large, values of \( L \) correspond to short, respectively long, waves. Explain this by investigating the relation between the functions \( f \) and \( g \) that are related by
\[
f(x) = g(X), \quad \text{i.e.} \quad f(x) = g(Lx);
\]
relate the derivatives of \( f \) to those of \( g \) and see the effect of the value of \( L \).

3. All three parameters \( \alpha, \beta, \gamma \) in (1.2) contain the parameter \( L \), and therefore it is not immediate what the effect is of taking a limiting value of \( L \). Therefore, argue as follows. The appearance of \( L \) in \( \alpha \) can be scaled away easily by taking \( \tau = L \), or, nicer from a dimensional point of view, \( \tau = \frac{L}{c} \); then
\[
\alpha = 1, \quad \beta = \frac{1}{6} \left( \frac{H}{L} \right)^2, \quad \gamma = \frac{3}{2} \left( \frac{a}{H} \right).
\tag{1.5}
\]
This is a good result from a dimensional point of view: Show that if \( L \) is given the dimension of length, that \( \tau \) will have the dimension of time, and that the variables \( x, t \) and \( u \) and the parameters \( \alpha, \beta, \gamma \) are dimensionless.

4. Continuing with the previous argument, observe that now \( L \) only appears in the coefficient \( \beta \). Keeping all other parameters fixed, look at the limit for long waves, and explain the statement: "The third order spatial derivative in the KdV equation describes effects that are due to the length of the waves under consideration; the longer the waves the less this term contributes.". Find the equation obtained in the limit for infinite long waves of finite amplitude\(^4\).

5. Consider the limit of infinitesimally small, infinitely long, waves. Show that by taking the limits \( a \to 0, L \to \infty \), in (1.2), the equation formally becomes:
\[
\partial_t u + u_x = 0.
\]
The solutions of this equation are easy: Show that for any ‘wave profile’ \( f \),
\[
u(x, t) = f(x - t)
\]
is a solution, representing the profile \( f \) translated undisturbed in shape with velocity 1 in the direction of the positive \( x \)-axis. (For this reason, this equation is called the translation equation.) Translate this result to the physical problem and conclude that: infinitesimal waves of long wave length propagate with speed \( c = \sqrt{g \lambda} \).

6. Now, return to (1.2) with the scaling such that the parameters are given by (1.5):
\[
\partial_t u + u_x + \frac{H^2}{6L^2} u_{xxx} + \frac{3a}{2H} \frac{a}{L^2} u u_x = 0. \tag{1.6}
\]
In the reasoning of Korteweg and de Vries, the last two terms are improvements of the translation equation in the sense that (some) effects of the nonlinearity and dependence on

\(^3\)The general solution of this linear equation can be written down using Fourier transformation techniques. This will be done in Chapter 4.

\(^4\)In Chapter 3 we will investigate this equation in more detail; from the behaviour of its solutions it will become clear why this equation is called the breaking wave equation.
wave length are taken into account. Being ‘corrections’, the coefficients should be small compared to 1, say small of order \( \varepsilon \ll 1 \). KdV takes both effects into consideration and assume that both effects are equally important, say of the same order \( \varepsilon \). Show that this means that the KdV equation describes waves of amplitude \( a \) and wavelength \( L \) that are such that \(^5\)

\[
\frac{a}{H} = O(\varepsilon), \quad \frac{H}{L} = O(\sqrt{\varepsilon}).
\]

Waves for which amplitude and wavelength are related in this way are said to satisfy the Boussinesq assumption.

7. For experiments in a towing tank as can be found in hydrodynamic laboratories, wave heights (twice the amplitude) up to 10 % of the depth are considered, for instance waves of amplitude 0.3m on a depth of 6m. Determine the order of wavelength of such waves that satisfy the Boussinesq assumption.

### 1.3 Exercises

#### 1.3.1 Prandtl-Blasius problem for flat plane

The following description of flow above a flat plate is a simple model to study for instance

- the disturbance of the air by a thin airfoil of an airplane that cruises at constant altitude with constant speed \( U \),
- the disturbance of a strong, uniform, wind by vegetation, buildings, etc.,
- the disturbance of water flowing over a rough area, etc.

In all these cases the flow (air, water) near the surface experiences resistance from the presence of the airfoil, obstacles, etc. In an idealised way, we model the airfoil or the ground by a flat plate (standing still) and let air (water) flow over it, assuming the flow is uniform before it reaches the plate.

Hence, let a semi-infinite plane \((x \geq 0, \ z = 0)\) be placed in a uniform flow \( U \) in the positive \( x \)-direction. The fluid has constant mass density and is viscous with viscosity \( \nu \), for which the dimension is \([\nu] = L^2 T^{-1}\). The viscosity causes the flow to be decelerated and to stick to the plate. Let the velocity of the flow be denoted by \((u, w)\) with \( u \) the velocity in the \( x \)-direction, \( w \) the velocity in the normal \( z \)-direction. Hence, far away from the plate, \((u, w) = (U, 0)\). The so-called shear rate at the plate (skin friction) \( S \) is the variation of the horizontal velocity in the normal direction. This shear rate will depend on the distance from the edge of the plate, and it is this dependence that we want to investigate.

Except the dependence on the distance \( x \) from the edge, \( S \) will also depend on the viscosity and the velocity \( U \) and we can write generally:

\[
S = S(x, \nu, U).
\]

In the following steps we want to find this relationship as precisely as possible, using (first) dimensional analysis and then (with more information about the governing equations) much more specific.

1. Determine two dimensionless variables.

2. Show that for some function \( f \)

\[
S = \frac{U^2}{\nu} f\left(\frac{U x}{\nu}\right).
\]

3. The actual equations of motion are given for steady, i.e. time independent, flow:

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \\
\frac{u}{\nu} \frac{\partial u}{\partial x} + \frac{u}{\nu} \frac{\partial u}{\partial z} &= \nu \frac{\partial^2 u}{\partial z^2}
\end{align*}
\]

The first equation may be recognized as the continuity equation (divergence of velocity field vanishes), the second equation is balance of forces between convection (at left hand side) and viscosity (at right hand side).

Using these equations, verify the dimension of \( \nu \) as given above.

---

\(^5\) Observe that the quotients \( \frac{a}{H} \) and \( \frac{H}{L} \) are dimensionless and that all length scales that are relevant for measuring the two effects depend on the (only!) physical length \( H \) that is present in the problem.
4. Show that upon writing

\[ u_s = \frac{u}{U}, \quad w_s = \frac{w}{W}, \quad x_s = \frac{x}{\alpha}, \quad z_s = \frac{z}{\beta} \]

with \( \alpha, \beta, W \) scaling parameters, the equations above reduce to a normalised form, i.e. that the undisturbed flow is normalized, \( U = 1 \), and that the equations are normalized with \( \nu = 1 \), provided two conditions are satisfied:

\[ \frac{\alpha W}{\beta U} = 1, \quad \frac{\alpha \nu}{\beta^2 U} = 1. \]

Given \( \beta \), the second condition determines \( \alpha \), after which \( W \) follows from the first condition. Hence these two conditions do not uniquely define the scaling to normalised form. Determine a scaling for which all scaled variables are dimensionless.

5. Now exploit the non-uniqueness of the scaling (keep \( \beta \) arbitrary) to find an explicit expression for the stress as follows.

From the definition of shear rate, first realize that at the plate the scaled shear rate is defined by

\[ S_s \equiv \frac{\partial u_s}{\partial x_s}(x_s, 0). \]

Hence, at the plate the shear rate only depends on \( x_s \), i.e. for some function \( h \) it holds that

\[ S_s = h(x_s). \]

Determine the dimension of \( S_s \) and \( x_s \), expressed in the dimension of \( \beta \). Then apply dimensional arguments to show that for some constant \( \sigma > 0 \)

\[ h(x_s) = \frac{\sigma}{\sqrt{x_s}} \]

(Observable that this argument can only be applied because \( \beta \) was kept arbitrary; for the choice of \( \beta \) such that all quantities are dimensionless, this relation cannot be derived!)

6. Now, translating to the original quantities, show that

\[ S(x, \nu, U) = \sigma \sqrt{\frac{U^3}{\nu x}} \]

which is the final, desired, relation.

### 1.3.2 Heat conduction in a bar

Temperature is a measurable quantity; well known units in which the temperature is measured are degrees Centigrade or Kelvin. In the following we will denote the dimension of temperature by \( \Theta \). This is an example of a non-mechanical dimension: it cannot be expressed in the mechanical dimensions, and is therefore a fundamental dimension.

Heat is intimately related to temperature, but is a mechanical quantity, it has dimension of energy. Heat is the amount of energy in a material of certain temperature, and is related to temperature by an expression like

\[ \text{heat} = \rho c \text{ temp} \]

where \( \rho \) is the mass density of the material and \( c \) is the so-called specific heat.

To make matters more specific, we consider heat conduction in an infinite bar, i.e. we restrict to a one-dimensional material. If we denote by \( x \) is the distance along the rod, \( t \) is the time, and \( u \) the temperature, the dimensions are

\[ [x] = L, [t] = T, [\rho] = \frac{M}{L}, [u] = \Theta \]

Since heat is an energy-like quantity, its dimension is

\[ [\text{heat}] = \frac{M}{L} \left( \frac{L}{T} \right)^2. \]

From this it follows that the dimension of the specific heat, which is the coefficient that relates the mechanical quantities with the temperature, has dimension

\[ [c] = \frac{L^2}{T^2 \Theta}. \]

When the temperature in the bar is not uniform, heat will flow from places where the temperature is high to places where the temperature is lower. The ‘flux’ \( Q \) of the heat has dimension

\[ [Q] = [\text{heat}] \frac{L}{T}, \]

heat times ‘velocity’. A simple relation between heat flux and the temperature gradient is given by Fourier’s law, which describes a linear relation between these quantities

\[ Q = -D \frac{\partial u}{\partial x} \]

with \( D > 0 \) the heat conduction coefficient.
1. Determine the dimension of $D$.

2. For the following it is simpler to introduce $\kappa = \frac{D}{\rho c}$, the so-called thermal conduction coefficient (thermal conductivity). Show that

$$\kappa^4 = \frac{L^2}{T}.$$ 

In the following we will study heat conduction in the bar. We assume that $\rho, c, D$ are given material properties, and assume that initially the total heat in the bar is given, say $S$, with $[S] = \text{[heat]}L$; let $s = \frac{S}{\rho c}$.

3. Determine the dimension of $s$.

Then the temperature $u$ at later times depends on previous quantities

$$u = f(x, t, \rho, c, D, S).$$

First we investigate how the temperature depends on combinations of the other quantities involved and then find a more precise relation by using an explicit equation.

4. Show that there are 7 quantities and 4 dimensions involved. In determining 3 dimensionless quantities, choose two of the dimensionless quantities, to be denoted by $\xi, \tau$, such that these are a scaling of $x$ (independent of $t$ and $u$) and a scaling of $t$ (independent of $x$ and $u$) respectively. Show that

$$x = \frac{k^2}{sc}, t = \frac{k^3}{c^2 s^2} \tau.$$ 

5. Show that for some function $F$ the temperature is given by

$$u = \frac{k^2}{\kappa^2} F(\xi, \tau).$$

6. Now we use more specific information about the temperature evolution, and consider a specific case of initial heating.

The actual equation that describes the temperature in the bar is a partial differential equation, the temperature conduction equation, that reads

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \quad (1.7)$$

where $\kappa$ is the thermal conductivity. Show that the dimension of $\kappa$ derived above agrees with the dimension as follows from the equation as given above.

7. Now consider the temperature distribution from an initial heat source (situated at $x = 0$), i.e. the solution of (1.7) with initial condition

$$u(x, 0) = q \delta(x) \quad (1.8)$$

where $\delta(x)$ is Dirac’s function at the origin. Determine the dimension of $q$. [Hint: integrate the initial condition over some interval including $x = 0$ to get rid of the delta-function.]

8. Perform a scaling in time and temperature such that in the above formulation both $\kappa$ and $q$ are scaled away:

$$\frac{\partial u_1}{\partial t_1} = \frac{\partial^2 u_1}{\partial x^2}, \quad u_1(x, 0) = \delta(x).$$

9. Then use dimensional analysis (determine dimensionless quantities) to show that the solution of (1.7, 1.8) is given by

$$u(x, t) = \frac{q}{\sqrt{\kappa t}} g\left(\frac{x}{\sqrt{\kappa t}}\right)$$

for some function $g$.

Remark 15 By substituting this form of the solution in the equation, the function $g$ is seen to satisfy an ordinary differential equation; this equation can be solved in such a way that also the initial condition is satisfied; in doing so, the following interesting result is found:

$$u(x, t) = \frac{q}{\sqrt{4\pi \kappa t}} e^{-x^2/4\kappa t},$$

i.e. at each time the spatial temperature distribution is a Gaussian function in $x$, with mean 0 and standard deviation determined by $\kappa t$, that decays with increasing time $t$. 

Chapter 2

Conservation and Balance Principles

Ordinary differential equations (ode’s) are of the form
\[ \frac{dx}{dt} = f(x) \]
where the state variable is a finite dimensional vector \( x(t) \in \mathbb{R} \) and the vector field \( f \) is a given function on the state space. For this reason systems described by ode’s are often called finite dimensional, or discrete, systems. In this chapter partial differential equations (pde’s) for various systems from Mathematical Physics are considered. For pde’s the state variable of the system is determined by an element from an infinite dimensional space, for instance a function of spatial variables defined on a certain domain; therefore pde’s describe infinite dimensional, or continuous, systems.

We will derive pde’s for various problems by using simple arguments of balance laws. In general terms, a suitable density (for instance the state itself) is integrated over a spatial domain; this quantity then depends only on time, and the change in time must be caused by a ‘flux’ through the boundary of the domain and by creation/annihilation within the domain; this is the ‘global’ description of the system. For smooth solutions this global description can be transformed to a pde that holds locally: changes in the state variable at time \( t \) and position \( x \) are determined by partial derivatives with respect to these time and space variables. For non-smooth solutions, such as when shocks appear in certain models for car flows and for supersonic flows around airfoils, the integrated form of the equations is more suitable.

Finally, it is briefly shown how the balance form of an equation may be exploited to design consistent discretizations of the equations that can be used in numerical codes for the simulation of the system.

2.1 PDE’s from balance laws

In the following examples we will exploit ‘balance’ arguments to derive equations for various important phenomena from mathematical physics. In all these cases the balance arguments are applied to the state variable itself: the change of the state variable is due to a flux from neighbouring points. Formulated in a more ‘global’ way, this can be expressed by saying that the change in time of the state integrated over a spatial domain is solely due to flux (flow) through the boundary. It turns out that many diverse phenomena are described by equations with such a structure, and that expressions for the flux – its relation to the state variable – depends on the physical problem at hand. In this chapter we will concentrate on the derivation of the equations, more than on properties of the solutions. We will first explain the idea of a balance law from a model for car flow.

2.1.1 Car flow

Consider a long, one lane freeway with cars driving in one direction. The velocity of each car is supposed to depend on the density near that car (when the density is higher, a careful driver will drive slower than when the density is lower) or density variations (when driving anticipating: changing the velocity when changes in the density are observed). The aim is to derive a model for the car density, assuming that no cars leave the freeway.

Conservation from first principles

Let \([a,b]\) be part of the road, and \(N([a,b], t)\) the number of cars in the interval \([a,b]\) at time \(t\). In
fact, we will consider \( N \) to be a positive real number instead of restricting to integers. From the assumption of one lane, the only access is at \( x = a \) and the only exit at \( x = b \). Fixing \([a, b]\) for the moment, the change in the number of cars in a small time interval \( \Delta t \) is given by

\[
N([a, b], t + \Delta t) - N([a, b], t)
\]

This change can only be caused by a difference of cars flowing into the interval at \( x = a \) and leaving the interval at \( x = b \). To make this more specific, introduce \( Q(x, t) \) as the flux of the cars, i.e. the number of cars passing the point \( x \) of the road at time \( t \) “per unit time interval”. Since \( Q(x, t)\Delta t \) is the number of cars passing the point \( x \) in a time interval \( \Delta t \), the change of the cars in the interval \([a, b]\) is given by

\[\left[-Q(b, t) + Q(a, t)\right] \Delta t\]

Conservation of cars implies that

\[
N([a, b], t + \Delta t) - N([a, b], t) = \left[-Q(b, t) + Q(a, t)\right] \Delta t.
\]

Dividing by \( \Delta t \) and taking the limit \( \Delta t \to 0 \), assuming differentiability of \( N \) with respect to \( t \), there results

\[
\frac{d}{dt} N([a, b], t) = -Q(b, t) + Q(a, t).
\]

(2.1)

This relation between \( N \) and the flux \( Q \) holds for each interval \([a, b]\) of the road; we will now exploit this property to derive a relation at a single point, instead of on an interval.

**Global and local car conservation law**

First define a *car density* \( \rho(x, t) \): the number of cars per unit length at position \( x \) at time \( t \). Clearly \( \rho \) is nonnegative, and is bounded above, say \( \rho \in [0, \rho_m] \). (This formulation as a *continuum description* is in fact not without problems for the interpretation; it can be reasonable if the density is not too low or too high.) Then \( N \) can be expressed like

\[
N([a, b], t) = \int_a^b \rho(x, t) \, dx
\]

and the left hand side of (2.1) can be written as \( \frac{d}{dt} \int_a^b \rho(x, t) \, dx \). Assuming that the flux is differentiable with respect to the spatial variable, the right hand side can be written like

\[
-Q(b, t) + Q(a, t) = -\int_a^b \frac{\partial Q(x, t)}{\partial x} \, dx.
\]

Taken together, the conservation law above becomes

\[
\frac{d}{dt} \int_a^b \rho(x, t) \, dx = -\int_a^b \frac{\partial Q(x, t)}{\partial x} \, dx.
\]

(2.2)

Assuming smoothness (interchanging the time derivative with the integration over \( x \)) this can be rewritten like

\[
\int_a^b \left\{ \frac{\partial \rho(x, t)}{\partial t} + \frac{\partial Q(x, t)}{\partial x} \right\} dx = 0.
\]

(2.3)

This is the so-called *integral (or global) car conservation law*.

Next we will exploit the property that the integral expression vanishes for each interval \([a, b]\). To that end we use the following fundamental lemma.

**Lemma 16 (Lagrange)** Let \( f \) be a continuous function defined on an interval \([A, B]\). If it holds that

\[
\int_A^B f(x) \, dx = 0
\]

for each subinterval \([a, b]\) \( \subset [A, B] \), then \( f \) vanishes identically: \( f(x) = 0 \) for each \( x \in [A, B] \).

**Exercise 17**

1. Prove the Lemma.

2. Generalize the Lemma to functions of more variables.

Using this Lemma, and assuming continuity (!!) of the integrand in brackets in (2.3), the integral formulation leads to the so-called *local car conservation law*:

\[
-\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial Q(x, t)}{\partial x} = 0.
\]

(2.4)

**Models for the car flux**

The flux density \( Q \) that appears in the expressions above will now be considered in detail. Despite the notation used above, \( Q(x, t) \), should not be interpreted as an a priori given function of \( x, t \). Instead, the flux will depend in fact on the density
2.1. PDE’s from Balance Laws

$\rho$, and possibly on its spatial derivatives. Below we will briefly describe two simple models for the flux. From the description it will become clear that specifying the relation between $Q$ and $\rho$ is an essential part of the modelling of this problem. From its interpretation as car flux, it is simplest to interpret $Q$ in terms of the car velocity $v_{\text{car}}$ according to

$$Q = \rho v_{\text{car}}, \quad (2.5)$$

and to model the flux by making reasonable assumptions for this car velocity.

**Flux for non-anticipating drivers** Assuming that a driver will adjust the car velocity depending on the situation around him (i.e. depending only on the density at the position of the car), the velocity will depend on the density. A simple model for $v_{\text{car}}$ is a linear dependence on $\rho$:

$$v_{\text{car}} = \beta (\rho_m - \rho); \quad (2.6)$$

the parameter $\beta$ is as yet unspecified, but has a clear interpretation by observing that $v_{\text{car}} \max \equiv \beta \rho_m$ is the maximal car velocity that is achieved on an empty road ($\rho = 0$). Furthermore, the relation implies that the velocity vanishes if the density reaches its maximum value $\rho_m$.

Substituting $Q(\rho) = \beta \rho (\rho_m - \rho)$ in the local conservation law $\partial_t \rho + \partial_x Q(\rho) = 0$, there results

$$\partial_t \rho + V(\rho) \partial_x \rho = 0, \quad (2.7)$$

where the so-called **density velocity** $V$ appears which is defined by

$$V(\rho) = \frac{dQ(\rho)}{d\rho} = \beta (\rho_m - 2\rho). \quad (2.8)$$

It is interesting to observe that the density velocity $V(\rho)$ is smaller than the car velocity; although the car velocity is always positive, the density velocity is negative for $\rho > \frac{1}{2} \rho_m$ (for heavy traffic) and positive for light traffic, for $\rho < \frac{1}{2} \rho_m$.

**Exercise 18** Here we will not discuss the governing equation (2.7) in much detail but only investigate the following aspects.

1. First observe that, irrespective the precise functional relation between $Q$ and $\rho$, any constant car density is a solution. Give the interpretation of this observation.

2. Assume that the car velocity is constant (independent of $\rho$). Let the car density be observed at a specific time, say $t = 0$, and given by a function $r$:

$$\rho(x, t = 0) = r(x).$$

Then determine the density at later times, and give the interpretation of the result (e.g. on an infinitely long road).

3. Now investigate a situation that the density is almost uniform, say $\rho_0$, with a small ‘perturbation’ that may depend on space and time:

$$\rho(x, t) = \rho_0 + \eta(x, t).$$

For ease of interpretation assume that (initially) the perturbation is a ‘localized hump’, a small increase of car density on part of the road. Then the equation for $\eta$ is approximately the ‘linearized equation’ (ignoring higher order terms in $\eta$)

$$\partial_t \eta + V_0 \partial_x \eta = 0 \quad (2.9)$$

where $V_0 \equiv V(\rho_0)$ is constant (see (2.8) for some $Q(\rho)$). This is called the translation equation.

Show that if $\eta_0(x)$ is the initial perturbation (at $t = 0$), the density at later times is given by a translation $\eta(t)$ with the constant velocity $V_0$:

$$\eta(x, t) = \eta_0(x - V_0 t).$$

For the model described above, the translation is to the right or to the left depending on the sign of $V_0$. Explain from the solution the following interpretation: a locally higher car density decreases locally the density velocity; then the higher car density propagates forward or backwards depending on whether $V_0$ is positive (light traffic) or negative (heavy traffic). Give the corresponding interpretation of a perturbation for the car velocity.

**Flux for anticipating drivers** In the above model for the flux, the car velocity was supposed to depend on the local density only. It is known, however, that most drivers (fortunately) do not only take the local density into consideration, but
also adapt their speed when they observe density variations ahead of them: when the density is larger in front, they will reduce speed, and conversely. A simple model to take this effect into account is to let the car velocity also depend on the spatial derivative $\rho_x$. So, to model only this aspect, the car velocity could be expressed like:

$$v_{car} = -\alpha \rho_x$$

where $\alpha$ is a constant.

**Exercise 19** 1. Show that the coefficient $\alpha$ should be positive to model the effect of anticipatory driving. (Note that cars will move backwards in this simple model when there is heavy traffic ahead of them.)

2. Show that the resulting equation can be written like

$$\partial_t \rho = \frac{\alpha}{2} \partial_x^2 \left( \rho^2 \right). \tag{2.10}$$

The linearization of this equation around a constant density $\rho_0$ will be studied in the next Chapter, where it will be shown that the result is a modification of (2.9) and given by

$$\partial_t \eta = \alpha \rho_0 \partial_x^2 \eta$$

This is the diffusion equation; argue what will happen if initially the perturbation is a ‘localized hump’; compare with the previous exercise.

3. Taking together both effects discussed above, show that the resulting equation is given by

$$\partial_t \rho + V(\rho) \partial_x \rho = \frac{\alpha}{2} \partial_x^2 \left( \rho^2 \right). \tag{2.11}$$

with $V(\rho)$ as in (2.8).

### 2.1.2 Falling snow: shocking

Consider snow falling down and forming a layer of snow on the earth. We are interested to find the increase of the thickness of the layer with time.

In a first approach, making everything as simple as possible, the following modelling assumptions seems natural:

- Assume that during falling the snow is uniformly distributed in the air, say density $\rho_0$, and that it is falling with constant vertical velocity, say $v_0$.

- Assume that within the layer the snow is also uniformly distributed (rather unrealistic, see below for an improvement); the density in the layer will be larger than in the air. Say the density in the layer is $\rho_1 > \rho_0$.

Using only simple arguments (first principles), it is possible to find the evolution of the thickness as follows.

Denote the layer thickness by $s(t)$. To increase the thickness by $\varepsilon$ requires an (additional) amount of snow equal to $\varepsilon (\rho_1 - \rho_0)$. Hence, in a time interval $\delta t$ the following amount of snow is added to the layer:

$$\rho_1 v_0 \delta t.$$  

We observe that the snow added to the layer in that time interval is the snow in the air with a distance less than $v_0 \delta t$ from the layer, carrying a total amount of snow approximately equal to

$$\rho_0 v_0 \delta t.$$  

Snow conservation leads to the conclusion that these expressions should be equal; taking the limit for $\delta t \to 0$ there results an expression for the time derivative $\frac{ds}{dt} = \dot{s}$ which is the growth velocity of the layer:

$$\dot{s} = \frac{\rho_1 - \rho_0}{\rho_1 - \rho_0} v_0. \tag{2.12}$$

This result shows that the layer grows linearly in time.

**Remark 20** The formula above give rise to the following remarks.

1. Consider again the car flow model, and suppose (rather morbid) that a heavy wall is placed on the road, with cars colliding into the wall, piling up before the wall when more and more cars are colliding. By collision, the cars are deformed, and the density increases abruptly. This situation can be described in the same way as the snow fall.
2.1. PDE’S FROM BALANCE LAWS

2. The assumption that the density is constant within the layer is not so realistic: lower parts of the layer carry a larger load of snow, experience a larger pressure, which leads to a higher density. (All this, assuming the snow doesn’t melt.) Therefore, it makes sense to let the ‘fall’ velocity depend on the density: the higher the density, the lower the velocity. If this assumption is used, for instance with the velocity (2.6) the car flow model (2.7) is obtained again!

3. Taking the z-axis vertically, with the bottom at $z = 0$, let $\rho(z), v(z)$ denote the density and velocity of the snow at height $z$ respectively. The situation above describes a discontinuous change in density and velocity across $z = s(t)$:

$$
\rho(z) = \begin{cases} 
\rho_1 & \text{for } z < s(t), \\
\rho_0 & \text{for } z > s(t) 
\end{cases},
$$

$$
v(z) = \begin{cases} 
0 & \text{for } z < s(t), \\
v_0 & \text{for } z > s(t) 
\end{cases}.
$$

This discontinuous change in density is a simple example of what is commonly described as a shock, with $s$ the shock velocity.

4. It is clear that for such discontinuous functions describing a shock, a local description in terms of partial differential equations is not so easy. However, a global formulation should be possible. Hence consider the local conservation law as described above for the car flow model:

$$
\frac{\partial}{\partial t} \int_0^b \rho \, dz = - \left[ Q(\rho) \right]_0^b
$$

where $b$ is a point far away above the layer, and $Q$ is the snow flux. The integral in the lhs. is split in two parts,

$$
\frac{\partial}{\partial t} \left\{ \int_0^{s(t)} \rho \, dx + \int_{s(t)}^b \rho \, dx \right\} = - \left[ Q(\rho) \right]_0^b.
$$

The discontinuous change in $\rho$ causes no problem in this formulation, and the lhs. becomes $(\rho_1 - \rho_0) s$ and so

$$
\dot{s} = \frac{Q(\rho_1) - Q(\rho_0)}{\rho_1 - \rho_0}. \quad (2.13)
$$

Since $Q(\rho_1) = 0, Q(\rho_0) = -\rho_0 v_0$, the same expression as above for the shock velocity is recovered. Formula (2.13) for the shock velocity, with $\rho_0, \rho_1$ the densities on different sides of the shock, is widely applicable. The result has the nice interpretation that, compared to the density velocity $V(\rho) = \frac{dQ(\rho)}{d\rho}$ that is applicable for smooth solutions, the shock velocity is the differential quotient (2.13).

2.1.3 Heat conduction

Diffusion is important in many physical processes: dye will diffuse in water, smoke and pollutants will diffuse in air, heat will be conducted in solids, etc. Although the physical appearance may be different, the governing equations are quite similar and have a characteristic form.

To emphasize the diffusive properties, we assume in the following that the medium in which the diffusion takes place is stationary (time independent).

As a characteristic example, we will consider heat conduction. For simplicity, we assume that the conduction takes place in a uniform solid for which the density $\rho$ and the so-called specific heat $c$ are constant. With $T$ the temperature, the combination $\rho c T$ has the interpretation of heat energy.

First the one-space dimensional model is investigated, then generalized to more dimensions.

Conduction in a rod (1D)

As a model for conduction in one spatial dimension, imagine a thin rod positioned along the $x$-axis. Let $T(x, t)$ denote the temperature, depending on position and time. Consider an interval $\Omega = [a, b]$ corresponding to a part of the rod. The heat energy in $\Omega$ is given by

$$
\int_{\Omega} \rho c T(x, t) \, dx.
$$

The change of the energy in $\Omega$ can be caused by two effects:

- Energy can be produced locally, say by a source $S(x, t)$, for instance by heating the rod with a flame, or by chemical reactions inside the rod. In a time interval $\delta t$ the change in energy is given by $\delta t \int_{\Omega} S(x, t) \, dx$.

- Another cause for a change is a flow of energy through the boundaries; we will denote this energy flux by $Q$.
Taking both causes into account, the change in time of the heat energy in $\Omega$ is given by

$$\frac{d}{dt} \int_{\Omega} \rho c T(x,t) \, dx = - \left[ Q(x,t) x=a \right] + \int_{\Omega} S(x,t) \, dx.$$

Repeating the arguments as in the example for car flow, this can be rewritten (assuming smoothness) as a global balance law:

$$\int_{\Omega} \{ \partial_t (\rho c T) + \partial_x Q(x,t) - S(x,t) \} \, dx = 0.$$ 

Assuming the integrand to be continuous, using Lagrange’s Lemma, the following local balance law for heat energy results:

$$\partial_t (\rho c T) + \partial_x Q = S$$

In the absence of external sources, $S = 0$, this balance law becomes a conservation law: the energy change in time is solely due to energy flux:

$$\partial_t (\rho c T) = -\partial_x Q.$$

**Heat energy flux: Fourier’s law**

It remains to relate the energy flux $Q$ to the temperature $T$. This depends very much on the material properties of the solid. However, the following observations from every day experience give motivations for a reasonable model.

- In a solid of constant temperature there will be no transport of heat: if $\partial_x T = 0$ then $Q = 0$.
- Heat flows from places of high temperature to places with lower temperature.

These observations support an empirical relation that is often used, and called Fourier’s law:

$$Q = -D \partial_x T$$

where $D$ is the so-called heat conductivity coefficient. The minus sign before $D$ appears for historic reasons; in view of the direction of heat flow, the heat coefficient can then be taken to be positive. The coefficient $D$ may in fact still depend on $T, \nabla T$, etc., but in the simplest cases it is a ‘material constant’. Inserting Fourier’s law in the balance law, the so-called heat equation results:

$$\partial_t (\rho c T) = \partial_x (D \partial_x T) + S.$$ 

In the special case that $\rho, c$ and $D$ are constant, the equation can be written simpler like

$$\partial_t T = \kappa \partial_x^2 T + s$$

where $\kappa = \frac{D}{\rho c}$ is the thermal conductivity coefficient, and $s = \frac{S}{\rho c}$. In the absence of external sources, the resulting equation

$$\partial_t T = \kappa \partial_x^2 T$$

is simply called the linear diffusion (or heat) equation (see also Section 1.3.2.)

**Boundary conditions**

The equation derived above holds inside the rod where flow of energy is possible. At the end points of the rod, the material is in contact with the surrounding. Several cases can be distinguished:

- **Insulated end point.** In that case it is assumed that there is no energy exchange with the surrounding. Hence the energy flux must vanish there: $Q = 0$; with Fourier’s law this means (when $D$ is strictly positive) that $\partial_x T = 0$ at that end point.

- **Prescribed energy flux**, when energy is added or extracted from the rod to a known extent, $Q$ is prescribed (possibly depending on time or even on the (unknown) temperature as in Newton’s law of cooling $\partial_x T = k(T_0-T)$, with $T_0$ the temperature of the surroundings).

- **Prescribed temperature**, in which case energy is added or extracted to such an extent that the temperature is prescribed (for instance, placing the end point of the rod in a large bath of water of a constant temperature).

**Exercise 21**

1. Consider a rod of length $\pi$, $x \in (0, \pi)$, of which both endpoints are kept at a fixed temperature $T = T_0$; assume that the conduction is described by (2.14) with constant $\kappa$ (so no heat source is present).

    (a) Show that the constant temperature distribution $T(x,t) = T_0$ is a solution, the so-called equilibrium solution.
2.1. PDE’S FROM BALANCE LAWS

(b) Suppose that initially the temperature distribution is given by

\[ T(x,t=0) = T_0 + A \sin(mx) \]

for some integer \( m \). Find the temperature distribution for increasing time (by searching for a solution of the form \( T(x,t) = T_0 + f(t) \sin(mx) \)). What is the effect of the value of \( m \)? Can you explain why the temperature approaches the equilibrium solution faster for larger values of \( m \)? (Investigate the value of the flux.)

2. Consider a rod of length \( L \), of which the left endpoint is kept at a fixed temperature, and the right endpoint is insulated. Assume that \( \kappa \) is constant. Suppose that a constant local heat production \( s \) is present and that it is observed that the temperature distribution inside the rod is time independent. What is in this case the heat flux through the left endpoint? (Define \( q = Q/\rho c \) and determine the equation for \( q \). Can you determine the temperature distribution?)

3. Continuing the previous exercise, the left endpoint serves as a heating (or cooling) device for the surroundings. To investigate its effect, assume another rod is attached to the left, which has conductive properties described by a thermal conductivity coefficient \( \kappa_1 \). Determine the temperature gradient in the second rod near the joint. Show that this temperature gradient is not continuous across the joint.

**Conduction in more dimensions**

Now consider a conducting material in more spatial dimensions, and a bounded domain \( \Omega \) in the interior of the material, say \( x \in \Omega \subset \mathbb{R}^n \); usually \( n = 2 \) (for a material with vanishingly small thickness) or \( n = 3 \). Let \( n \) denote the outward pointing normal at a point of the boundary \( \partial \Omega \) of \( \Omega \). The heat energy inside the domain can be defined as a direct generalization of the one dimensional case:

\[ \int_{\Omega} \rho c T(x,t) \, dx. \]

The energy flux is now an \( n \)-vector \( Q \), its direction indicates the direction of the energy flow. The amount of energy that flows through the boundary depends on the normal component \( Q \cdot n \) at a point of the boundary, and the flux through the whole boundary is given by

\[ \text{flux through boundary: } \int_{\partial \Omega} Q \cdot n \, dA \]

Repeating the arguments of the one-dimensional case, in the presence of a source density \( S \), a global balance law can be written down for the heat energy:

\[ \frac{d}{dt} \int_{\Omega} \rho c T(x,t) = - \int_{\partial \Omega} Q \cdot n + \int_{\Omega} S(x,t) \]

To derive a local relation from this global one, we try to generalize the reasoning from the one-dimensional case. An essential step in this process is to replace the boundary integration by a volume integration. Here one has to use a well known integral relation, Gauss’ theorem (partial integration in more dimensions). Specifically: the flux through the boundary can be replaced by a volume integral according to Gauss’ theorem as follows:

\[ \int_{\partial \Omega} Q \cdot n \, dA = \int_{\Omega} \text{div} Q \, dx \]

where it is recalled that the divergence of a vector field \( Q = (Q_1, Q_2, Q_3) \) is defined (in Cartesian coordinates) as

\[ \text{div } Q = \partial_x Q_1 + \partial_y Q_2 + \partial_z Q_3. \]

Then the global balance law can be written like

\[ \int_{\Omega} \left\{ \partial_t (\rho c T(x,t)) + \text{div } Q - S(x,t) \right\} \, dx = 0. \]

Using the more dimensional version of Lagrange’s Lemma, and assuming continuity of the integrand, the following local balance law is obtained:

\[ \partial_t (\rho c T) + \text{div } Q = S. \]

This is the generalization to more dimensions of the local balance law obtained above for \( n = 1 \). If \( S = 0 \), the balance law becomes a conservation law.

To relate the flux to the temperature distribution in more dimensions, Fourier’s law takes the form

\[ Q = -D \nabla T \]
where $D$ is the (scalar) heat conductivity coefficient; recall that $\nabla T$ is the gradient of $T$, a vector pointing in the direction of steepest ascent of $T$, and (in Cartesian coordinates) defined by

$$\nabla T = (\partial_x T, \partial_y T, \partial_z T).$$

Inserting this model for the flux in the local balance law leads to the following heat conduction equation:

$$\partial_t (\rho c T) = \text{div} \ (D \nabla T) + S.$$  \hspace{1cm} \text{(2.15)}

In the special case that $\rho, c$ and $D$ are constant, the equation becomes

$$\partial_t T = \kappa \Delta T + s$$

where $\kappa$ is again the thermal conductivity, $s = S / (\rho c)$, and $\Delta$ is the so-called Laplace operator

$$\Delta \equiv \text{div} \ \nabla = \partial_x^2 + \partial_y^2 + \partial_z^2$$

(sometimes denoted by $\nabla^2$).

2.1.4 Eulerian Fluid Dynamics

Continuum description

A fluid, gas, etc., can mathematically be idealized by a continuum description with an infinite number of identical, infinitesimally small, ‘particles’ that move around in space. If $\mathbf{x}$ denotes a point in space, and $t$ the time, the mass density $\rho(\mathbf{x}, t)$ determines the mass of the medium in a macroscopic domain $\Omega$ by $\int_{\Omega} \rho(\mathbf{x}, t) \, d\mathbf{x}$. The Eulerian velocity $\mathbf{v}(\mathbf{x}, t)$ is by definition the velocity of the ‘particle’ that is at time $t$ at the position $\mathbf{x}$. The relation between position and velocity of a single particle is then described by

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t).$$

Mass conservation: continuity equation

Mass conservation leads to the so-called continuity equation

$$\partial_t \rho + \text{div} \ (\rho \mathbf{v}) = 0.$$  \hspace{1cm} \text{(2.16)}

(Relate to (2.4) for car flow.) Although this is a fundamental physical law, the continuity equation is a kinematic relation: it is a consequence of describing the continuum with density $\rho$ and Eulerian velocity $\mathbf{v}$, and does not depend on any law that determines the dynamics of the fluid. Expained differently, by using the macroscopic quantities $\rho$ and $\mathbf{v}$, the individual particle motions are discarded, and the kinematic relation is a consequence of this reduction in the description. As a consequence, when in any model description a relation like (2.16) appears, the quantities $\rho$ and $\mathbf{v}$ can be interpreted as ‘density’ and ‘velocity’ of a continuum respectively.

Using the identity that holds for any scalar function $\rho$ and any vector field $\mathbf{v}$

$$\text{div} \ (\rho \mathbf{v}) = \rho \text{div} \mathbf{v} + \mathbf{v} \cdot \nabla \rho,$$

the continuity equation can be described as

$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho + \rho \text{div} \mathbf{v} = 0.$$

The first two terms can be taken together by using the notion of material time derivative (or total time derivative), which is the time derivative of a quantity $f(x, t)$ evaluated along a particle trajectory

$$\frac{Df}{Dt} = \partial_t f + \mathbf{v} \cdot \nabla f$$

where $\mathbf{v} \cdot \nabla f$ is the so-called convective term that is present because of the flow of the medium. Then the continuity equation can be rewritten like

$$\frac{D\rho}{Dt} + \rho \text{div} \mathbf{v} = 0.$$

A special case is a fluid with constant density; then necessarily the fluid is incompressible:

$$\text{div} \ v = \partial_x v_1 + \partial_y v_2 + \partial_z v_3 = 0.$$

Momentum equation

The continuity equation should be accompanied by at least one other equation to describe the complete evolution of the continuum since up to now the velocity $\mathbf{v}$ is not yet specified. It turns out that the additional equation is a variant of Newton’s equation for the momentum. The local momentum density of the fluid is given by $\rho \mathbf{v}$; its change should annihilate any other forces. Arguing in this way, the change in time of the momentum is determined by its flux (giving a convective term that is quadratic in the velocity) and the forces that act
as a source density. Using the continuity equation, the change in momentum can be interpreted using the material derivative:

$$\frac{D\mathbf{v}}{Dt} = \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}.$$

Then generalizing Newton’s idea leads to

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \text{div} \, \rho \mathbf{v} \mathbf{v} = \frac{D\mathbf{v}}{Dt} = \text{forces}$$

where the forces may be of different nature, depending on the continuum under consideration. When the forces can be expressed solely in terms of $\rho$ and $\mathbf{v}$, a closed system of equations is obtained. An example are the so-called **Navier Stokes equations** for viscous fluids,

$$\frac{\rho D\mathbf{v}}{Dt} = -\nabla p + \mu \Delta \mathbf{v} + \mathbf{F}$$

with hydrostatic pressure $p$, viscosity $\mu$ and external forces $\mathbf{F}$ (e.g. gravity); here a relation between pressure $p$ and density $\rho$ must be given in an equation of state.

More complicated sets of equations are obtained if also an energy balance equation (heat exchange) and/or magnetic effects (magnetohydrodynamics) are taken into account. We will not specify these effects at this place.

### 2.1.5 Shallow surface waves

Consider a layer of fluid, for instance water. A deviation of the still water level will produce surface waves that evolve under the influence of gravity. We will derive the basic equations under some simplifying assumptions.

- Consider only one horizontal direction (x-axis). The vertical direction (the z-axis) is in the (opposite) direction of gravity. The still water level is taken at $z = 0$, the flat bottom (assumed to be impermeable) is at $z = -H$, so that $H$ is the depth of the layer. In the following we will assume that the bottom is flat, so $H$ constant, and that the layer is shallow, meaning that $H$ is small.

- The water is assumed to have constant density, normalized to $\rho = 1$.

- Effects of surface tension or any other external forces on the free surface are neglected.

The free surface will be described by a function $z = \eta(x, t)$; in the following, also the depth below the surface will be used: $h(x, t) = H - \eta(x, t)$. Inside the fluid, the horizontal and vertical component of the fluid velocity are denoted by $u(x, z, t), w(x, z, t)$ respectively; at the free surface, the velocity is denoted by $U, W$.

**Mass balance: kinematic free surface relation**

First consider mass conservation, which will give a dynamic equation for the free surface. Considering the mass between two vertical lines, a change in time of mass between the lines can take place by a change of the free surface, and can only be caused by a flow of fluid through the lines.

Taking the lines close to each other, and applying first principles of mass conservation (no flux through the fixed boundary nor through the free surface) during a small time interval, a limit procedure leads to the following result:

$$\partial_t \eta(x, t) = -\partial_x \int_{-H}^{\eta} u(x, z, t) \, dz$$

(2.17)

The horizontal derivative in the right hand side has effect on the integrand and on the upper boundary in the integral:

$$\partial_x \int_{-H}^{\eta} u(x, z, t) \, dz = \int_{-H}^{\eta} \partial_x u(x, z, t) \, dz + \partial_x \eta(x, t) u(x, \eta(x, t), t).$$

The second term in this right hand side is expressed in quantities at the free surface only; also the first term can be reduced by using the incompressibility assumption as follows. Using $\text{div} \, (u, w) = \partial_x u + \partial_z w = 0$, the first term in the right hand side can be simplified as follows:

$$\int_{-H}^{\eta} \partial_x u(x, z, t) \, dz = - \int_{-H}^{\eta} \partial_z w(x, z, t) \, dz$$
\[ = -w(x, \eta(x,t), t) + w(x,-H, t). \]

Since the bottom is impermeable, \( w(x, -H, t) = 0 \), it follows that \( \int_{-H}^{0} \frac{\partial}{\partial z} u(x, z, t) \, dz = -W(x, t) \).

Taken together, the following equation is found to hold at the free surface:

\[ \partial_t \eta = -U \partial_x \eta + W. \tag{2.18} \]

This is a kinematic relation, valid without any approximation, and can be seen as a variant of the continuity equation.

Another description makes it clear that the right hand side is the component of the fluid velocity at the surface perpendicular to the surface:

\[ \partial_t \eta = (U,W) \cdot (-\partial_x \eta, 1). \]

Note that the normal direction is not normalized!

**Remark 22** An alternative derivation of the kinematic surface relation is equally appealing and can be described as follows. In fact this formulation makes it clear that incompressibility is not even required for the result. When looking at the motion of a fluid particle, let the position of a single particle be denoted by \( x(t), z(t) \). Then its velocity is given by the Eulerian velocity

\[ \frac{d}{dt}(x(t), z(t)) = (u(x,t), w(x,t)). \]

Now, specializing this for a particle at the free surface, observe that \( z(t) = \eta(x(t), t) \). Then \( \frac{\partial x}{\partial t} = U \frac{\partial z}{\partial t} = W \) and

\[ \frac{d}{dt} z(t) = \partial_x \eta(x,t) \frac{dx}{dt} + \partial_t \eta \eta = \frac{D\eta}{dt}. \]

Taken together, it follows that \( W = \partial_x \eta(x,t) U + \partial_t \eta \), and the kinematic relation is recovered.<

### Approximate free surface equation for shallow layers

We will now consider the above expression in the approximation that the depth \( H \) is small, as is the case for so-called shallow layers. For shallow layers it is reasonable to neglect the \( z \)-dependence of the horizontal fluid velocity\(^1\). The horizontal fluid velocity will then be denoted by \( u = u(x,t) \), and we review the balance law for the mass to see the relation between \( \eta \) and \( u \) in this approximation. Observe that now the horizontal component of the fluid velocity at the free surface is also given by \( u \).

A direct integration of the rhs. of the conservation law (2.17) is possible now since \( u \) is taken to be \( z \)-independent; doing so there results

\[ \partial_t \eta + \partial_x ((\eta + H) u) = 0, \tag{2.19} \]

or, equivalently,

\[ \partial_t h + \partial_x (h u) = 0; \tag{2.20} \]

this is the kinematic relation for shallow water.

### Momentum equation

The restriction to a shallow layer is particularly useful to derive another relation between \( \eta \) and \( u \); then the set of equations will be closed. This other relation is the momentum equation, like in Eulerian fluid dynamics.

The equation for the momentum (i.e. for \( u \)) is rather complicated, but can be derived by first introducing the ‘pressure’ \( p = p(x,z,t) \) in the interior of the fluid. The pressure force on a vertical line at position \( x \) is given by

\[ \int_{-H}^{0} p(x, z, t) \, dz \]

while the total momentum in a water column of unit length in the \( x \)-direction is given by

\[ \int_{-H}^{0} \rho u(x,t) \, dz = \rho h u(x,t) \]

Then balance of momentum leads to (investigate)

\[ \partial_t \left\{ \int_{-H}^{0} \rho u \, dz \right\} = -\partial_x \left\{ \int_{-H}^{0} \{\rho u^2 + p(x,z,t)\} \, dz \right\} \]

i.e.

\[ \partial_t (\rho h u) = -\partial_x \left\{ \rho h u^2 + \int_{-H}^{0} \{p(x,z,t)\} \, dz \right\} \]

An expression for the pressure is still needed; in general this is quite complicated, but in a shallow layer the pressure is well approximated by the hydrostatic pressure distribution

\[ p = \rho g (\eta - z) + p_{atm} \]
where $g$ is the gravitational acceleration, and $p_{atm}$ is the atmospheric pressure. For a pressure free atmosphere, $p_{atm} = 0$, the governing equation becomes

$$
\partial_t (h u) = -\partial_x \left\{ h u^2 + \frac{1}{2} g h^2 \right\}.
$$

By using the continuity equation (2.20) derived above, this can be simplified somewhat to

$$
\partial_t u + u \partial_x u = -g \partial_x h \tag{2.21}
$$

which can equivalently be written in a conservation form like

$$
\partial_t u = -\partial_x \left\{ \frac{1}{2} u^2 + g h \right\}.
$$

The two equations (2.20, 2.21) form a closed set of equations and describe the desired evolution of the surface waves in the approximation for shallow layers.

**Exercise 23** Consider the nonlinear surface wave equations above for elevations of small amplitude.

1. Show that the full equations have as trivial solution

$$
u(x, t) \equiv 0, \ \eta(x, t) \equiv 0.
$$

2. The linearized equations can be written down and lead to a second order wave equation for $\eta$ (and just as well for $u$):

$$
\partial_t^2 \eta - c^2 \partial_x^2 \eta \text{ with } c = \sqrt{g H}.
$$

This is the standard second order wave equation; check that the general solution is given by

$$
\eta(x, t) = f(x - ct) + g(x + ct)
$$

and that it consists of waves travelling to the right ($f(x - ct)$) and waves travelling to the left ($g(x + ct)$) with velocity $c$.

3. Show that the full equations have also special solutions of the form

$$
u(x, t) \equiv U, \ \eta(x, t) \equiv 0,
$$

corresponding to a uniform flow in the $x$-direction with constant velocity $U$ in a layer with flat surface.

## 2.1.6 Recapitulation local and global balance laws

Here we summarize in general terms the main notions and ideas that were treated above in several examples.

### Global conservation law

Let the state of a dynamic system be described by a function $u = u(x, t)$ of the spatial variable $x$ and time $t$. Let $e = e[u]$ be some local density, i.e. an expression that depends on $u$ in such a way that the value of $e$ at $(x, t)$ is determined by $u(x, t)$ and a finite number of $x$- derivatives of $u$ at $(x, t)$. We write $e = e[u](x, t)$ for its value at $(x, t)$.

The integral of $e$ along a given spatial interval $[a, b]$ defines the integrated quantity $E_{[a,b]}$ that will depend on $u$ and time:

$$
E_{[a,b]}[u](t) \equiv \int_a^b e[u](x, t) \, dx
$$

In specific cases, depending on the evolution equation and the density, the time derivative may depend only on the value of a (local) flux density $Q$ at the end points of the interval:

$$
\frac{d}{dt} E_{[a,b]}[u](t) = -Q[u](b, t) + Q[u](a, t). \tag{2.22}
$$

This is then called a global conservation law for the density $e$, and $Q$ is called the flux density corresponding to $e$.

### Local conservation law

Rewriting the right hand side

$$
-Q[u](b, t) + Q[u](a, t) \equiv -\int_a^b \partial_x Q[u](x, t) \, dx
$$

and (assuming differentiability) the left hand side

$$
\frac{d}{dt} E_{[a,b]}[u](t) \equiv \int_a^b \partial_t e[u](x, t) \, dx,
$$

the global law can be rewritten like

$$
\int_a^b \left\{ \partial_t e[u](x, t) + \partial_x Q[u](x, t) \right\} \, dx = 0.
$$
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If this holds for any interval of integration \([a, b]\) (within some given interval), it follows by Lagrange’s Lemma, assuming continuity of the expression in brackets in the integral, that

\[
\partial_t e[u](x, t) + \partial_x Q[u](x, t) = 0.
\]

(2.23)

This expression is called the local conservation law for the density \(e\).

As described above, the local conservation law is derived from the global conservation law; the reverse is equally true (and simpler): integrating (2.23) along an interval \([a, b]\), there results (2.22). In the special case that the evolution equation for the state variable itself is of the form of a local conservation law

\[
\partial_t u(x, t) = -\partial_x Q[u](x, t)
\]

it is custom to talk about the evolution equation in conservation form.

**Balance law**

For a given evolution equation, and a local density \(e\), the existence of a local/global conservation law is special; in general the density will satisfy a local balance law of the form:

\[
\partial_t e[u] + \partial_x \Phi[u] = S[u]
\]

where \(S\) is the so-called local source density.

**More space dimensions**

For a system in more spatial dimensions, write \(x \in \Omega \subset \mathbb{R}^n\) and let \(\Omega\) be a bounded or unbounded domain with \(n\) the outward pointing normal at the boundary \(\partial \Omega\). The flux of a (scalar) density \(e\) is an \(n\)-vector \(Q\) and the flux through the boundary is determined by the normal component:

\[
\text{flux through boundary: } \int_{\partial \Omega} Q \cdot n \, dA
\]

With a source density \(S\), the global balance law reads

\[
\frac{d}{dt} \int_{\Omega} e(x,t) \, dx = -\int_{\partial \Omega} Q \cdot n \, dA + \int_{\Omega} S(x,t) \, dx
\]

Using Gauss’ theorem:

\[
\int_{\partial \Omega} Q \cdot n \, dA = \int_{\Omega} \text{div} \, Q \, dx,
\]

and assuming smoothness, a local balance law is obtained in the form

\[
\partial_t e + \text{div} \, Q = S.
\]

If \(S = 0\), the balance law becomes a conservation law.

**Exercise 24** Write the KdV equation in conservation form; conservation of what physical quantity is expressed by this equation?\(<\)

**Exercise 25** The diffusion equation (without source) is in conservation form (directly from the way it is derived); consider a normalized formulation like

\[
\partial_t u = \partial_x^2 u.
\]

Consider the density \(e(u) = u^2\) and observe that for solutions of this equation this density satisfies the following balance equation:

\[
\partial_t (u^2) = 2u \partial_t u = 2u \partial_x^2 u = 2 \left[ \partial_x (u \partial_x u) - (\partial_x u)^2 \right].
\]

Write down the flux of this density and the source term. (Can you give arguments to show that this density is not in conservation form, i.e. that the source cannot be written as a spatial derivative of a local quantity?)\(<\)

**2.1.7 Finite volume discretizations**

In general, when a system is modeled by a (continuous) equation (like an ode or pde), numerical calculations in an algorithmic way can only be performed provided that the equation is ‘discretized’. It is a good attitude to consider the resulting numerical code, just as the continuous equation, as a model of the system (in this case a discrete model).

In fact, in some cases it is even possible (or more natural) to derive the discrete description directly from modelling the system.

In each case it is important to design the discretization in such a way that the main properties of the system are also present in the model. In particular, in the above sections, we have derived equations from balance laws; starting from
2.1. PDE’s from Balance Laws

integral formulations, we derived differential equations when the solutions are smooth. The integrated form of an equation is very well suited for constructing numerical schemes that have the desired balance properties. We will illustrate that in the following. For simplicity we consider spatial discretization, keeping the time derivative as a continuous operation, and consider only the simplest case of a scalar state variable depending on a single spatial variable.

The starting point is a conservation law for a state variable \( u(x,t) \) of one spatial variable \( x \) in an interval. With \( Q \) the flux, the integrated form of the local conservation law reads

\[
\partial_t \int_a^b u \, dx = -Q(b) + Q(a)
\]

for any subinterval \((a,b)\).

Now choose an equidistant grid, with grid points \( x_k \) and mesh size \( h \). Take an arbitrary grid point, the index of which will be denoted by an asterisk, \( x_* \); we assume that this point is in the interior, together with all its neighbours that will be needed in the following (boundary points require a special treatment, depending on the boundary condition). Taking for the interval \((a,b) = (x_* - \frac{1}{2} h, x_* + \frac{1}{2} h)\), i.e. a symmetric interval around \( x_* \) of length \( h \), the conservation law can be written like

\[
\partial_t U_* = \frac{-Q(x_* + \frac{1}{2} h) + Q(x_* - \frac{1}{2} h)}{h}
\]

where

\[
U_* \equiv \frac{1}{h} \int_{x_* - \frac{1}{2} h}^{x_* + \frac{1}{2} h} u \, dx
\]

Observe that in this way the conservation property is retained if we break up an interval in several parts: for any larger interval the contributions of the fluxes in interior points cancel; for instance for the interval \([x_* - (m + \frac{1}{2}) h, x_* + (n + \frac{1}{2}) h]\) there results:

\[
\partial_t \int_{x_* -(m+\frac{1}{2}) h}^{x_* +(n+\frac{1}{2}) h} u \, dx = -Q(x_* + (n + \frac{1}{2}) h) + Q(x_* - (m + \frac{1}{2}) h)
\]

Of course, to obtain a closed set of equations in the variables \( U_* \), it remains to express the flux at the intermediate points \( x_* + \frac{1}{2} h \) in the variables \( U_* \). Characteristic in this approach, which is called a finite volume method, is that a staggered grid is used: the grid points which are used to calculate \( U_* \) are different from the points \( x_* + \frac{1}{2} h \) where the fluxes are calculated. These variables \( U_* \) are approximately the values of \( u \) at the grid points:

\[
U_* = u(x_*) + O(h^2),
\]

which can be exploited when expressions for the flux are sought.

We now give several examples.

**Example 26** 1. First suppose that \( Q \) depends algebraically on \( u \): \( Q = q(u) \). Then one reasonable approximation for the flux can be derived as follows:

\[
Q(x_* + \frac{1}{2} h) = q(u(x_* + \frac{1}{2} h))
\]

\[
\approx q\left(\frac{1}{2}(u(x_*+1) + u(x_*))\right)
\]

which can in its turn be approximated by

\[
Q(x_* + \frac{1}{2} h) \approx q\left(\frac{1}{2}(U_{x_*+1} + U_*)\right)
\]

leading to a set of equations given by

\[
\partial_t U_* = -\frac{q\left(\frac{1}{2}(U_{x_*+1} + U_*)\right) + q\left(\frac{1}{2}(U_*+U_{x_*-1})\right)}{2h}
\]

2. Observe that the right hand side can be approximated using

\[
q\left(\frac{1}{2}(U_{x_*+1} + U_*)\right) \approx q(U_* + \frac{1}{2} q(U_*) (U_{x_*+1} - U_*)
\]

so that

\[
\partial_t U_* \approx -q'(U_*) \frac{U_{x_*+1} - U_{x_*-1}}{2h}
\]

where the right hand side is recognized as the central difference discretization of \( -\partial_x q(u) \equiv -q'(u) u_x \) when \( U_* \) and \( u(x_*) \) are identified.

3. For a diffusive equation with \( Q = -D u_x \) an expression for the derivative at \( x_* + \frac{1}{2} h \) can be obtained using central difference,

\[
Q(x_* + \frac{1}{2} h) \approx -D \frac{U_{x_*+1} - U_*}{h}
\]

\[
\approx -D \frac{U_{x_*+1} - U_*}{h},
\]

leading to a discretization in which the central difference for the second derivative \( u_{xx} \) can be recognized:

\[
\partial_t U_* = D \frac{U_{x_*+1} - 2U_* + U_{x_*-1}}{h^2}.
\]
2.2 Exercises

2.2.1 Sedimentation

To model sedimentation of suspended particles in a viscous fluid, suppose that all particles are identical. With the z-axis pointing upwards, let \( C(z, t) \) denote the concentration, and \( w(z, t) \) the fall velocity in the fluid.

1. Derive the law of mass conservation.

2. Let \( u_0 \) be the fall velocity of a single particle in the fluid. For dilute concentration the fall velocity will be reduced. A simple model would be to take

\[
w = w_0(1 - \alpha C)
\]

(Observed the value of the concentration at which the sediment is packed maximally.) Derive the governing equation for the concentration. Do you recognize this equation?

3. Let the bottom at \( z = 0 \) be impermeable. Suppose that initially the concentration is uniform: \( C = C_0 \) (with \( C_0 \) not too large) at \( t = 0 \) for all \( z > 0 \). Describe this initial situation as a discontinuous concentration profile with discontinuity at \( z = 0 \).

4. Determine the evolution for \( t > 0 \) of the density profile as a shock structure; derive the velocity of the shock.

5. Derive the equation when also diffusive effects are taken into account.

2.2.2 Momentum conservation for surface waves

Consider the KdV equation in scaled variables

\[
\partial_t u + c \partial_x u + \partial_x^3 u + u \partial_x u = 0.
\]

1. Show that the equation itself is of the form of a local conservation law. What conservation property is expressed?

2. Now show that the density \( u^2 \) is a conserved density too; derive the corresponding flux density.

3. For solutions decaying at infinity sufficiently fast, derive the global quantity that is conserved; this global quantity is actually the horizontal momentum of the waves.

2.2.3 Porous media: water table

A porous medium can be thought to consist of densely packed grains with fluid between the grains that fills the pores. Instead of describing flow through such a medium on a granular scale, one usually considers a macroscopic description.

As a specific example, consider a (shallow) aquifer (one horizontal direction, the x-axis) that is saturated with water above a horizontal bottom at \( z = 0 \). The water table is positioned at \( z = h(x, t) \).

1. Let \( u(x, t) \) denote the horizontal fluid velocity; since we assume the aquifer to be shallow, it is supposed that this velocity is independent of the height \( z \). Derive the equation expressing mass balance.

2. To obtain an equation in \( h \) alone, a relation between \( u \) and \( h \) is required. A standard relation is given by the so-called Darcy’s law

\[
u = -\kappa \partial_x h(x, t)
\]

where (in the simplest cases) \( \kappa \) is a constant; note the (mathematical) resemblance of this law with Fourier’s law for diffusive systems. Derive the governing equation for \( h \).

3. Assume that the aquifer is enclosed at the left endpoint at \( x = a \) by a solid, impermeable rock, and at the right endpoint \( x = b \) is connected to a lake. Derive the governing boundary conditions at both endpoints.

2.2.4 Decay rate of heat conduction

Consider the standard heat conduction equation with constant heat conducting coefficient \( D \) in the form

\[
\partial_t u = D \partial_x^2 u.
\]

1. Derive the equation for the density \( u^2 \).

2. Consider this equation for a rod of length \( \pi \) with fixed temperature at the endpoints

\[
u(0, t) = u(\pi, t) = 0.
\]
Show that the following norm
\[ N(u) := \int_0^\pi u(x,t)^2 \, dx \]
is monotonically decreasing in time for any nontrivial solution. Conclude from this that the temperature \( u \) decays to zero for \( t \to \infty \).

3. Now consider a rod of length \( \pi \) that is insulated
   at the endpoints
   \[ u_x(0) = u_x(\pi) = 0. \]
   Investigate the change of \( N(u) \) for this case; does the solution decay to zero in this case?

4. Give the interpretation of your results.

2.2.5 Finite volume method

Derive a finite volume discretization for the diffusion equation (with flux \( Q = -D \nabla u \)) for a two-dimensional domain \( \Omega \). Consider the cases that \( \Omega \) is the rectangle \( \Omega = (a, b) \times (c, d) \) and the circle \( \Omega = \{ |x| < R \} \).
Chapter 3

Linearization, Perturbation, Sensitivity

We have derived various equations from conservation and balance principles for systems that depend on time and on spatial variables. In general terms, if the state variable (which may be a vector) is denoted by \( u = u(x, t) \), with \( t \) the time, and \( x \) the spatial variable (which may be a vector), these equations are of the form of a so-called non-linear evolution equation. When written as a first order in time equation, the general form of the equation reads

\[
\partial_t u = K(u).
\]

Here the right hand side \( K(u) \) depends on the state and its spatial derivatives (but not on temporal derivatives anymore), say \( K(u) = F(x, u, \partial_x u, \ldots) \) for some function \( F \) of the variables:

\[
\partial_t u = F(x, u, \partial_x u, \ldots).
\]

Writing the equation in this form is motivated by comparing with simpler ordinary differential equations, which for a (vector) function \( u(t) \) would be written for a vector field \( f \) like

\[
\dot{u} = f(u).
\]

For this reason, the rhs. of (3.2) is also often called the vectorfield of the non-linear evolution equation. A specific example of possible terms in the vectorfield \( F \), restricted to those of simple form, is given in the equation below:

\[
\partial_t u = \alpha u_x + \beta u_x + \gamma u_{xx} + \delta u_{xxx} + \nu u^2 + \omega u u_x
\]

The coefficients \( \alpha, \ldots, \omega \) that appear here usually represent some physical quantity that is specific for the problem at hand; for a moment we just interpret them as given constants.

For a vector field in which only one term is present, the equation is rather simple and the solutions can mosttimes be studied. This is for instance the case for the following equations:

\[
\begin{align*}
\partial_t u = \alpha u & \quad \text{exponential growth (} \alpha > 0 \text{)} \\
\alpha u + \nu u^2 & \quad \text{restricted growth (} \alpha > 0, \nu < 0 \text{)} \\
\beta u_x & \quad \text{translation (wave)} \\
\gamma u_{xx} & \quad \text{diffusion (} \gamma > 0 \text{)} \\
\delta u_{xxx} & \quad \text{dispersion} \\
\omega u u_x & \quad \text{nonlinearity: "wave breaking"}
\end{align*}
\]

Remarkable is the great variety of phenomena that correspond to the various individual terms as indicated in this table by the description of the phenomenon.

It is rather difficult in general to understand the behaviour of the solution when various of these terms are combined. Much research has been spent to study special combinations. We list the most important special combinations, with the name of the equation:

\[
\begin{align*}
\partial_t u = \gamma u_{xx} + \alpha u + \nu u^2 & \quad \text{reaction-diffusion eqn (} \gamma, \alpha > 0, \nu < 0 \text{)} \\
\alpha u_x + \beta u_{xx} + \gamma u u_x & \quad \text{Korteweg-de Vries eqn} \\
\gamma u_{xx} + \omega u u_x & \quad \text{Burgers eqn (} \gamma > 0 \text{)} \\
\gamma u_{xx} + \delta u_{xxx} + \omega u u_x & \quad \text{KdV-Burgers eqn}
\end{align*}
\]

When combining various physical effects in one equation by adding different terms in the vector field, the question presents itself what the relative importance of each of these terms is. In many cases we often neglect many effects in modelling: any model is only a limited description of the phenomenon it is supposed to describe. For instance, writing down the simple pendulum equation, effects of air-resistance, friction in the joint, extensi-
bility of the cord, etc. are neglected. Sometimes it can be ‘expected’ beforehand that such effects will be negligible, which motivates to discard these. However, it turns out that this can be rather misleading, and depends essentially on the purpose for which the model is used. For instance, neglecting the air resistance in the pendulum motion may be justified for small time intervals, but is unacceptable when you want to know how long it takes a real pendulum to slow down to its downward position. Neglecting or including this effect of air resistance in the model can be done by adding a friction term in the vector field depending on a (physical) parameter: when the parameter vanishes, the effect is neglected, when the parameter is positive, air resistance is included (to an extent determined by the magnitude of the parameter). This leads one to study the effect of the value of the parameter on the solutions. Depending on the aim (purpose) for which the model is designed, the effect may indeed be negligible or not. More generally speaking, to study the “reliability” of a model, it is necessary to study perturbations of the model. Roughly speaking the question is

Do small causes (perturbations) have small consequences?

In this sentence, the words ‘causes’ and ‘consequences’ depend on the specific phenomenon under consideration, and the two words ‘small’ can have a different meaning: an affirmative or negative answer to the question may depend crucially on the precision of these notions, as we have demonstrated above for the pendulum example.

Roughly speaking, when the answer is affirmative, the perturbation will be called regular, else singular.

The well known “butterfly of Lorenz” is a warning that the answer may not be as one expects.

For evolution equations as we consider, it is common to distinguish between two kinds of perturbations:

- **perturbation in the initial condition(s)**: what is, in one given model equation, the effect of perturbing the initial state of the system? Is the effect on the solution ‘small’ or ‘large’?
  When the effect of a perturbation is investigated on an infinite time interval, such investigations belong to stability theory;

- **perturbation of the vector field**: what is the effect on solutions of perturbations in the parameters. Investigating such questions asks for the structural stability of the system.

To illuminate and motivate these concepts, consider the following examples of spring models

**Example 27 Spring models**

Consider the equation for a spring described with a potential energy function $V$ and with damping linear in velocity:

$$
\ddot{x} = -\delta \dot{x} - \frac{dV(x)}{dx};
$$

for a linear spring the potential is that of the harmonic oscillator: $V(x) = \frac{1}{2}x^2$, for the nonlinear spring we take $V(x) = \frac{1}{2}x^2 + \frac{1}{3}x^3$.

The characteristic properties of the solutions of these equations depend essentially on the parameters, as can be seen from the phase portraits in figure 3.1.

**Exercise 28** 1. For all these systems, consider perturbations of the equilibrium solution $x(t) = 0$. Motivate that this equilibrium solution is called stable.

2. Consider in the non-linear systems the other equilibrium solution $x(t) = -1$. Is this solution stable or unstable?

3. What is the effect of damping in the linear model? Give arguments why you would call a small damping a regular perturbation when you are interested in the motion on a finite time interval, but a singular perturbation when you are interested in long time intervals.
3.1. Linearization

3.1.1 Basic idea

The basic geometric idea of the derivative of a function $f$ of one variable $x$ is the fact that the derivative determines the tangent line at the graph of the function; analytically this means that near the point $(x_0, y_0)$, with $y_0 = f(x_0)$, the function values $f(x)$ are well approximated by the linear relation of the tangent line $y - y_0 = f'(x_0)(x - x_0)$. Stated differently

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(|x - x_0|);$$

$f'(x_0)$ determines how much the $y$-value changes when small changes in $x$ are considered.

Technically speaking, when it is defined, the derivative at $x_0$ is defined by the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

The ‘linear’ approximation $f(x_0) + f'(x_0)(x - x_0)$ of the function $f$ near $x_0$ is of course in general simpler to study than the (nonlinear) function itself.

This idea may therefore also be exploited when more difficult relations are considered; situations where the scalar $x$ is replaced by a vector, or even by a function itself, and the ‘function’ $f$ is more general a mapping, also called ‘operator’. Formally speaking, a mapping, or operator is a prescription $\mathcal{F}$ that assigns to an element of a set $U$ (the domain) one unique element from another set $V$, $\mathcal{F} : U \to V$. Standard terminology is as follows: when $U = \mathbb{R}^n$ and $V = \mathbb{R}^m$, $\mathcal{F}$ is an $m$-vector-function of $n$-variables; when $U$ is infinite dimensional, and $V = \mathbb{R}$, then $\mathcal{F}$ is called a functional and when both $U$ and $V$ are infinite dimensional, $\mathcal{F}$ is called an operator.

Example 29 Below we present various examples of operators/mappings.

1. The standard problem in Linear Algebra can be written as $\mathcal{F}(x) = 0$ with

$$\mathcal{F}(x) = A x - b$$

2. The pendulum equation with normalized time can be written like $F(\varphi(t)) = 0$, with

$$\mathcal{F}(\varphi(t)) = \ddot{\varphi} + \sin(\varphi)$$

3. The KdV-equation in normalized variables reads $F(u(t)) = 0$, with

$$\mathcal{F}(u(t)) = \partial_t u + u_{xxx} + u u_x$$

4. The $L^2$-norm (squared) of a function defined on $[0, 1]$ is defined with the functional

$$\mathcal{F}(u) = \int_0^1 u(x)^2 \, dx$$

while the energy of a KdV-solution is given by the functional $E(u)$ where

$$E(u) = \int \left[ \frac{1}{2} u_x^2 - \frac{1}{3} u^3 \right] \, dx.$$
3.1.2 Directional derivative, formal Frechet derivative

Let $\mathcal{F} : U \to V$ be a mapping and $u_0$ the point of interest; we want to investigate the change in $\mathcal{F}$ for $u$ near $u_0$. Instead of considering a full neighbourhood of $u_0$ (which needs some precise definitions), let us start to restrict the problem to a line through the point $u_0$. That is, let $v$ be any other element such that for small $\varepsilon$ the line element

$$R \ni \varepsilon \mapsto u_0 + \varepsilon v$$

is in the domain of $\mathcal{F}$; such element $v$ is called the direction of variation. Restricting $\mathcal{F}$ to this line element leads to a function of the scalar variable $\varepsilon$, with values in $V$:

$$f(\varepsilon) = \mathcal{F}(u_0 + \varepsilon v) \in V.$$

**Definition 30** If it makes sense, the derivative of $f(\varepsilon) = \mathcal{F}(u_0 + \varepsilon v)$ with respect to $\varepsilon$ at $\varepsilon = 0$ defines the so-called *directional derivative of $\mathcal{F}$ at the point $u_0$ in the direction $v$*, and is denoted by $D\mathcal{F}(u_0, v)$, or simply $\mathcal{F}'(u_0; v)$:

$$\mathcal{F}'(u_0; v) = \left( \frac{d}{d\varepsilon} \mathcal{F}(u_0 + \varepsilon v) \right)_{\varepsilon=0}$$

$$= \lim_{\varepsilon \to 0} \frac{\mathcal{F}(u_0 + \varepsilon v) - \mathcal{F}(u_0)}{\varepsilon}.$$

This formula should be used as the *algorithm* to calculate the directional derivative. The algorithm for the directional derivative can be executed for all directions $v$ for which $R \ni \varepsilon \mapsto u_0 + \varepsilon v$ is in the domain of $\mathcal{F}$.

In many cases the resulting expression $\mathcal{F}'(u_0, v)$ is linear in the direction of variation $v$; in that case one writes

$$\mathcal{F}'(u_0; v) \equiv \mathcal{F}'(u_0) v$$

and the linear mapping $\mathcal{F}'(u_0)$ is called the *formal Frechet derivative*\(^2\). Also the notation $\delta \mathcal{F}(u_0)$ or $\delta_{u_0} \mathcal{F}$ is used. The restriction of the mapping to the line can then be approximated by a linear function:

$$\mathcal{F}(u_0 + \varepsilon v) = \mathcal{F}(u_0) + \varepsilon \mathcal{F}'(u_0) v + O(\varepsilon).$$

**Example 31**

1. If $U = \mathbb{R}^3, V = \mathbb{R}^m$, and all partial derivatives of $\mathcal{F}$ exist, then the formal Frechet derivative is the well known *Jacobi matrix*

$$\mathcal{F}'(u_0) = \nabla \mathcal{F}(u_0).$$

(If all elements depend continuously on $x_0$ this is the Frechet derivative.)

2. If the operator $\mathcal{F}$ is linear, the derivative at each point is the same, and is in fact given by the operator itself:

$$\mathcal{F}'(u_0) = \mathcal{F}$$

if $\mathcal{F}$ is linear.

3. If $u_0 \in L_2(0,1)$ then for any $v \in L_2(0,1)$ the (formal) Frechet derivative of $\mathcal{F}(u) = \int_0^1 \frac{1}{2} u^2$ at $u_0$ is defined and given by

$$\mathcal{F}'(u_0) v = \int_0^1 u_0(x) v(x) \, dx.$$

4. For the functional $E(u) = \int \left( \frac{1}{2} u_x^2 + \frac{1}{3} u^3 \right) \, dx$

the directional derivative is linear in $v$ and given by

$$E'(u) v = \int \left[ u_x v_x + u^2 v \right] \, dx.$$

5. For the differential operator of the pendulum equation $\mathcal{F}(\psi) = \ddot{\psi} + \sin(\psi)$ the directional derivative is linear and given by

$$\mathcal{F}'(\psi) \psi = \ddot{\psi} + \cos(\psi) \psi.$$

3.2 Linearization around equilibrium

Now consider again an evolution equation, written generally like before: $\partial_t u = K(u)$. Especially when the equation is nonlinear, solutions cannot be found in general. An exception may be some special (and possibly important) solutions. Suppose that we can find an *equilibrium solution*, i.e. a solution $U$ that does not depend on time:

$$K(U) = 0.$$
3.2. LINEARIZATION AROUND EQUILIBRIUM

Then it is natural to investigate what happens with solutions ‘near’ this known solution. For solutions in a neighbourhood of $U$ we write symbolically $u(t) = U + \varepsilon \xi(t)$, where $\varepsilon$ is introduced as a ‘small’ parameter, indicating that $\varepsilon \xi(t)$ is small as long as $\xi(t)$ remains bounded (which may not be the case for large $t$). Since we want $u(t)$ to be a solution it should hold that

$$\partial_t(U + \varepsilon \xi(t)) = K(U + \varepsilon \xi(t)).$$

This can be rewritten, if we use the facts that $\partial_t U = 0$, and $K(U) = 0$, like

$$\partial_t \xi(t) = \frac{K(U + \varepsilon \xi(t)) - K(U)}{\varepsilon}.$$

In the limit $\varepsilon \to 0$ the rhs. becomes the directional derivative; assuming it to be linear, there results

$$\partial_t \xi(t) = K'(U) \xi(t)$$

where $K'(U)$ is (a differential operator in spatial variable) acting linearly on $\xi(t)$.

**Definition 32** The linear equation for the ‘perturbation’ $\xi(t)$

$$\partial_t \xi(t) = K'(U) \xi(t)$$

is called the **linearization** of the equation $\partial_t u = K(u)$ around the equilibrium solution $U$.

The above procedure, and name, is a generalization of the known linearization procedure for ordinary differential equations.

Observe that linearization procedure can be seen as a perturbation procedure where the initial condition is perturbed: for the same equation, consider different solutions by perturbing the initial condition

$$u(0) = U + \varepsilon \xi_0 \text{ with } \xi_0 = \xi(0).$$

**Example 33 1.** Consider the pendulum equation in normalized variables:

$$\ddot{\varphi} + \sin(\varphi) = 0$$

Considering a perturbation (of the initial value), the linearization about a given solution $\varphi(t)$ reads

$$\ddot{\psi} + \cos(\varphi(t)) \psi = 0$$

meaning that $\varphi(t) + \varepsilon \psi(t)$ will be a solution that is ‘close’ to $\varphi(t)$ if $\varepsilon$ is small and as long as $\psi$ is bounded.

Recall from stability theory for ode’s that if all solutions $\psi$ are bounded for all positive times, the solution $\varphi$ is called **linearly stable**, while if at least one solution $\psi$ is unbounded, the solution $\varphi$ is called **linearly unstable**. Then the equilibrium solution $\varphi \equiv 0$ is linearly stable, while the equilibrium solution $\varphi \equiv \pi$ is unstable. This agrees with the global picture of the solutions of the pendulum equation that is shown in the phase portrait.

2. **Infinitesimal surface waves.**

Consider the KdV equation as model for the waves on the surface of a layer of fluid. In terms of the surface elevation $u(x,t)$, the equation is given by:

$$\partial_t u + c (\partial_x u + \beta \partial_x^3 u + \gamma u \partial_x u) = 0.$$

Clearly, a flat fluid surface is a solution of this equation, in fact an equilibrium solution: $u(x,t) = 0$. Looking for small surface elevations, so-called infinitesimal waves, is precisely the linearisation procedure above: writing $u(x,t) = \varepsilon v(x,t)$, the linearised KdV equation reads

$$\partial_t v + c (\partial_x v + \beta \partial_x^3 v) = 0. \quad (3.3)$$

**Exercise 34** Study some properties of the linearised KdV equation (3.3) as follows.

1. For $\beta = 0$, the equation is the **translation equation**: any initial profile $f$ is translated with the speed $c$:

$$v(x,t) = f(x - ct) \text{ if } v(x,0) = f(x).$$

2. For $\beta \neq 0$, solutions of this linear equation can be found by Fourier transformation techniques, or (as a generalisation of a known technique for linear ordinary differential equations) by looking for solutions in the form of (complex) exponentials. Show that special solutions exist of the form

$$v(x,t) = A e^{i(kx - \omega t + \phi)}$$
for an arbitrary amplitude $A$, and an arbitrary phase factor $\phi_0$, provided the so-called wave number $k$, and the frequency $\omega$, are related in a specific way by the so-called dispersion relation. Derive this dispersion relation. Solutions of this form are called monochromatic solutions. (Note that since the equation is linear and real, the real and imaginary parts give real solutions.)

3. At a fixed position $x_0$, $\omega$ determines the frequency of the disturbance of the surface. Explain the meaning of the wave number $k$ in terms of the spatial period of the wave profile (the wave length).

4. The monochromatic solution translates undisturbed in shape with the so-called phase velocity:

$$V_{\text{phase}} = \frac{\omega}{k}.$$

Show that waves of different wave length have different phase velocity. What is the velocity for infinitely long waves?

3.3 Sensitivity: Regular and Singular perturbations

We start with a general description of investigating perturbations of a certain ‘system’. After that we will specify to perturbations of model equations we are interested in.

Let $u$ be the state variable of a certain system; $u$ may be a scalar or vector variable of one or more variables. Let a model (of a certain phenomenon) of the system be described in an abstract way as

$$E(u, p) = 0. \quad (3.4)$$

This expression is symbolic notation, for instance for a (partial) differential operator, together with boundary conditions and initial data.

The presence of model parameters is indicated explicitly by $p$. If the “unperturbed” problem corresponds to the value $p_0$, which in general is a problem that is known, or special, problems $E(u, p) = 0$, with $p$ near $p_0$ are considered as “perturbed” problems.

A ‘good’, reliable, model would be one for which the solutions of the perturbed model agree quite well with those of the unperturbed model. In many cases, the unperturbed system, corresponding to the value $p_0$, has a unique solution, to be denoted by $u_0$:

$$E(u_0, p_0) = 0;$$

$u_0, p_0$ will be called the solution of the unperturbed system. For $p$ near $p_0$ we consider (3.4) as a perturbed problem, and distinguish two cases:

Definition 35 The problem $E(u, p) = 0$ is called a regular perturbation of the particular solution $u_0, p_0$ if for small changes in $p$ there exists a unique solution, to be denoted by $u(p)$, such that the difference $u(p) - u_0$ is “small” (such that $\lim_{p \to p_0} u(p) = u_0$).

Definition 36 The problem $E(u, p) = 0$ is called a singular perturbation of the so-called limit problem $E(u, p_0) = 0$ if either

- this limit problem has no solution (while for $p$ different from $p_0$ there are solutions),
- or if there are more solutions of $E(u, p) = 0$ near the unique solution $u_0, p_0$,
- or if $\lim_{p \to p_0} u(p) \neq u_0$.

3.3.1 Linearization from perturbation of initial conditions

For a given evolution equation $\partial_t u = K(u)$, the study of the effect of perturbing the initial condition can be formulated in the way as above by including the initial condition in the definition of the model:

$$E(u, u_0) = \left( \frac{\partial_t u - K(u)}{u(0) - u_0} \right) = 0$$

An equilibrium solution $U$ for which $K(U) = 0$, corresponds to $E(U, U) = 0$, which can be considered as the limit problem, while the perturbed problem investigates the solution when a perturbation of the initial condition is considered:

$$E(u, U + \varepsilon v_0) = \left( \frac{\partial_t u - K(u)}{u(0) - (U + \varepsilon v_0)} \right) = 0$$

The previous section can thus be rephrased in this terminology.
3.3. **SENSITIVITY: REGULAR AND SINGULAR PERTURBATIONS**

3.3.2 **Perturbation and sensitivity of the model equation**

In many of the equations considered up to now, various parameters turn up, and with perturbation theory we investigate such questions as: if a coefficient is large/small, is the corresponding phenomenon dominant/negligible?

Let us write the equation in the general form

\[ \mathcal{P}_p : \quad \partial_t u = K_p(u) \]  

(3.5)

where the dependence of the vector field on parameters is indicated by the subscript \( p \). Suppose that \( p = 0 \) corresponds to a `standard' problem for which we want to study perturbations. Then the limit problem is given by

\[ \mathcal{P}_0 : \quad \partial_t u = K_0(u) \]  

(3.6)

and the interest is to find the relation between solutions of \( \mathcal{P}_p \) and those of \( \mathcal{P}_0 \); in particular: if \( \mathcal{P}_0 \) has a (known) solution, what is the relevance for solutions of \( \mathcal{P}_p \) with small \( p \)?

**Examples**

In the following we present various examples to give simple illustrations of these notions.

1. **Algebraic equation**
   
   For \( \varepsilon x^2 + x - 1 = 0 \) the solutions are given by
   
   \[ x_{1,2} = -1 \pm \sqrt{1 + 4\varepsilon} \]
   
   i.e. \( x_1 = 1 + O(\varepsilon) \), \( x_2 = -O(\frac{1}{2}) \). The limit problem (\( \varepsilon \rightarrow 0 \)) is given by \( x - 1 = 0 \), with the only solution \( x = 1 \), while for \( \varepsilon \neq 0 \) there are two solutions. (Make graphs.)

2. **Resonance**
   
   Consider the forced harmonic oscillator, with a time periodic forcing, a simple model for a swing. Formulated as an initial value problem, choosing a specific initial condition, the problem reads
   
   \[ \partial_t^2 u + \omega_0^2 u = \varepsilon \sin \omega t, \quad u(0) = 1, \quad \partial_t u(0) = 0. \]
   
   The limit problem for \( \varepsilon = 0 \) has the bounded solution
   
   \[ u_0(t) = \cos \omega_0 t, \]
   
   while for \( \varepsilon \neq 0 \) the solution will depend critically on the frequency of the forcing.
   
   - When \( \omega \neq \omega_0 \), the solution is given by
     \[ u_e(t) = \cos \omega_0 t - \frac{\varepsilon}{\omega^2 - \omega_0^2} \sin \omega t. \]  
     (3.7)
   
   This solution shows that the periodic forcing causes an effect that has the same frequency as the driving frequency; the amplitude depends on the strength \( \varepsilon \) of the forcing, and on the difference of the frequencies. This is the so-called non-resonant case. In the terminology of above, for small \( \varepsilon \) the forcing is a regular perturbation of the limit problem.
   
   - When \( \omega = \omega_0 \) the solution is given by
     \[ u_e(t) = \cos \omega_0 t - \frac{\varepsilon}{2\omega_0} t \cos \omega_0 t. \]  
     (3.8)
   
   It is observed that on the half line \( t \in [0, \infty) \) the solution is unbounded. No matter how small \( \varepsilon \) is, the forcing has an effect that grows linearly in time. This phenomenon is known as resonance. In the terminology of above, for small \( \varepsilon \) the forcing is a singular perturbation of the limit problem.
   
   - Note that the resonant solution (3.8) can in fact be found from the non-resonance solution (3.7) by taking the limit \( \omega \rightarrow \omega_0 \).

3. Consider again the simple harmonic oscillator with frequency \( \omega \):
   
   \[ \ddot{x} + \omega^2 x = 0. \]
4. \textit{Boundary layers}

The equation for non-negative $\varepsilon$, the problem

$$\varepsilon \partial_x u + u = 0, \quad x > 0, \quad u(0) = 1,$$

is a singular perturbation of the limit problem:

$$u = 0, \quad x > 0, \quad u(0) = 1$$

since this limit problem has no regular (continuous) solution. The exact solution for $\varepsilon > 0$:

$$u(x, \varepsilon) = e^{-x/\varepsilon}$$

shows the phenomenon of a boundary layer: for small $\varepsilon$, a fast transition (for $x$ in an interval of length $O(\varepsilon)$) from the boundary value ($u(0) = 1$) to the value of the limit function $u = 0$.

5. \textit{Global effects}

For the equation with non-negative $\varepsilon$

$$\varepsilon^2 \partial_x^2 u + u = 0, \quad x > 0,$$

and initial values

$$u(0) = 0, \quad u_x(0) = 1$$

the solution of the limit problem is not defined. The exact solution for $\varepsilon > 0$:

$$u(x, \varepsilon) = \varepsilon \sin(x/\varepsilon),$$

is small in sup-norm, and

$$\lim_{\varepsilon \to 0} |u(x, \varepsilon)| = 0,$$

but, since $\partial_x u(x, \varepsilon) = \cos(x/\varepsilon)$,

$$\lim_{\varepsilon \to 0} |\partial_x u(x, \varepsilon)|$$

does not exist.

This example does not show the phenomenon of a boundary layer: even for small $\varepsilon$ the solution behaves very different on any interval $[0, X]$.

6. \textit{Bifurcation}

Consider the ode for the scalar function $u(t)$:

$$\dot{u} = f(u, \lambda) \equiv u(\lambda - u^2) \quad (3.9)$$

Now we want to investigate the effect of a perturbation in the frequency, keeping the initial data fixed, say

$$x(0) = 1, \quad \dot{x}(0) = 0.$$

For given value $\omega_0$ the solution is given by $x_0(t) = \cos(\omega_0 t)$. A perturbation $\delta \omega$ in the frequency, $\omega = \omega_0 + \delta$ leads one to study the linearized equation:

$$\ddot{\xi} + \omega_0^2 \xi = \varepsilon x_0(t)$$

with $\varepsilon = -2 \omega_0 \delta$, and $\xi(0) = 0, \dot{\xi}(0) = 0$. Since $x_0(t)$ is periodic with period $\omega_0$, resonance occurs in the eqn. for $\xi$. In fact, the solution $\xi$ reads

$$\xi(t) = -\delta t \sin(\omega_0 t)$$

and leads to the approximate solution

$$x(\omega, t) = \cos(\omega_0 t) - t \delta \sin(\omega_0 t) + \ldots$$

Compare this with the actual solution:

$$x(\omega, t) = \cos((\omega_0 + \delta) t),$$

and observe the essential difference: while the actual solution is periodic (with period related to the perturbed frequency $\omega_0 + \delta$) the approximation is unbounded in time. This can be understood from the observation that the approximation consists of the first two terms of the Taylor expansion of the exact solution

$$\cos((\omega_0 + \delta) t) = \cos(\omega_0 t) - t \delta \sin(\omega_0 t) + \ldots$$

The approximation with only two terms is valid provided the product $t \delta$ is small, which is only true as long as $t$ remains sufficiently small. Hence, the approximation constructed in the way above is only valid for ‘small’ time scales, say $t \delta \ll 1$. In general we are interested in results that are valid on larger time scales, and other methods have to be used, so-called asymptotic methods. In chapter 7 asymptotic results are found for two typical types of problems for oscillators, using the \textit{WKB method}, and \textit{self-excitation}. 

Now we want to investigate the effect of a perturbation in the frequency, keeping the initial data fixed, say

$$x(0) = 1, \quad \dot{x}(0) = 0.$$
where $\lambda$ is a real parameter.
First consider the equation for the equilibrium solutions, i.e. the solutions of the algebraic equation

$$ u(\lambda - u^2) = 0. $$

There is only one real solution for $\lambda \leq 0$, namely $u_0 = 0$, while there are three different real solutions for $\lambda > 0$:

$$ u_0 = 0, \text{ and } u_\pm = \pm \sqrt{\lambda} $$

This is an example of bifurcation: the zero solution “bifurcates” at the bifurcation point $\lambda = 0$ into three solutions, two of which are non-zero. This particular branching is the so-called pitch fork bifurcation, the simplest type of bifurcation.

Now investigate the stability of these equilibrium solutions found above. A simple analysis gives the following results.

- The (only) equilibrium $u_0 = 0$ for $\lambda < 0$ is stable.
- For $\lambda > 0$ the trivial equilibrium $u_0$ is unstable, while both nontrivial equilibria $u_\pm = \pm \sqrt{\lambda}$ are stable.

This phenomenon is commonly referred to as an exchange of stability: the new equilibrium solutions that bifurcate from the trivial solution are stable, while at the bifurcation point, when $\lambda$ crosses zero, the stable equilibrium looses its stability. In fact, the situation here is the characteristic example of super-critical bifurcation\(^3\).

**Conclusion 38** The conclusion from these simple examples should be a warning to be careful with neglecting ‘small’ terms; in some cases (regular perturbations) neglecting the small term does not affect the solution very much; for singular perturbations, neglecting a small term may drastically change (the solution of) the problem; which case is encountered will in general also depend on what property is investigated (which norm is used).

\(^3\)Bifurcation at the bifurcation point $\lambda_0$ of a solution $u_0$ is defined to be super-critical if $u_0$ is a stable solution of $f(\cdot, \lambda_0)$ and sub-critical when $u_0$ is unstable.

### 3.4 Exercises

#### 3.4.1 Linearised car flow

Consider the model for car flow, for anticipating drivers:

$$ \partial_t \rho + V(\rho) \partial_x \rho = \frac{\alpha}{2} \partial_x^2 (\rho^2) \quad (3.10) $$

with $V(\rho) = \frac{d}{d\rho} (\rho v_{\text{car}}(\rho))$ and a simple linear relation for $v_{\text{car}}$ as in Chapter 2. Take $\alpha$ as a given nonnegative constant, $\alpha \geq 0$.

1. Show that any constant density $\rho_0$ is a solution of this equation. Determine the density velocity and the car velocity at that density. Linearise the equation around the constant car density $\rho_0$.
2. First consider the case $\alpha = 0$; give the general solution for the equation. Explain the meaning of the solution corresponding to an initial situation of a localised higher car density (distinguish the cases that the density velocity is positive and negative).
3. Can you give in the case of problem 3, the motion of an individual car? Explain the solution you get.
4. Now consider $\alpha > 0$ in the linearised equation. Introduce a moving frame of reference moving with the density velocity $V(\rho_0)$. Write down the linearised equation in this frame of reference. Do you recognise the equation?
5. Suppose that initially, in the moving frame above, the perturbation is described by a harmonic function with small amplitude. Determine the evolution of this perturbation. Explain your result. Are there cars that keep the same velocity all the time in this situation?

#### 3.4.2 Population growth and migration

We study population dynamics of a species that can migrate in one direction (the $x$-axis).

1. Population growth without migration. Suppose that when the population is uniformly distributed in space, the growth is exponential with growth factor $\alpha$ when the population is small, but that there is a bound on the
maximal size of the population. Define a normalised density of the population \( u \) such that \( u \in [0, 1] \) with 1 corresponding to the maximal population size. Show that the following ode is a simple model for the population growth:

\[
\dot{u} = \alpha u(1 - u)
\]

and determine the solution with initial value \( u_0 \in [0, 1] \).

2. Find the equilibrium population densities, and determine their stability.

3. Now suppose that a spatially non-uniform population \( u(x, t) \) will migrate in a way as modelled by Fourier’s law for diffusion (heat conduction). Show that the following equation is then a simple model for the migration:

\[
\partial_t u = \kappa \partial_x^2 u
\]

where \( \kappa \) is a constant; what name would you assign to \( \kappa \)?

4. Now combine both effects by simply adding the two terms in the vector field:

\[
\partial_t u = \kappa \partial_x^2 u + \alpha u(1 - u)
\]

This equation has as equilibrium densities \( U_0 = 0, U_1 = 1 \); determine the linearization of the equation around each equilibrium solution.

5. Investigate harmonic solutions of the linearized equation around \( U_1 \); derive the dispersion relation and conclude that this equilibrium is linearly stable.

6. Investigate the linearized equation around \( U_0 \); derive the dispersion relation. Conclude that perturbations with long wave length will grow in time (which implies that this equilibrium is linearly unstable).

7. Suppose that the spatial domain is bounded, and that no migration can take place outside the interval. Investigate what happens with an arbitrary small, concentrated initial population. Will your conclusion remain true when at the boundaries of the interval the population density is kept at a zero level (there is a deep valley in which migrating individuals disappear); investigate.

### 3.4.3 Lindstedt’s method

We consider the oscillations of a pendulum-type equation

\[
\frac{d^2 x}{dt^2} + x - \varepsilon x^3 = 0.
\]

1. Make a phase diagram for \( \varepsilon = 0 \), for a soft spring \( \varepsilon > 0 \), and for a hard spring \( \varepsilon < 0 \).

2. The solution \( x_\varepsilon(t) \) with initial conditions

\[
x(0) = A, \quad \frac{dx}{dt}(0) = 0,
\]

is periodic if \( A \) is small enough. Determine the solution \( x_0(t) \) and its frequency.

Due to the cubic term, the frequency of this unforced vibration will depend on the amplitude \( A \).

3. Substitute a regular expansion

\[
x_\varepsilon(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + ...
\]

and show that already in first order in \( \varepsilon \), resonance is observed (which means that the series does not approximate the periodic \( x_\varepsilon(t) \) uniformly on \( t \geq 0 \)).

4. Anticipating that \( x_\varepsilon(t) \) will be periodic, scale time \( t \) with the (unknown) frequency \( \omega \) of \( x_\varepsilon(t) \), \( \tau = \omega t \), and write \( \omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + ... \)

Derive the scaled equation using \( \tau \) and substitute the series for \( x_\varepsilon \) and \( \omega \).

5. Show that \( \cos^3 t = \frac{3}{4} \cos t + \frac{1}{4} \cos 3t \).

Now solve the equations in 4. up to first order in \( \varepsilon \). Return to the variable \( t \) to find (the first order approximation of) the periodic solution \( x_\varepsilon \) and show that

\[
\omega = 1 - \frac{3}{8} \varepsilon A^2 + O(\varepsilon^2)
\]

gives the dependence of frequency on amplitude.
Chapter 4

Basic concepts and phenomena 1

Abstract

When dealing with pde’s from Mathematical Physics that describe various phenomena from nature, such as wave phenomena, several concepts turn up regularly that describe some characteristic properties:

- **linearity and nonlinearity.** When the equation is linear, superposition of solutions produces another solution, which makes Fourier methods applicable; this leads to the study of monochromatic modes, $e^{ikx-\omega t}$, which are the pde-analogous of the simple exponentials that are substituted in linear ode’s to find a basis of independent solutions. When the equation is nonlinear, the superposition principle does not hold. Here we first consider linear equations.

- **dissipation.** When the monochromatic mode has frequency $\omega$ that is not real for real wave number $k$, the time factor $e^{-i\omega t}$ is exponentially increasing (amplification) or exponentially decreasing (dissipation). The amplification/dissipation may depend on the wave number, which leads to a different decay of different modes, a simple source of “self-organization”.

- **dispersion.** When the monochromatic modes are both periodic in space and time, i.e. when $\omega$ is real for real $k$, ‘dispersion’ refers to the property that the propagation speed of monochromatic modes depends on the wave length of the modes; a wave consisting of a superposition of various modes shows a complicated pattern. Concepts like dispersion relation, phase velocity and group velocity will be dealt with.

In “Basics 1” we introduce and illustrate the concepts of dispersion and dissipation for linear equations from scratch. For clarity we consider the simplest possible equations, in particular we restrict to equations of first order in time, and with one space variable. Also we restrict ourselves to equations and solutions on the whole real line, avoiding boundary conditions.

The description of the concepts is necessarily brief and cannot replace a good introductory text.

Accompanying Maple Work Sheets

The concepts to be introduced here can be visualized clearly using Maple; for instance the profiles of a wave group, the propagation with the group velocity, can be plotted and animated rather easily. The Maple Work Sheet (MWS) “Basics1.mws” deals with the topics of this text; all figures are taken from this work sheet, which contain many more and the animations. For an optimum use of the mws, a good understanding of the text is required.

4.1 Monochromatic modes, dispersion relation

For a function $u = u(x,t)$ with $x$ the space variable and $t$ the time, consider a linear evolution equation of the form

$$\partial_t u = Lu \quad (4.1)$$

Here, $L$ is a differential operator in $x$, with constant coefficients. That means that for a certain
polynomial $p = p(\zeta)$ the effect of $L$ on the function $u$ is the function given by
\[(Lu)(x) = p(\partial_x)u(x)\]
For instance, for
\[p(\zeta) = 7\zeta + 3\zeta^2 + \zeta^3,\]
the corresponding operator is given by
\[Lu = 7\partial_x u + 3\partial_x^2 u + \partial_x^3 u \equiv 7u_x + 3u_{xx} + u_{xxx}.\]
In particular it holds that
\[Le^{ikx} = p(ik)e^{ikx},\]
i.e. each $x$-derivative is just multiplication by $ik$ for this monochromatic function.

The function $p$ is called the *symbol* of the linear operator $L$ (called ‘pseudo-differential operator’ when $p$ is not a polynomial)\(^1\).

The evolution equation (4.1) is a *linear* equation: superposition of solutions leads to another solution. As one consequence of this, the real and imaginary part of complex valued solutions are solutions as well; this makes it possible to look for complex-valued solutions when that is easier for notation, as we shall do for monochromatic solutions.

Just as for linear ODE’s with constant coefficients, one may look for a basis of solutions that consists of exponentials. In the case of (4.1) this Ansatz leads to consider functions of the form
\[e^{i(kx-\omega t)};\]
here $k$ is the *wave number*, which is related to the wave length $\lambda$ by
\[\lambda = \frac{2\pi}{k},\]
and $\omega$ the *frequency.* Usually we will take $k$ real; $\omega$ may be complex.

Substitution in (4.1) leads to
\[\partial_t u - Lu = [-i\omega - p(ik)]e^{i(kx-\omega t)}.\]
This is a solution provided $\omega$ and $k$ are related such that the expression between square brackets vanishes:
\[\omega = \Omega(k) \equiv i p(ik).\]
This is the so-called *dispersion relation*: the relation to be satisfied between the *wavenumber* $k$ and the *frequency* $\omega$ in order that $e^{i(kx-\omega t)}$ is a solution.
For instance, for the example above, one finds:
\[\Omega(k) = -7k - 3ik^2 + k^3.\]
The corresponding solutions
\[e^{i(kx-\Omega(k)t)}\]
are called *monochromatic solutions* (or *modes*) since only one wave number is present.

Since the functions $x \mapsto e^{ikx}, k \in \mathbb{R}$ form a basis (in $L_2$-sense) of $(L_2)$ functions (according to Fourier-integral theory), any initial value $u_0(x)$ can be expanded:
\[u_0(x) = \int_{-\infty}^{\infty} \hat{u}_0(k) e^{ikx} \, dk\]
where $\hat{u}(k)$ denotes the Fourier-transform of $u_0(x)$, the *spectral density* or spectral function. From linearity, the solution of (4.1) can then be written down in principle:
\[u(x,t) = \int_{-\infty}^{\infty} \hat{u}_0(k) e^{i(kx-\Omega(k)t)} \, dk \quad (4.2)\]
(provided the integral makes sense).

This shows that the solution of the initial value problem for (4.1) can be written down explicitly in terms of a Fourier integral. The properties of the solution will be determined by the specific dispersion relation, i.e. by the function $k \mapsto \Omega(k)$.
It turns out that essentially two different cases can be considered:

---

\(^1\) *Pseudo-differential operators* are the generalization to operators for which $p$ is not necessarily a polynomial. For arbitrary function $p$ the operator with $p$ as symbol is defined as follows:
If $\hat{u}$ denotes the Fourier transform of $u$, then the relation between the operator $L$ and its symbol $p$ is given by
\[L\hat{u}(k) = p(ik)\hat{u}(k)\]
4.3. **DISPERSION**

- $\Omega(k)$ is real valued for real $k$.
  In that case the monochromatic solutions are periodic in time, not damped, and each one is a translation in space; this is the conservative case, and the solutions behave like waves.
- $\Omega(k)$ is not real for real $k$.
  In particular, when $\Omega(k)$ is purely imaginary, the mode with wave number $k$ is exponentially decaying or amplified, depending on the sign. When decaying, this is the dissipative case, as appears for instance in diffusion problems.
  When $\Omega$ has also a real part, a combination of translation and dissipation is present.

In the following sections we will investigate these cases in detail.

### 4.2 Translation equation as simplest wave equation

We start with the most simple equation, TransE

$$\text{TransE} := \partial_t u(x, t) + c \partial_x u(x, t) = 0$$

Solutions of this equation are just translations of a given (initial) profile function $f$ with velocity $c$; that is, for every $f$,

$$u(x, t) = f(x - ct)$$

is a solution.

This solution can be interpreted as a *travelling wave*: the wave profile $f$ is translated with velocity $c$ along the $x$-axis: in the positive direction if $0 < c$, and in the negative direction if $c < 0$.

A good view of the motion is possible by plotting the wave profile at different times in the same plot; to indicate the increase of time, we shift the profile vertically with increasing time. For instance:

![Graph of travelling waves](image)

For reasons that will be clear, TransE is called the translation equation: its mathematical properties show that it can be viewed to describe the simplest wave phenomena when we give the interpretations of wave profile and velocity as has been done above.

**Exercise 39**

1. The dispersion relation is given by $\Omega(k) = ck$. Derive the solution with initial condition $f$ using the solution (4.2).

2. Animate the motion for various wave profiles and various values of $c$.

### 4.3 Dispersion

Now we consider the phenomenon of dispersion, present in somewhat more difficult (and more realistic, although linear) wave equations.

Simply stated, *dispersion* is the effect that monochromatic modes with different wave number (i.e., different wave length) propagate with different speed (the phase speed). As a consequence, although each mode travels undisturbed in shape, a ‘wave’ that consists of a superposition of two (or more) different modes, shows a profile that changes constantly.

Another interesting phenomenon is visible when superpositions of waves with only slightly different wave numbers are considered; then interference will happen at each time and this may cause the spatial profile to be very small outside a certain region. However, due to the slightly different phase speeds of the constituent waves, the interference pattern will translate; the speed of this translation is approximately the group velocity, a second and most relevant velocity for dispersive equations.

For the examples and plots we will take in the following the typical equation

$$\partial_t u + c \partial_x u + a \partial_x^3 u = 0$$

The dispersion relation is given by

$$\Omega = ck - a k^3$$

and the (real) monochromatic modes are

$$\sin(k x - (ck - a k^3) t)$$
4.3.1 Phase velocity

The monochromatic mode with wave number $k$ can be written like

$$e^{i(kx-\Omega t)} = e^{ik(x-V_{ph}t)}$$

where

$$V_{ph}(k) = \frac{\Omega(k)}{k}$$

Since we assume $\Omega(k)$ to be real, $V_{ph}$ is real, and the monochromatic mode is a translation with speed $V_{ph}$, the so-called phase velocity.

For the translation equation, the phase velocity is independent of $k$, but for more general equations the phase velocity is different for different wave numbers.

For the standard example the phase velocity is plotted below.

This graph of the phase velocity (for $c = 1, a = 1$) shows that the phase velocity can be positive or negative, depending on the wave number. For instance, for $k = 1$ the velocity is positive (wave travels to the right), while for $k = 4$ the velocity is negative (wave travels to the left).

**Superposition:** from the linearity of the equation, it follows that a superposition of monochromatic solutions is a solution again. Although the constituent modes are simple, the superposition shows a pattern that varies in a rather complicated way, already for two waves.

**Exercise 40**

1. Animate the superposition of two monochromatic modes, e.g. one with wave number 1 and amplitude 1 and one with wave number 2 of amplitude 1/2.

2. Take as initial spectrum a block function (nonzero for $1/2 \leq k \leq 2$). Animate the initial profile and the evolution. (Approximate the Fourier integral by a Riemann sum.)

3. Do the same when the block function is nonzero for $1 \leq k \leq 5$.

4.3.2 Group velocity

The velocity of one monochromatic mode, the phase velocity, has a clear interpretation: the speed at which the profile is translated. For a superposition (of already two) modes with different wave numbers (which typically have different phase velocities), there is no clear notion of velocity in general.

Nevertheless, the concept of group velocity turns out to describe a kind of collective velocity for superposition of many waves. We will describe three cases in which the group velocity is relevant.

**Definition 41** For a problem with dispersion relation $\omega = \Omega(k)$, the group velocity is defined to be

$$V_{gr}(k) := \partial_k \Omega(k).$$
4.3. DISPERSION

Bichromatic waves: beats

Consider the (special) superposition of two monochromatic waves:
\[ u(x, t) = e^{ik_1 x - \omega_1 t} + e^{ik_2 x - \omega_2 t} \]
(or the real part if desired). Introduce the averaged wavenumber and frequency
\[ k_0 := \frac{1}{2} (k_1 + k_2); \quad \omega_0 := \frac{1}{2} (\omega_1 + \omega_2); \]
and the differences:
\[ \Delta k := \frac{1}{2} (k_1 - k_2); \quad \Delta \omega := \frac{1}{2} (\omega_1 - \omega_2). \]

Then it is easily verified that
\[ u(x, t) = e^{ik_0 x - \omega_0 t} \times 2 \cos(\Delta k \ast (x - \frac{\Delta \omega}{\Delta k} t)); \]
this expression shows that the solution consists of the multiplication of the carrier wave \( e^{ik_0 x - \omega_0 t} \) with a modulation \( \cos(\Delta k \ast (x - \frac{\Delta \omega}{\Delta k} t)) \). For small \( \Delta k \), the modulation has long wave length \( \approx \frac{1}{\Delta \omega} \), and propagates with the group velocity of the center wave number:
\[ \frac{\Delta \omega}{\Delta k} \approx \partial_k \Omega(k_0). \]

If \( \Delta k \) is small compared to \( k_0 \), the modulation has the optical effect that the amplitude of the fast oscillation with \( k_0 \) changes sinusoidal with the longer wave length \( \frac{1}{\Delta \omega} \). At each time the composition looks like a chain of “beats”; in time, the beats propagate along the chain with the group velocity. A simple illustration is shown below, with \( k_0 = 15, \Delta k = 1 \).

![Beat Pattern](image)

Exercise 42 Animate the evolution with Maple. (Warning: see the effect of changing the option “numpoints” in the animation!!).

Wave groups

A wave group is a superposition consisting of a collection of waves with wave numbers centered around one wave number, say \( k_0 \), i.e. the spectral function \( \hat{f}(k) \) vanishes (or is vanishingly small) outside a small neighbourhood of \( k_0 \).

Exercise 43 A typical example of a spectral function is a Gaussian, with standard deviation \( \sigma \):
\[ \hat{G}(k) = \frac{1}{\sigma \sqrt{2 \pi}} e^{-\frac{(k-k_0)^2}{2\sigma^2}} \]

Determine the corresponding spatial profile \( G \). Investigate for \( k_0 = 0 \) how the value of \( \sigma \) determines the width of \( \hat{G} \) and the spatial extension of \( G \). Investigate the effect of \( k_0 \neq 0 \).

A wave group centered around a wave number \( k_0 \) that is sufficiently large compared to the extension of the spectral function, is a superposition of constituent waves for which the wave length is small compared to the large spatial extension of the envelope; the envelope of such a wave group travels approximately with the group velocity. Analytically this can be seen in a rough way as follows.

The dispersion relation is approximated by a few Taylor terms around \( k_0 \):
\[ \Omega(k) = \Omega(k_0) + V_{gr}(k_0)(k - k_0) + \mathcal{O}((k - k_0)^2). \]

Provided the spectral function \( \hat{f} \) is sufficiently peaked at \( k_0 \), and if \( V_{gr}(k_0) \) doesn’t vanish, the contribution of the higher order terms can be neglected in the expression for the solution:
\[ u(x, t) = \int_{-\infty}^{\infty} \hat{f}(k) e^{i(k x - \Omega(k) t)} dk \]
\[ \approx e^{-i(\Omega(k_0) - k_0 V_{gr}(k_0)) t} \int_{-\infty}^{\infty} \hat{f}(k) e^{i(k x - V_{gr}(k_0) t)} dk \]
and hence
\[ u(x, t) \approx e^{-i(\Omega(k_0) - k_0 V_{gr}(k_0)) t} f(x - V_{gr}(k_0) t). \]

This shows that the initial profile \( f \) is translated with the group velocity \( V_{gr}(k_0) \), at the same time being subject to a time periodic modulation.
Below is shown a representation of the evolution of a wave packet; the straight line (following the highest peak in time) indicates the motion of the packet as a whole with the group velocity (here negative: \( V_{gr} = -2.75 \)); the almost vertical translation of each constituent mode (indicated for example by the line just left to the vertical axis) reflects the fact that in this case the phase velocity is ten times smaller than the group velocity.

\[
\partial_t X(t) = \frac{\int V_{gr}(k)|\hat{f}(k)|^2 dk}{\int |\hat{f}(k)|^2 dk};
\]

the weight function is \(|\hat{u}(k)|^2 = |\hat{f}(k)|^2\), where \(\hat{f}\) is the spectral function of the initial profile. This velocity is in fact constant in time.

### 4.4 Dissipation

The simple pde (actually an ode)

\[
\partial_t u = \alpha u
\]

has dispersion relation \(\Omega(k) = i\alpha\). In fact, each initial profile, and hence each mode, has a uniform damping/amplification:

\[
u(x, t) = u(x, 0)e^{\alpha t}.
\]

The linear diffusion equation

\[
\partial_t u = D\partial_x^2 u, \ D > 0,
\]

has monochromatic modes

\[
e^{ikx-Dt^2}i
\]

that decay with a rate \(-D k^2\) that depends on the wave number: the smaller the wave length, the faster it decays. This is called selective dissipation, and can lead to self-organization when there is an upperbound on the wave length (for instance, on a space periodic interval).

The evolution of an initial condition is again given by (4.2); a characteristic example is the diffusion of a Gaussian initial profile, shown below.

**Exercise 44 Maple**

1. Animate the evolution.

2. Investigate the difference between the ‘exact’ solution and the approximation given above.

### Centro velocity

It is possible to give an alternative interpretation of the concept of group velocity as follows.

For a profile function \(u(x)\), consider the center of gravity of \(u^2\):

\[
X := \frac{\int xu^2dx}{\int u^2dx};
\]

\(X\) can be viewed as a measure of the position of the profile. When \(u\) depends on time, \(X\) will depend on time. The time derivative is a measure of the velocity of the profile:

\[
\partial_t X(t) := \text{centro-velocity}.
\]
4.5 Exercises

1. Derive the dispersion relation for the following dispersive wave equation (the linear BBM-equation)

\[ (1 - \partial_x^2) \partial_t u = -\partial_x u \quad (4.3) \]

Show that the phase velocity is always positive; determine the wave lengths for which the group velocity is positive. Investigate the limit cases for long and ultra-short wave lengths.

2. Consider the following second order in time wave equations.

Derive in each case: the dispersion relation, and calculate (plot) for each of the branches \( \omega = \Omega_{\pm}(k) \) the phase and group velocity. Investigate in each case whether the equation is conservative or dissipative.

(a) Bi-directional translation equation

\[ \partial_t^2 u = c^2 \partial_x^2 u \]

(b) \( \partial_t^2 u = c^2 \partial_x^2 u + au \)

(c) \( \partial_t^2 u = c^2 \partial_x^2 u + \alpha \partial_x^4 u \)

(d) Telegraph equation

\[ \partial_t^2 u + a \partial_t u + bu + c \partial_x u + d \partial_x^2 u = 0 \]

3. The initial value problem for a second order equation describes the initial profile and the initial velocity

\[ u(x, 0) = f(x), \quad \partial_t u(x, 0) = g(x). \]

Write down in Fourier integrals the solution of the IVP (initial value problem) for

\[ \partial_t^2 u = c^2 \partial_x^2 u + \partial_x^4 u \]

Animate, and interpret the solution for

\[ f(x) = e^{-x^2}, \quad g(x) = 0. \]

Observe, and explain, that the solution remains symmetric about \( x = 0 \) for all time.

Exercise 45 Dissipation combined with dispersion.

1. Animate and interpret the evolution of the following dissipative dispersive wave equation

\[ \partial_t u + a \partial_x^3 u = D \partial_x^2 u. \]

2. Investigate the initial profiles that are bounded for all (positive) times for the following equation

\[ \partial_t u + a \partial_x^3 u = D \partial_x^2 u + \partial_x^4 u \]
Chapter 5

Basic concepts and phenomena 2

When dealing with pde’s that are nonlinear, the most important characteristic property is that general solutions can not be written down. Only in special cases, special solutions can be written down in closed form.

- **Linearity and nonlinearity.** When the equation is nonlinear, the superposition principle does not hold: a superposition of two solutions is not a solution in general. Characteristic for nonlinear eqns is interaction between various modes: nonlinear mode interaction. This can lead to waves becoming steeper without bound: the breaking of waves.

- **Travelling waves and fronts.** Solutions for nonlinear equations can not be found in general. However, special solutions may sometimes be found; these solutions are usually characteristic for certain phenomena. They can also be exploited for theoretical reasons in various ways, for instance they can be compared with solutions of a numerical code to check the accuracy of the code.

Examples of such solutions are travelling waves and fronts: spatial profiles that are translated at fixed speed undeformed in shape. This can be the case if dispersive and/or dissipative effects and nonlinear effects balance each other. This is the case (and most easily studied) in the famous Burgers eqn. (for fronts), the Korteweg - de Vries eqn. (for solitons) and the reaction-diffusion equation. With the Ansatz of undisturbed translation, the PDE transforms to an ODE in which the speed of translation appears as a parameter. The analysis of the ODE is possible in these cases with phase plane analysis, and explicit expressions can be found.

- In the last section we recall the notions of conservation and balance laws. This leads to concepts of conservation and dissipation as treated already in Chapter 2.

- The plot and calculational properties of Maple are exploited in ’Basics2.mws’ to illustrate the material covered in this text.

5.1 Nonlinearity: ”Breaking” waves

Consider the following prototype of a nonlinear equation:

\[
\text{NonlE} := \partial_t u(x, t) + u(x, t) \partial_x u(x, t)
\]

The solution with initial value \( f \) can be written analytically in an implicit way as follows. First observe that on a curve \( t \rightarrow x(t) \) the total derivative of \( u \) is given by

\[
\frac{d}{dt} u(x(t), t) = \partial_x u(x(t), t) \frac{d}{dt} x(t) + \partial_t u(x(t), t).
\]

From this it follows that

\[
\frac{d}{dt} u(x(t), t) = 0 \quad \text{on} \quad \frac{d}{dt} x(t) = u,
\]

and hence

\[
u(x, t) = \text{constant on } \frac{d}{dt} x(t) = u.
\]

Therefore, the ‘curve’ with \( \frac{d}{dt} x(t) = u \), which is called a characteristic curve, is in this case a straight line with slope \( u \). From this the construction becomes clear by using the information of the
initial value. In detail:
Let \( y \) be a point at the \( x \)-axis of the \( x, t \) plane; at this point \( u \) has the value \( f(y) \). The characteristic through this point is the straight line with tangent \( f(y) \):
\[
ch(t, y) := y + f(y)t
\]
and on this line the function \( u \) has the value \( f(y) \), and so
\[
 u(y + f(y)t, t) = f(y)
\]
is the solution in implicit form.

**Exercise 46** For a Gaussian function as example of an initial profile, the solution can be plotted easily using the parametric representation; a succession of profiles at different times is given below, together with a different presentation that is obtained by shifting the profiles vertically with increasing time.

1. Explain what you see! Draw characteristic curves in the \( x, t \)-plane.
2. Determine the first time \( t^* \) at which “breaking” occurs: \( \partial_t u(x, t^*) = \infty \).

Hint: Note that
\[
\frac{d}{dy}u(y + f(y)t, t) = \frac{\partial u}{\partial x}(x, t) \left( 1 + t \frac{d}{dy}f(y) \right)
\]
and show that from this it follows that on the characteristic through \( y \) this is the case for the time (if it exists) such that ...., etc; complete the argument.

### 5.2 KdV-eqn: combined dispersion and nonlinearity

#### 5.2.1 Korteweg-de Vries equation (1895)

Korteweg and de Vries derived in 1895 a model equation for the motion of waves on the surface of a layer of fluid above a flat bottom. Restricting to rather low, rather long waves, that travel mainly in one direction, the equation for the surface elevation \( \eta \), in a coordinate system moving with the velocity of small waves (which is \( \sqrt{gH} \), where \( g \) is the gravitational acceleration and \( H \) the depth of the layer), and in normalized variables, reads:
\[
\partial_t \eta(x, t) + \eta(x, t) \partial_x \eta(x, t) + \partial_x^3 \eta(x, t) = 0
\]
This equation became well known in the sixties since it turned out that from a mathematical point of view it was the first partial differential equation shown to be completely integrable, leading to a huge extension of the theory of nonlinear pde’s. It also became clear that many problems in physics and technics are modelled by this equation. Being an evolution equation, first order in time, the initial value problem requires to find the evolution of the surface profile from a given initial profile. This initial value problem for KdV is not easy to solve; for arbitrary initial profiles, numerical calculations have to be used to find the subsequent wave profiles; the complete integrability makes it possible in principle to write down the time-asymptotic profile.
5.2. KDV-EQN: COMBINED DISPERSION AND NONLINEARITY

5.2.2 Travelling waves of KdV

Introduction

The motivation for KdV to study the problem of surface waves, was to settle a dispute that continued throughout the nineteenth century about the existence of travelling waves: is it possible that a wave exists that doesn’t change in time, merely travelling at a fixed speed?

They showed, by deriving the equation and analyzing it, that the answer is affirmative; more so, it is possible to write down the wave shapes and speeds explicitly. This is quite unexpected at first sight, since KdV combines nonlinearity (leading to “breaking”-phenomenon) and dispersion (“spreading” of initial profile). The remarkable property is that these combined effects make it possible that there exist travelling waves, waves with a specific profile, say if, that travel undisturbed in shape at a specific speed, say $V$. That is, a solution of the form

$$\eta(x, t) = f(x - Vt)$$

just as in the translation equation, but now only for specific profiles $f$ and velocity $V$.

To find the wave profile $f$ and the velocity $V$, we substitute this form in the KdV-equation; the pde then becomes an ode for the function $f$ in which $V$ enters as a parameter to be determined together with the profile. We shall see that, in fact, there is a whole family of such waves; the higher the amplitude, the larger the velocity.

Writing $\xi := x - Vt$, the equation becomes

$$-V \partial_\xi f(\xi) + f(\xi) \partial_\xi f(\xi) + \partial^2_\xi f(\xi) = 0$$

A solution of this equation, for certain $V$, produces the wave profile $f$ of the wave that travels undisturbed in shape at speed $V$.

Analysis of solitary wave profiles

To find the solution we have to distinguish two cases:

- **space-(and time) periodic solutions**, for which $f$ is a periodic function of $\xi$, the so-called cnoidal waves (since the profile is expressed with the elliptic cnoidal function), and

- **solitary wave solutions**: wave profiles of a single hump that decay, together with all derivatives, sufficiently fast at infinity (“almost confined”, exponentially small outside a certain interval).

We will concentrate on the solitary wave profiles. Then by integrating the equation above once, noticing that the constant of integration has to vanish as a consequence of the decay at infinity, leads to the second order ode for the profile:

$$-V f(\xi) + \frac{1}{2} f(\xi)^2 + \partial^2_\xi f(\xi) = 0$$

This equation can be solved in a standard way by observing the mechanical analogue:

when $\xi$ is interpreted as the time, and $f$ as the position, the equation describes the motion of a particle of unit mass subject to a potential force with potential energy $U$ according to Newton’s law:

$$\partial^2_\xi f(\xi) + \frac{dU}{df} = 0 \quad (5.1)$$

with potential energy

$$U(f) = -\frac{1}{2} V f^2 + \frac{1}{6} f^3.$$ 

The plot of $U$ is qualitatively as shown below, at the left for positive values of $V$, at the right for negative values:

Looking for a solitary wave profile $f$ that decays to zero for $\xi$ tending to $-\infty$, $\infty$, we look for the solution that is nontrivial and connects the origin with itself: a *homoclinic orbit*. Clearly, this can only be achieved for positive values of $V$.

In more detail, for the profile equation mechanical-energy conservation holds. Multiplying (5.1) with $\partial_\xi f$ and integrating the equation again:

$$\frac{1}{2} |\partial_\xi f|^2 + U(f(\xi)) = E.$$
Since \( E \) should be zero for any solitary wave profile, the eqn becomes

\[
\frac{1}{2} \left[ \partial_\xi f(\xi) \right]^2 - \frac{1}{2} V f(\xi)^2 + \frac{1}{6} f(\xi)^3 = 0.
\]

This is a first order equation for the profile function and its solution can be given explicitly. This solution is a solitary wave profile: for each \( V \), with \( 0 < V \), it is given by

\[
f(\xi, V) = \frac{3V}{\cosh(\frac{3V}{2}\sqrt{\xi})}
\]

Two profiles, for \( V = .2 \) and \( V = 1 \), are shown below:

**Observe:** with \( V \) the velocity, the amplitude is proportional to \( V \), and the width proportional to \( \frac{1}{\sqrt{V}} \): the larger the amplitude, the more confined the wave, and the larger its speed.

The solution above can be found in the phase plane \( z = f, w = \partial_\xi f(\xi) \) in the following way. The curves of constant energy, given by

\[
\frac{1}{2} w^2 + U(z) = E
\]

are sketched below, and in this phase portrait the solitary wave corresponds to the homoclinic orbit which is the level curve through the origin (for which \( E = 0 \)).

### 5.3 Burgers’ eqn: combined dissipation and nonlinearity

#### 5.3.1 Normalized form of Burgers’ eqn

Consider Burgers equation in its normalized form: for a scalar state variable \( u \) depending on time \( t \) and one spatial variable \( x \)

\[
\partial_t u(x, t) + u(x, t) \partial_x u(x, t) = \nu \partial_x^2 u(x, t).
\]

Here \( \nu \) is a positive parameter; it could be scaled away but is illustrative to show the balance between the dissipative term in the right hand side and the nonlinearity, which was studied in the breaking equation, in the left hand side.

The equation was derived by Burgers as a simple model to study some aspects of turbulence that appear in three dimensional Navier-Stokes equations.

To start, observe that any constant \( C \) is a (stationary) solution: \( u(x, t) = C \); the linearization about this constant solution is given by

\[
\partial_t v(x, t) + C \partial_x v(x, t) = \nu \partial_x^2 v(x, t);
\]

this is just the linear diffusion equation in a frame of reference moving with velocity \( C \).

#### 5.3.2 Travelling front solutions

Again, it is not easy to write down solutions of this nonlinear equation. Just as for the wave equations,
one can investigate travelling ‘wave’ profiles. However, we will see that now, necessarily, the profiles cannot have the same limit value at $-\infty$ and at $\infty$; the profiles to be found will be monotone transitions between limiting values, and the velocity is related to the difference of these limit values. Profiles of this form are called fronts. The dissipative term prevents the solution of the breaking equation to become infinitely steep and overturn; the fronts can be considered to be the smoothened versions of these solutions.

Looking for solutions that are traveling waves, one substitutes

$$u(x, t) = f(x - V t)$$

in the Burgers eqn, leading to

$$-V \partial_x f + f \partial_x f = \nu \partial_x^2 f.$$ Integrating this eqn over the whole real axis, assuming the derivatives to vanish at infinity, shows that

$$V = \frac{1}{2} f(\infty) + \frac{1}{2} f(-\infty)$$

which relates the asymptotic values and the velocity.

In particular, for $V \neq 0$, no solutions of solitary-wave type (decaying for $x \to \infty$ and for $x \to -\infty$) exist. In fact, any bounded solution will be monotone, with values between $f(-\infty)$ to $f(\infty)$. These are solutions of the form of a front: tending to a constant value at infinity, but different values at $-\infty$ and at $\infty$.

Integrating the equation there results

$$\frac{1}{2} (f - 2V) f = \partial_x f + \text{constant}.$$ Without restriction, we will take the constant to be zero, and look for positive solutions, vanishing at $\infty$: $f(\infty) = 0$; then necessarily $V$ is positive, and this front solution is monotonically decreasing with asymptotic value $f(\xi) \to 2V$ for $\xi \to -\infty$.

Any shift of a solution is a solution again; the front can be ‘positioned’ at zero by taking as additional condition $f(0) = V$. The explicit formula for this centered front reads

$$f(\xi) = V - V \tanh \frac{V \xi}{2 \nu}$$
as can be verified easily.

**Exercise 47** 1. Make a plot (sketch) of the solution, and indicate with an arrow the direction of propagation.

2. Give a measure of the width of the front.

   How does the width and the ‘amplitude’ depend on $V$ and how on $\nu$?

   Explain the dependence on $V$ by scaling arguments (compare to KdV-eqn).

3. How can this front solution be related to a solitary wave profile? (Consider the derivative; what is the equation for the spatial derivative?)

4. Integrate the eqn over the whole real axis to determine the change of the “total mass” $\int_{-\infty}^{\infty} u(x, t) \, dx$ for an arbitrary solution of Burgers eqn. Specify this for the front solution and interpret the result.

**Exercise 48 Localised perturbation of a front.**

We will call the point $p$ for which $f(p) = V$ the ‘position of the front’ in the following.

1. What happens with a localized disturbance of the front solution when the initial perturbation is at the point $x_0$ (suppose the position of the front is at the origin at $t = 0$). Distinguish the cases $x_0 < 0$ and $x_0 > 0$.

2. What happens to the “position” of the disturbance when measured by its distance to the position of the front.

3. Sketch an initial configuration of a front perturbed by two symmetrically positioned localized disturbance far from the position of the front; give a sketch of the configuration at a later time.

**Exercise 49 Cole-Hopf transformation**

The following remarkable result holds.
Proposition 50 \( u \) satisfies the Burgers equation (for \( \nu = 1 \)) if \( \eta \) satisfies the linear diffusion equation:
\[
\partial_t \eta = \partial_x^2 \eta
\]
when \( u \) and \( \eta \) are related by the so-called Cole-Hopf transformation:
\[
u(x, t) = -2 \partial_x \ln(\eta(x, t)) = -2 \frac{\partial_t \eta(x, t)}{\eta(x, t)}
\]
1. Prove this statement. Explain the common expression: “Cole-Hopf linearizes the Burgers equation”.
2. Is the transformation very useful in solving the initial value problem for Burgers eqn?
3. Find the solution of the linear diffusion equation that corresponds through the Cole-Hopf transformation to Burgers front solution.

5.4 Reaction diffusion eqn.

5.4.1 Chemical reaction eqn (ode)
A typical eqn for the concentration \( u \), for which \( 0 \leq u \leq 1 \), of a constituent in a chemical reaction is given by
\[
\partial_t u(t) = u(t) (1 - u(t))
\]
The solutions of this ode are easily investigated:
- the equilibrium solution \( U_0 = 0 \) is unstable,
- the equilibrium solution \( U_1 = 1 \) is stable,
- the solution for every initial condition \( \epsilon \in (0, 1) \) increases monotonically to limiting value 1.

5.4.2 Reaction-diffusion eqn
Spatial effects appear when the reaction product can diffuse in a solvent:
\[
\partial_t u(x, t) = \partial_x^2 u(x, t) + u(x, t)(1 - u(x, t))
\]
The spatially independent solutions are special solutions, so in particular the equilibria \( U_0 = 0 \) and \( U_1 = 1 \). To investigate their stability, consider the linearization.

The linearization around the equilibrium solution \( U_0 = 0 \) reads
\[
\partial_t u(x, t) = \partial_x^2 u(x, t) + u(x, t).
\]

Exercise 51
1. Derive the dispersion relation.
2. Argue that the spatial-independent solutions are the long-wave limits of such harmonic solutions.
3. For which wave lengths is \( U_0 \) linearly stable?
4. When the equation is considered on an interval with Dirichlet boundary conditions, \( u(0, t) = 0 \) and \( u(L, t) = 0 \), derive the maximum length \( L \) for which \( U_0 \) is linearly stable.
5. Verify that the linearization around the equilibrium solution \( U_1 = 1 \) is stable for all wave lengths.

5.4.3 Front solutions
We investigate the existence of front solutions
\[
u(x, t) = f(x - V t)
\]
along the following lines.
The equation for the profile function \( f \) reads
\[-V \partial_x f = \partial_x^2 f + f(1 - f).
\]
A mechanical analogue of this equation is a non-linear oscillator with damping (for \( V > 0 \)):
\[
\partial_x^2 f + V \partial_x f + \frac{dU}{df} = 0,
\]
with potential
\[
U(f) = \frac{1}{2} f^2 - \frac{1}{3} f^3.
\]
For \( V = 0 \) the phase portrait \( (f, w = \partial_x f) \) looks as follows:
- periodic solutions near the centre \((0, 0)\),
- an unstable equilibrium at \((1, 0)\), with a homoclinic orbit that separates the bounded, periodic motions, from the unbounded motions.

For \( V > 0 \) the equilibria remain the same: \((0, 0)\) and \((1, 0)\).

Exercise 52 Investigate the phase portrait along the following lines.
5.5. Conservation and Dissipation

1. Show that necessarily a solution of this type has limiting values \( f(-\infty) = 1 \) and \( f(\infty) = 0 \). Give the physical interpretation of such a solution.

2. Give a mechanical analogue of this eqn. Argue that necessarily \( V \) will be positive.

3. Plot the phase portrait for various values of \( V \), starting with \( V = 0 \) (use ‘with(DEtools)’, ‘dfieldplot’). Observe that the front solution we are after is a heteroclinic trajectory connecting the equilibrium solutions.

4. Give a detailed analysis of the origin in the phase plane; study its stability depending on the value of \( V \). Do the same for the equilibrium \( U_t \). (Do not rely on the graphs only; determine (un-) stable directions and remember concepts like sub- and super-critical decay for the mechanical analogue). Give a sketch of the linearized phase portraits for \( V = 1.5 \), for \( V = 2 \) and for \( V = 2.5 \).

5. Find the minimal value \( V_{\text{min}} \) of \( V \) for which you expect that a front solution of the reaction-diffusion eqn exists; sketch the front profile and indicate with an arrow the direction of propagation. Give a chemical interpretation of the solution.

6. Can you find the explicit formula for the front? If not, give an accurate plot using ‘dsolve’ (numerics).

7. You may have given convincing arguments that a front solution exists for \( V > V_{\text{min}} \), yet it has not been a formal existence proof. Referring to courses on ODE, a full proof can in fact be given based on two concepts: (i) the construction of a suitable invariant set (with the two critical points at the boundary) and (ii) the idea that there cannot be any closed trajectories inside this set (from divergence-property of the equation). Could you sketch (indicate) a candidate for an invariant set?

5.5 Conservation and dissipation

(See also Chapter 2)
Consider a dynamical system with state variable \( u \).
A local conservation law for the system is a relation between a (local) density \( e(u) \) (that depends on \( u \) and a finite number of its spatial derivatives) and a corresponding flux \( f(u) \) such that for all solutions it holds that

\[
\partial_t e(u) + \partial_x f(u) = 0;
\]

for a system with more spatial variables, the generalization is a vector flux \( \mathbf{f} \) such that

\[
\partial_t e(u) + \text{div} \mathbf{f}(u) = 0.
\]

The interpretation of a local conservation law becomes clear by integrating over a fixed (in time) volume; applying Gauss’ theorem there results

\[
\partial_t \int_{\Omega} e(u) d\mathbf{x} = - \int_{\partial \Omega} \mathbf{f} \cdot \mathbf{n},
\]

showing that the change in time of \( \int_{\Omega} e(u) \) is only caused by the flux through the boundary \( \partial \Omega \): no annihilation or creation within \( \Omega \).

In particular, when boundary conditions are such that on the domain of consideration the integrated flux vanishes, the quantity \( \int_{\Omega} e(u) \) is a conserved quantity, also called constant of the motion

\[
\partial_t \int_{\Omega} e(u) = 0.
\]

Remark. The existence of a local conservation law is special; in general a density will satisfy a balance law of the form

\[
\partial_t e(u) + \text{div} \mathbf{f}(u) = S(u),
\]

where \( S \) is a source term. (The splitting between flux and source is, without further specification, not unique!)

All these notions become particularly relevant when for \( e(u) \) a physically motivated density is taken.
Example 53 1. KdV-eqn in normalized form:
\[ \partial_t u + u_{xxx} + uu_x = 0, \]
can be written as a conservation law:
\[ \partial_t u = -\partial_x \left[ u_{xx} + \frac{1}{2} u^2 \right], \]
and so \( u \) is a conserved density. In fact there are more conserved densities; three relevant conserved densities are given below, with the corresponding fluxes:
\[
\begin{align*}
\phi(u) &= u, \quad \text{mass} \\
\phi(u) &= u^2, \quad \text{momentum} \\
\phi(u) &= \frac{1}{2} u^2 - \frac{1}{6} u^3, \quad \text{energy}
\end{align*}
\]
\[ f(u) = - \frac{1}{2} u_{xxx} + u_x u_{xx} + u_x u_{xxx} + \frac{1}{2} u^2 u_{xx} - \frac{1}{8} u^4. \]
For solutions decaying (with all derivatives) at infinity (or for periodic solutions), the global conserved quantities are the integrals over the whole real line (or with the integrals over one period, respectively).

2. Burgers eqn in normalized form
\[ \partial_t u + uu_x = u_{xx} \]
can be written as a conservation law:
\[ \partial_t u = \partial_x \left[ -\frac{1}{2} u^2 + u_x \right] \]
and so an obvious conserved density is the ‘mass’
\[ \phi(u) = u, \quad f(u) = -u_x + \frac{1}{2} u^2. \]
For the density \( u^2 \) a local balance law can be written like
\[ \partial_t u^2 + \partial_x \left[ 2uu_x + \frac{2}{3} u^3 \right] = -2u_x^2. \]
For solutions that decay, together with their derivatives at infinity, this shows that the global quantity \( \int u^2 \) evolves like
\[ \int u^2 \, dx = -2 \int u_x^2 \, dx \]
and is decaying monotonically for any solution.

5.6 Exercises

1. Consider the nonlinear “breaking” equation
\[ \partial_t u + u \partial_x u = 0. \]
(a) Draw the profiles at two successive (positive) times of the solution with initial condition the positive step function:
\[ \text{step}(x) = \begin{cases} 
0 & \text{for } x < 0, \\
1 & \text{for } x > 0.
\end{cases} \]
(b) Do the same for the reversed step function, i.e. the initial condition \( \text{step}(-x) \).

2. Consider the so-called BBM eqn. (Benjamin, Bona & Mahony, 1972):
\[ (1 - \partial_x^2) \partial_t u = -\partial_x u - u \partial_x u \quad (5.2) \]
This model equation is a variant of the KdV eqn.
(a) Looking for travelling waves, \( u(x, t) = f(x - V t) \), write down the equation for the profile function \( f \); do you recognise this (form of the) equation?
(b) Find the solution.

3. Consider the Burgers eqn.
(a) Make a plot (sketch) of the front solution and indicate with an arrow the direction of propagation.
(b) Make also a plot of the front for a small positive value of \( \nu \). What is the relation between this front and the ‘solution’ of the breaking equation with the reversed step (multiplied by \( 2V \)) as initial function. (Sketch both solutions in the same figure; make animations).
(c) *** Find the solution of the linear diffusion equation that corresponds through the Cole-Hopf transformation to Burgers front solution.
Chapter 6

Motion of continua

Many equations from Mathematical Physics are partial differential equations; they describe so-called continuous systems: systems in which besides the (continuous) time, also spatial variables as independent variables for the state variables appear. The assumption that the system can be described with spatial variables that are continuous means that a continuum hypothesis is underlying the model. That is, instead of looking at the phenomena at a microscopic level, which would require to describe the behaviour (motion) of the individual molecules, a macroscopic description is exploited. The mathematical description as a continuum has several consequences that are irrespective of the underlying physical aspects: the so-called kinematic relations; we will investigate these in the first section. (Besides that, the process of translating the microscopic physical processes to a macroscopic description should be derived and justified, for instance the modelling of the viscous effects in the Navier-Stokes equations; we will not go into the details of these derivations.)

In the second section we consider briefly elastic media, actually only one-dimensional elastics. It gives a good opportunity to get accustomed to Lagrangian descriptions. In Section 3 we describe the motion of (compressible and incompressible) fluids and gases; Newton’s equation, relating accelerations and applied forces, that is known for systems with a finite number of particles, can formally be generalized to continuous systems; the resulting equations are known as Euler equations (for inviscid fluids) or the Navier-Stokes equations. In section 4 we derive as a special case of Euler’s equations the description of the waves on a layer of inviscid, incompressible fluid, the so-called surface wave equations.

6.1 Lagrangian and Eulerian description

6.1.1 Kinematics of continua

In this subsection we consider how a continuum that is evolving in time can be described. Two different ways are possible: following the motion of each individual “particle”, the so-called Lagrangian description, or the Eulerian description in which all functions are considered as functions of laboratory coordinates. We present the two descriptions and show the relation between the two. Although fluid dynamics usually uses the Eulerian description, the Lagrangian description is important for the motion of elastic materials, and since nowadays many numerical techniques (partly) use this description (in hybrid methods).

It should be observed that we only deal with kinematics in this subsection: the relations we derive, such as the continuity equation, are only a consequence of the way how we describe the evolution; the specification of the evolution is not described and will be treated in the next subsections.

Particle trajectories and flow map

Consider the motion of $N$ masses in 3-dimensional space, a so-called discrete system. Assuming the masses to be confined to a single point, so-called mass-points, the position of the $k$-th mass point at time $t$ will be denoted by

$$q_k(t) \in \mathbb{R}^3.$$ 

An evolution of the system corresponds to a trajectory

$$t \mapsto (q_1, \ldots, q_N) \in (\mathbb{R}^3)^N$$
For given evolution, the velocity of each mass point is defined in the usual way
\[ \mathbf{v}_k(t) := \partial_t \mathbf{q}_k(t), \]
the velocity \( \mathbf{v}_k \) is the tangent to the trajectory of the \( k \)-th mass point.

A \textit{continuum} can best be thought of as a collection of infinitely many mass-points, which are often called particles. Formally this can be described by using, instead of the discrete label \( k \in \{1, 2, \ldots, N\} \) of a discrete system, a continuous label, which we will call \( \xi \). As is often done, we will use as “label” the position in space at a specific time, for which we take some initial time \( t = 0 \). If the spatial position at a subsequent time \( t \) is denoted by \( \mathbf{x} = \mathbf{X}(\xi, t) \), the label is characterized by
\[ \xi = \mathbf{X}(\xi, 0). \]

Fixing the label \( \xi \),
\[ t \mapsto \mathbf{X}(\xi, t) \]
describes the trajectory in \( \mathbb{R}^3 \) as function of time of the particle with label \( \xi \): the \textit{particle trajectory}.

From now on we assume that initially the continuum occupies a domain \( \Omega_0 \subset \mathbb{R}^3 \), and so \( \xi \in \Omega_0 \). Furthermore, we will consider evolutions of the continuum that are smooth, i.e. differentiable with respect to \( t \) and with respect to \( \xi \).

Looking at the collection of particles, at every time the mapping from the initial position to the position at time \( t \)
\[ \Omega_0 \ni \xi \mapsto \mathbf{X}(\xi, t) \subset \mathbb{R}^3 \]
defines a “flow” of the continuum (the collection of particles) in \( \mathbb{R}^3 \).

The domain \( \Omega(t) \) occupied by the continuum at time \( t \) is described by
\[ \Omega(t) = \{ \mathbf{x} = \mathbf{X}(\xi, t) \mid \xi \in \Omega_0 \} \subset \mathbb{R}^3. \]

Following a particle as it transfers its particle trajectory between an initial time \( t = 0 \) and some time \( T \), the fact that it is a smooth trajectory of one and the same particle implies that the mapping \( \mathbf{X} \) should have the property that the total trajectory can be interpreted as being composed of consecutive trajectories: for all \( s \)
\[ \mathbf{X}(\xi, T) = \mathbf{X}(\mathbf{X}(\xi, s), T - s); \]
a consequence is that the flow map is invertible (take \( T = 0, s = t \)):
\[ \mathbf{X}(\xi, t)^{-1} = \mathbf{X}(\xi, -t). \]

A map \( \mathbf{X} \) with these properties defines a so-called one-parameter (with \( t \) as parameter) \textit{Lie-group} of mappings.

Just as for a discrete system, the \textit{particle velocity} is defined in the usual way as the velocity of the particle under consideration:
\[ \mathbf{v}(\xi, t) := \frac{\partial}{\partial t} \mathbf{X}(\xi, t). \]

Here, in the notation we express by a bar the fact that the differentiation with respect to time \( t \) is for fixed particle:
\[ \frac{\partial}{\partial t} \equiv \left. \frac{\partial}{\partial t} \right|_{\xi}. \]

### Eulerian velocity and material derivative

In a Eulerian description of a continuum and its flow, the basic quantities are considered as function of the laboratory coordinates, i.e. the time \( t \) and the spatial position \( \mathbf{x} \in \mathbb{R}^3 \); these are usually called Eulerian coordinates. Hence, instead of the role as dependent variable in the Lagrangian description, now \( \mathbf{x} \in \mathbb{R}^3 \) becomes an independent variable.

For example, let \( \bar{f} \) be a function depending on Lagrangian variables:
\[ \bar{f} = \bar{f}(\xi, t) \]

For given flow there corresponds a function \( f \) in Eulerian variables:
\[ f(\mathbf{x}, t) := \bar{f}(\xi, t) \text{ for } \mathbf{x} = \mathbf{X}(\xi, t); \]
in words: at time \( t \), and position \( \mathbf{x} \), \( f(\mathbf{x}, t) \) is the value of \( \bar{f} \) at time \( t \) for that particle that is at \( t \) at the position \( \mathbf{x} \).

A same relation can be considered for vector fields, in particular for the particle velocity:
Definition 54 The Eulerian velocity \( u = u(x, t) \) of the flow is defined to be the particle velocity expressed in the Eulerian coordinates:

\[
u(x, t) = \mathbf{v}(\xi, t) \equiv \frac{\partial}{\partial t} x(\xi, t) \quad \text{for} \quad x = X(\xi, t).
\]

This also leads to the notion of material derivative \( D_t \equiv \frac{D}{Dt} \). For a function of Eulerian variables, like \( f \), the material time derivative is the time derivative of the corresponding function in Lagrangian variables, i.e. the time derivative at fixed particle:

\[
D_t f(x, t) \equiv \frac{D}{Dt} f(x, t) := \frac{\partial}{\partial t} f(\xi, t)
\]

which leads with the Eulerian velocity to

\[
D_t f(x, t) = \partial_t f(x, t) + \mathbf{u}(x, t) \cdot \nabla_x f(x, t)
\]

The term

\[
\mathbf{u}(x, t) \cdot \nabla_x f(x, t) = \sum_{k=1}^{N} u_k(x, t) \frac{\partial}{\partial x_k} f(x, t)
\]

is the so-called “convective” contribution, the effect that when fixing the particle, the derivative with respect to time in Eulerian coordinates contains a contribution of the motion of the particle. The material derivative of the Eulerian velocity is given by

\[
D_t \mathbf{u}(x, t) = \partial_t \mathbf{u}(x, t) + (\mathbf{u}(x, t) \cdot \nabla_x) \mathbf{u}(x, t).
\]

Mass density, continuity equation

Attaching to each particle a mass, say \( \rho_0(\xi) \), the mass of an infinitesimal domain is given by \( \rho_0(\xi) d\xi \); under the flow this mass remains the same but occupies a different domain, with a possibly different volume. So, let \( \Omega_0 \) denote the domain in the particle space, and \( \Omega(t) \) the corresponding domain in the physical space; then the mass in this domain is given by

\[
\text{mass}(\Omega(t)) = \int_{\Omega_0} \rho_0(\xi) d\xi
\]

Defining the mass density in Eulerian coordinates as

\[
\rho(x, t) := \rho_0(\xi) \det \left( \frac{\partial x}{\partial \xi} \right)
\]

we have the interpretation

\[
\rho(x, t) d\mathbf{x} = \rho_0(\xi) d\xi
\]

and can write the mass in \( \Omega(t) \) as

\[
\int_{\Omega(t)} \rho(x, t) d\mathbf{x} = \int_{\Omega_0} \rho_0(\xi) d\xi
\]

If we consider a perturbation of the flow map that is given in Eulerian variables by \( \eta(x, \xi) \), the corresponding change in the mass density follows from flow through the boundary. In particular, the time derivative of the density for a flow with Eulerian velocity \( \mathbf{u} \) is found by

\[
\int_{\Omega(t)} \partial_t \rho(x, t) d\mathbf{x} = - \int_{\partial \Omega(t)} \rho(x, t) \mathbf{u}(x) \cdot \mathbf{n} \, dS
\]

which can be rewritten with Gauss’ theorem

\[
\int_{\Omega(t)} \partial_t \rho(x, t) d\mathbf{x} = - \int_{\Omega(t)} \text{div} (\rho(x, t) \mathbf{u}(x)) d\mathbf{x}
\]

and hence

\[
\partial_t \rho(x, t) = - \text{div} (\rho(x, t) \mathbf{u}(x, t))
\]

or shortly

\[
\partial_t \rho + \text{div} (\rho \mathbf{u}) = 0.
\]

Expanding the action of the divergence, there results

\[
\partial_t \rho + \mathbf{u} \cdot \nabla \rho + \rho \text{div} \mathbf{u} = 0.
\]

This result is known as the continuity equation; the derivation above clearly shows that it is just a consequence of the definitions of Eulerian flow velocity and mass density: it is a kinematic relation that holds no matter what the details of the flow are.

Proposition 55 For a continuum with mass density \( \rho(x, t) \) and Eulerian velocity \( \mathbf{u}(x, t) \), the following continuity equation holds:

\[
\partial_t \rho + \text{div} (\rho \mathbf{u}) = 0.
\]

Mass conservation

The continuity equation above has been derived in a straightforward way from the primitive definitions of the Eulerian quantities \( \rho \) and \( \mathbf{u} \) by following a specified set of particles. Another interpretation is obtained by integrating the equation over an arbitrary but fixed Eulerian domain \( D \subset \mathbb{R}^3 \). Then the result can be rewritten as follows

\[
\frac{d}{dt} \int_D \rho \, d\mathbf{x} = \int_D \partial_t \rho \, d\mathbf{x} = - \int_D \text{div} (\rho \mathbf{u}) \, d\mathbf{x}
\]
With Gauss’ theorem, the volume integral of the divergence reduces to the surface integral of the normal component, and there results

$$\frac{d}{dt} \int_D \rho \, d\mathbf{x} = - \int_{\partial D} \rho \mathbf{u} \cdot \mathbf{n} \, dS$$

(by assumption, the normal \( \mathbf{n} \) at the boundary points outward).

The interpretation of this equation provides once more a clarification of the concepts of mass density and Eulerian velocity: taking a domain \( D \) that is fixed in time in the laboratory,

- \( \int_D \rho \, d\mathbf{x} \) is the total mass within domain \( D \);
- since \( \mathbf{u} \cdot \mathbf{n} \) is the velocity component perpendicular to the boundary, \( \rho \mathbf{u} \cdot \mathbf{n} \) is the mass that flows out of the domain through the boundary per unit time and unit surface;
- the change of this mass in time, \( \frac{d}{dt} \int_D \rho \, d\mathbf{x} \), is solely a result of mass transfer through the boundary \( \partial D \) of the domain.

This shows that the continuity equation can be interpreted as mass-conservation: no mass can be annihilated or created; the change of mass in a domain is solely a consequence of mass-flow through the boundary.

**Incompressible flow**

Intuitively speaking, the flow will be called incompressible if the same set of particles occupy the same volume at each instant. This means that for a domain \( \Omega(t) \) moving with the flow

$$\frac{d}{dt} \int_{\Omega(t)} \mathbf{x} = 0$$

and hence

$$\int_{\partial \Omega(t)} \mathbf{u} \cdot \mathbf{n} \, dS = 0$$

which, upon using Gauss’ theorem, leads to the conclusion that \( \text{div} \, \mathbf{u}(\mathbf{x}, t) = 0 \).

**Proposition 56** A flow with Eulerian velocity \( \mathbf{u}(\mathbf{x}, t) \) is incompressible if and only if \( \mathbf{u} \) is divergence free:

$$\text{div} \, \mathbf{u}(\mathbf{x}, t) = 0.$$

For an incompressible flow, the continuity equation reduces to

$$D_t \rho(\mathbf{x}, t) = \partial_t \rho(\mathbf{x}, t) + \mathbf{u} \cdot \nabla \rho(\mathbf{x}, t) = 0$$

expressing the fact that the mass density is constant on flow trajectories.

**Exercise 57** 1. With \( \Omega(t) \) the domain occupied by a part of the continuum moving with the particles, show that for any scalar function \( f \) it holds that

$$\frac{d}{dt} \int_{\Omega(t)} f(\mathbf{x}, t) \, d\mathbf{x} \equiv \int_{\Omega(t)} \left\{ \partial_t f + \text{div} \, [f \, \mathbf{u}] \right\} \, d\mathbf{x}$$

2. Show that for all time

$$\int_{\Omega_0} \rho(\xi) \, d\xi = \int_{\Omega(t)} \rho(\mathbf{x}, t) \, d\mathbf{x}$$

Derive from this the continuity equation by differentiating with respect to \( t \).

3. Show that for any scalar function \( f \) it holds that

$$\frac{d}{dt} \int_{\Omega(t)} \rho(\mathbf{x}, t) f(\mathbf{x}, t) \, d\mathbf{x} \equiv \int_{\Omega(t)} \rho(\mathbf{x}, t) \frac{D}{Dt} f(\mathbf{x}, t) \, d\mathbf{x}$$

**6.1.2 Newtonian dynamics**

In the previous section we described the kinematics of a continuum: no matter what causes the flow of the continuum, the concepts of mass density and (Eulerian) velocity can be defined (assuming smoothness), and should be related by the continuity relation.

Now we turn to equations for the dynamics: what kind of forces can be considered and to what flow do these forces lead.

We find the motivation for the dynamics of continua from the classical theory of Newtonian dynamics for discrete systems.

**Discrete systems**

According to Newton, for mass points with (constant) masses \( m_k \), a force \( \mathbf{F} \) that is present changes the momentum \( p_k = m_k v_k \) of the particles according to

$$m_k \, \partial_t v_k = \mathbf{F}_k.$$
6.1. LAGRANGIAN AND EULERIAN DESCRIPTION

The force $F_k$ acting on the $k$-th particle will depend in general on the positions of all the particles at that time. A special case is described by so-called conservative forces. Then the force is obtained from a single scalar function, the potential energy

$$U = U(q_1, \ldots, q_N)$$

according to

$$F_k = -\frac{\partial}{\partial q_k} U(q_1, \ldots, q_N)$$

(the minus sign appears for historical reasons). Then Newton’s equations read

$$m_k \partial_t v_k = -\frac{\partial}{\partial q_k} U(q_1, \ldots, q_N).$$

(6.1)

Together with $v_k = \partial_t q_k$ this becomes a closed system of $3N$ ode’s of second order for the $3N$ position components

$$m_k \partial_t^2 q_k = -\frac{\partial}{\partial q_k} U(q_1, \ldots, q_N);$$

given initial positions of the mass points $q_1(0), \ldots, q_N(0)$ the successive evolution is determined.

Principle of virtual displacement

For generalizations to continua, it is better to interpret Newton’s equations through the principle of virtual displacements. This principles states that for infinitesimal displacements $\eta_k$ of the particles, the change in the so-called action:

$$\sum_k m_k \partial_t v_k \cdot \eta_k$$

should equal minus the change $\delta U$ of the potential energy $U$ due to the displacements. Taylor expansion of $U(q_k + \eta_k)$ shows that

$$\Delta U \equiv U(q_k + \eta_k) - U(q_k) = \sum_k \frac{\partial U}{\partial q_k} \cdot \eta_k$$

The principle of virtual displacement then requires

$$\sum_k m_k \partial_t v_k \cdot \eta_k = -\Delta U = -\sum_k \frac{\partial U}{\partial q_k} \cdot \eta_k$$

to hold for arbitrary displacements. From this it is readily seen that Newton’s equations follow from the principle of virtual displacement.

The principle can be generalized to continua as follows. Consider a displacement of the particles given by a function $\tilde{\eta}(\xi)$, and denote the corresponding Eulerian function by $\eta(x, t)$. Then the discrete expression $\Sigma_k m_k \partial_t v_k \cdot \eta_k$ generalizes to

$$\int_{\Omega_0} \rho_0(\xi) \partial_t v(\xi, t) \cdot \tilde{\eta}(\xi) d\xi,$$

which can be translated to the following expression in Eulerian variables

$$\int_{\Omega(t)} \rho(x, t) D_t u(x, t) \cdot \eta(x, t) dx.$$

Assuming that the continuum evolves under forces that are related to some potential energy $U$, let $\delta \tilde{U}$, resp. $\delta U$ denote the variations of the potential energy in the Lagrangian, resp., Eulerian description. Then the principle of virtual displacement leads to the dynamic equation in Lagrangian setting

$$\rho_0(\xi) \partial_t v(\xi, t) = -\delta \tilde{U},$$

and to the dynamic equation in the Eulerian setting

$$\rho(x, t) D_t u(x, t) = -\delta U.$$

Observe that the direct generalization of the discrete systems define $\delta U$ to be such that

$$\int_{\Omega} \delta U \cdot \eta(x) dx$$

describes the change in $U$ for the displacement $\eta(x)$.

Remains to specify the potential energy and to calculate its variation. From the example of discrete conservative systems, it is natural to restrict to potential energies that do not depend on the velocities, but merely on the actual position of the particles, and to potential energies that result

\[1\]In the Calculus of Variations, $\delta U$ is called the variational derivative of the functional $U$ with respect to variations in the displacement.
from energy densities. One of the simplest choices would be to take

\[ \dot{U} = \int_{\Omega_0} \rho_0 \dot{V}(\xi, \frac{\partial x}{\partial \xi}) d\xi; \]

in Eulerian variables this corresponds to

\[ U = \int_{\Omega} \rho V(x, \frac{\partial x}{\partial \xi}) dx. \]

Here \( \dot{V} \) and \( V \) are given functions of their arguments.
The dependence of \( V \) on \( x \) means that the position of the continuum in the laboratory is important; a simple example is the presence of gravity; acting in the negative \( z \)-direction, this is modeled by

\[ U_{\text{grav}} = \int_{\Omega} \rho g z dx, \]

where \( g \) is the gravitational acceleration.
The dependence of \( V \) on \( \frac{\partial x}{\partial \xi} \) gives the change in potential energy for internal deformations.

**Cauchy stress**

In general a deformation is associated with a force; any body that undergoes a deformation experiences an internal force. This is described above by the principal of virtual displacement, but here an alternative description is given.

Consider the domain moving with the particles, and consider the balance of momentum:

\[ \frac{d}{dt} \int_{\Omega(t)} \rho u dx = \int_{\partial \Omega(t)} \sigma \mathbf{n} dS \]

where \( \sigma \) is a tensor, components \( \sigma_{ik} \), and \( \mathbf{n} \) denotes the outward pointing normal at the boundary. \( \sigma \) is called the Cauchy stress tensor\(^2\).

Defining the divergence of the tensor \( \sigma \) as

\[ (\text{div } \sigma)_k \equiv \Sigma_i \partial_i \sigma_{ki} \]

the local expression of momentum conservation can be found by expanding the left hand side, leading to

\[ \rho D_t u = \text{div } \sigma; \]

the actual form of the stress \( \sigma \) depends on the material under consideration.

---

\(^2\)The corresponding expression in the particle description leads to the Piola-Kirchhoff stress.

### 6.2 Elastica

An elastic medium is characterised by the fact that the internal energy depends on the relative position of the material points; a virtual displacement will cost or release a certain amount of energy that depends on the displacement, and on the material properties, its ‘elasticity’. This can be expressed by stating that for an elastic medium the density of the internal potential energy (we suppress the bars in the notation above)

\[ U = \int_{\Omega_0} V(\xi; \frac{\partial x}{\partial \xi}, \frac{\partial^2 x}{\partial \xi^2}, \ldots) d\xi; \]

will in general depend on the particle, the components of the deformation

\[ \frac{\partial x}{\partial \xi} = \left[ \frac{\partial x_k}{\partial \xi_j} \right]_{(k,j)} \]

and higher order derivatives. For a uniform medium, the elastic properties are independent of the point under consideration, and \( V \) is independent of \( \xi \).

For the sake of simplicity in notation, as an illustrative treatment, we shall deal with 1D material. Specifically one can think of an elastic (rubber) band, stretched along the \( x \)-axis, say occupying the interval \([0, 1]\) and consider longitudinal motions, i.e., motions along the \( x \)-axis. We assume the left point to be fixed, and a deformation is described by a map

\[ X : [0, 1] \ni \xi \rightarrow x(\xi) \in [0, L(t)] \]

where \( x(\xi, t) \) is the position at time \( t \) of the mass point that was at \( \xi \) at \( t = 0 \). The scalar quantity

\[ \ell(\xi, t) = \frac{\partial x}{\partial \xi} \]

is called the specific length; being length per unit mass, it is the inverse of the mass density.

Let us take for the internal energy a simple expression of the form

\[ U = \int V(\frac{\partial x}{\partial \xi}) d\xi \]

where \( V \) is a smooth, non-negative, function. To apply the principal of virtual displacement, we have to calculate the variation of \( U \) for a given
6.2. ELASTICA

displacement \( \eta(\xi) \). To do so, first observe that in lowest order

\[
V(\frac{\partial (x + \eta)}{\partial \xi}) - V(\frac{\partial x}{\partial \xi}) = V'(\frac{\partial x}{\partial \xi}) \frac{\partial \eta}{\partial \xi}
\]

and hence, after one partial integration

\[
\Delta U = \int V'(\frac{\partial x}{\partial \xi}) \frac{\partial \eta}{\partial \xi} d\xi = \int -\eta \cdot \partial_\xi \left[ V'(\frac{\partial x}{\partial \xi}) \right] d\xi
\]

Then the principal of virtual displacement leads to the Newton type of equation

\[
\partial_\xi^2 x = \partial_\xi \left[ V'(\frac{\partial x}{\partial \xi}) \right] \quad (6.3)
\]

which is a partial differential equation for \( x(\xi, t) \), of second order in time. This equation is conservative in the sense that the total energy is conserved:

\[
E = \int \left( \frac{1}{2} (\partial_\xi x)^2 + V(\frac{\partial x}{\partial \xi}) \right) d\xi \quad (6.4)
\]

Expanding the derivative in the right hand side leads to

\[
\partial_\xi^2 x = \left[ V''(\frac{\partial x}{\partial \xi}) \right] \frac{\partial^2 x}{\partial \xi^2} \quad (6.5)
\]

**Example 58** For a linear elastic medium, the function \( V \) is quadratic

\[
V(\frac{\partial x}{\partial \xi}) = \frac{1}{2} Y \frac{\partial x}{\partial \xi} \frac{\partial x}{\partial \xi}
\]

where \( Y \) is the so-called Young’s modulus. Then the equation reads

\[
\partial_\xi^2 x = \partial_\xi \left[ Y \frac{\partial x}{\partial \xi} \right] \quad (6.6)
\]

which is the simple wave equation when \( Y \) is a constant.

**Remark 59** Using the specific length \( \ell = \frac{\partial x}{\partial \xi} \) the equations can be given another nice form. First introduce the (Lagrangian) velocity

\[
v(\xi, t) = \partial_\xi x(\xi, t)
\]

Then the trivial identity \( \partial_\xi \partial_\xi x(\xi, t) = \partial_\xi \partial_\xi x(\xi, t) \) leads to

\[
\partial_\xi \ell = \partial_\xi v \quad (6.7)
\]

and the dynamic equation becomes

\[
\partial_\ell v = \partial_\xi V'(\ell) \quad (6.8)
\]

These two first order in time equations can be transformed into a single second order equation

\[
\partial_\ell^2 \ell = \partial_\xi^2 V'(\ell) \quad (6.9)
\]

**Exercise 60** 1. Discretisation by particles with nearest neighbour interaction.

We have introduced a continuum as a limiting case of many particles, replacing a discrete particle label \( k \) by a continuous label \( \xi \). We will now show the reverse, a procedure very common when one wants to discretise a partial differential equation for numerical calculations on a computer. In the following we keep time continuous, but ‘discretise’ the label variable \( \xi \). To be precise, we investigate the equation for a material with internal energy given by (6.2). We neglect boundary conditions, and take a uniform grid along the \( \xi \)-axis, points \( \xi_k \) with mesh-width \( \varepsilon \). Letting

\[
x_k(t) = x(\xi_k, t),
\]

\( x_k(t) \) has the interpretation of the displacement of the \( k \)-th particle from its initial position \( \xi_k \). The derivative is replaced by a difference quotient

\[
\frac{\partial x}{\partial \xi}(\xi_k) \approx \frac{x_{k+1} - x_k}{\varepsilon}
\]

in the internal energy, which is then approximated:

\[
U = \int V(\frac{\partial x}{\partial \xi})d\xi = \sum_k \int_{\xi_k}^{\xi_{k+1}} V(\frac{\partial x}{\partial \xi})d\xi = \sum_k \varepsilon V\left(\frac{x_{k+1} - x_k}{\varepsilon}\right) = \tilde{U}(x_1, ..., x_{k-1}, x_k, x_{k+1}, ...)
\]

The internal energy has now become a function of all the displacements \( x_k \). A virtual displacement \( \eta_m \) of variable \( x_m \) leads to the variation of the internal energy (note, two terms!!)

\[
\delta_{x_m} \tilde{U} = \left[ V'(\frac{x_{m+1} - x_m - 1}{\varepsilon}) - V'(\frac{x_{m+1} - x_m}{\varepsilon}) \right]
\]
From the principal of virtual displacement, there results (take care of the $\varepsilon$, accounting for the mass between the grid points!)

$$
\varepsilon \partial_t^2 x_m = - \left[ V'(\frac{x_m - x_{m-1}}{\varepsilon}) - V'(\frac{x_{m+1} - x_m}{\varepsilon}) \right]
$$

Rewritten like

$$
\partial_t^2 x_m = - \frac{1}{\varepsilon} \left[ V'(\frac{x_m - x_{m-1}}{\varepsilon}) - V'(\frac{x_{m+1} - x_m}{\varepsilon}) \right]
$$

(6.10)

this is of the form of Newton’s equation for a system for which the potential energy for particle $m$ is only determined by its relative distance to its two neighbouring particles. This equation is the discretised form of (6.3).

Introducing the specific length at position $\xi_k$ as the central difference

$$
\ell_k = \frac{x_{k+1} - x_k}{2\varepsilon}
$$

the right hand side can be approximated, leading to

$$
\partial_t^2 x_m = -V''(\ell_m) \left[ \frac{x_{m+1} - 2x_m + x_{m-1}}{\varepsilon^2} \right]
$$

The expression in square brackets can be recognised as the discretisation of the second derivative $\frac{\partial^2 V}{\partial \xi^2}$; this result is a discretisation of (6.5).

It should be observed that the above derivation can be modified in various ways. For instance, when we had started with a central difference approximation for the derivative as it appears in the potential energy, a different discretisation is obtained, where one particle interacts with its second-nearest neighbours. All such discretisations are the same in the lowest order of approximation. Important in the derivation above is that equation (6.10) resembles the continuous equation (6.3) in the important aspect that the right hand side is the derivative of a certain potential energy. Stated differently, the discrete energy

$$
\sum_k \frac{1}{2} (\partial_t x_k)^2 + V(\frac{x_{k+1} - x_k}{\varepsilon})
$$

(6.11)

is exactly conserved for the dynamics (6.10) (check this!), and is at the same time a good discretisation of the continuous energy (6.4).  

2. Specialise the foregoing for the case of a linear elastic medium with constant Young’s modulus $Y$

$$
V(\ell) = \frac{1}{2} Y\ell^2
$$

3. Derive the equations of motion for a material for which the internal energy also depends on the second derivative:

$$
U = \int V\left( \frac{\partial x}{\partial \xi}, \frac{\partial^2 x}{\partial \xi^2} \right) d\xi
$$

Exercise 61: Transformation to Eulerian quantities

We will now show how the elastic equations above can be transformed to Eulerian description.

1. Define the mass density and Eulerian velocity by

$$
\rho(x, t) = \frac{1}{\ell(\xi, t)}, \quad u(x, t) = v(\xi, t)
$$

Show that

$$
\partial_t \rho = \ell \partial_x u, \quad \partial_t v = \partial_t u + u \partial_x u
$$

2. Calculate $\partial_t \rho$ and show that the continuity equation is obtained.

3. Show that the momentum equation becomes

$$
\rho D_t u = \partial_x V'(\frac{1}{\rho}).
$$

6.3 Inviscid gases and fluids

6.3.1 Gases and compressible fluids

For an elastic medium the internal potential energy will in general depend on all components of the deformation

$$
\frac{\partial \xi}{\partial x} = \left[ \frac{\partial x_k}{\partial x_j} \right]_{(k,j)}
$$

A fluid, or a gas, is by definition a continuum for which the internal energy depends only on the mass density, i.e. on $\det \left( \frac{\partial \xi}{\partial x} \right)$. Hence, if we

3So, actually, the one-dimensional elastica treated above, are in this definition also a fluid. In the treatment below, we restrict to isothermal flows; in general the internal energy depends also on the temperature, and an additional temperature equation is needed.
6.3. INVISCID GASES AND FLUIDS

let \( w = w(\rho) \) denote the energy density per unit mass, the so-called specific internal energy, and \( W(\rho) = \rho w(\rho) \) the local energy density, the potential energy is given by

\[
U_{int} = \int_{\Omega} W(\rho) d\mathbf{x}
\]

For ideal gases, the internal energy is given by

\[ W(\rho) = c \rho^\gamma \]

where \( \gamma (> 1) \) is the quotient of the specific heat at constant pressure and the specific heat at constant volume.

To calculate the variation of the internal potential energy for a virtual displacement \( \mathbf{\eta}(x) \), let \( \delta \rho(x) \) be the corresponding variation in the mass density. Then,

\[
U_{int}(\rho + \delta \rho) - U_{int}(\rho) = \int \partial_\rho W(\rho) \delta \rho d\mathbf{x};
\]

using the relation established before, \( \delta \rho = -\text{div}(\rho \mathbf{\eta}) \) there results

\[
\Delta U_{int}(\rho) = -\int \partial_\rho W(\rho) \text{div}(\rho \mathbf{\eta}) d\mathbf{x}
\]

which after using Gauss’ theorem becomes\(^4\)

\[
= \int \rho \nabla(\partial_\rho W(\rho)) \cdot \mathbf{\eta} d\mathbf{x};
\]

from this it follows that the variation in the potential energy is given by

\[
\delta U_{int} = \rho \nabla(\partial_\rho W(\rho)).
\]

Using the specific energy density \( w(\rho) \), a simple calculation shows that

\[
\rho \nabla(\partial_\rho W(\rho)) = \nabla p(\rho), \quad \text{with } p = \rho^2 \partial_\rho w;
\]

\( p \) is called the pressure, and its relation with \( w \) may be known from thermodynamics; for an ideal gas, \( p \sim \rho^\gamma \).

In the presence of gravity, along the same lines as above, one finds that

\[
\delta U_{grav} = \rho \nabla(g \mathbf{z}) = \rho g \mathbf{e}_z
\]

with \( \mathbf{e}_z \) the unit vector in the direction of gravity.

The principle of virtual displacement generalized to a continuum leads to the following dynamic equation

\[
\rho D_t \mathbf{u} = -\rho \nabla(\partial_\rho W(\rho)) - \rho g \mathbf{e}_z.
\]

Proposition 62 A gas or compressible fluid with specific internal energy \( w(\rho) \) in the presence of gravity (acting in the negative \( z \)-direction), is described by the so-called Euler equations, a closed system of equations for \( \rho \) and \( \mathbf{u} \):

the kinematic continuity equation

\[
\partial_t \rho + \text{div}(\rho \mathbf{u}) = 0
\]

and the dynamic momentum equation

\[
\rho D_t \mathbf{u} \equiv \rho \partial_t \mathbf{u} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p - \rho g \mathbf{e}_z
\]

where \( p = p(\rho) \) is the pressure, given in terms of the specific energy density \( w(\rho) \) by

\[
p(\rho) = \rho^2 \partial_\rho w.
\]

6.3.2 Incompressible fluids

When a fluid is incompressible, the internal energy is constant, and so \( \delta U_{int} = 0 \).

However, for incompressible flows, the virtual displacements should not violate the incompressibility condition, and therefore the virtual displacements \( \mathbf{\eta}(x) \) should be divergence free. Then the principle leads to the requirement that

\[
\int \rho D_t \mathbf{u} \cdot \mathbf{\eta}(x) d\mathbf{x} = 0, \quad \text{for } \text{div } \mathbf{\eta} = 0.
\]

From this follows that for some scalar function \( p = p(x, t) \) it should hold that \( \rho D_t \mathbf{u} = -\nabla p \). Again \( p \) is called the pressure, but unlike in the compressible case, now \( p \) is not simply related to the mass density.

Proposition 63 For incompressible flow the Euler equations are given by the incompressibility condition

\[
\text{div } \mathbf{u} = 0,
\]

and

\[
\rho D_t \mathbf{u} \equiv \rho \partial_t \mathbf{u} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p - \rho g \mathbf{e}_z.
\]
Exercise 64 Consider a 2-dimensional layer of (incompressible) fluid, \( x \) the horizontal direction, the \( z \)-axis in the vertical direction. Let \( H \) be the depth of the layer, and let \( P(x,t) \) be the atmospheric pressure at the water surface.

1. Assume the fluid is at rest. Under what condition on \( P \) is this possible? Determine in the case of a pressure free atmosphere \( (P \equiv 0) \) the pressure in the fluid as a function of the depth.

2. Consider a bar of rectangular form with length \( L \) and thickness \( d \) of total mass \( M \). Derive conditions such that the bar can float on the water.

3. A boat sailing with speed \( U \) creates a depression of the still water surface as if it sails in its own generated valley. Determine the distance from the bottom as a function of its speed. (Explain the advice to sail slowly when entering shallow water.)

6.3.3 Momentum balance law

The dynamic equations found above can be rephrased in a way that shows how the change of the integrated momentum is balanced by volume and boundary forces (Cauchy stress).

First observe that by using the continuity equation, for each component of \( D_t \mathbf{u} \) it holds that

\[
\rho D_t u_k = \partial_k (\rho u_k) + \text{div} (\rho u_k \mathbf{u})
\]

For a fixed domain \( \mathcal{D} \), applying Gauss' theorem, it follows that

\[
\int_{\mathcal{D}} \rho D_t u_k = \partial_k \int_{\mathcal{D}} \rho u_k + \int_{\partial \mathcal{D}} \rho u_k \mathbf{n} \cdot \mathbf{n}
\]

which can be written as a vector equality

\[
\int_{\mathcal{D}} \rho D_t \mathbf{u} = \partial_t \int_{\mathcal{D}} \rho \mathbf{u} + \int_{\partial \mathcal{D}} \rho \mathbf{u} (\mathbf{u} \cdot \mathbf{n})
\]

Here \( \rho \mathbf{u} \) is the momentum density of the fluid, the direct generalization of the momentum of a mass point of a discrete system.

For this reason the dynamic Euler equation, which is of the form

\[
\rho D_t \mathbf{u} = \mathbf{F}
\]

is usually called the momentum equation. (The equations for viscous fluids, the Navier-Stokes equations, will be of the same form.)

For the Euler equations, consider the pressure part in \( \mathbf{F} = -\nabla p \). The volume integral over the domain \( \mathcal{D} \) reduces also to an integral over the boundary. This is seen by considering one component, for instance the first one:

\[
\int_{\mathcal{D}} \frac{\partial p}{\partial x} = \int_{\mathcal{D}} \text{div} (p,0,0) = \int_{\partial \mathcal{D}} (p,0,0) \cdot \mathbf{n}
\]

Now use the divergence of a \( 3 \times 3 \)-matrix which has been introduced as the vector that has as components the divergence of the rows of the matrix. With \( I \) the identity matrix, it then follows that

\[
\int_{\mathcal{D}} \nabla p = \int_{\mathcal{D}} \text{div} (pI) = \int_{\partial \mathcal{D}} (pI) \cdot \mathbf{n}.
\]

Taken together, the momentum equation \( \rho D_t \mathbf{u} = -\nabla p \mathbf{c} \) can now be written like

\[
\rho D_t \mathbf{u} = -\text{div} p I.
\]

Comparing with the Cauchy stress introduced before in a general way, for this equation the stress is given by

\[
\sigma = -p I.
\]

Integrating over a fixed domain \( \mathcal{D} \) leads to the integrated form of the continuity equation:

\[
\partial_t \int_{\mathcal{D}} \rho \mathbf{u} = \int_{\partial \mathcal{D}} \rho \mathbf{u} (\mathbf{u} \cdot \mathbf{n}) = \int_{\partial \mathcal{D}} (pI) \cdot \mathbf{n}.
\]

This is recognized as a global balance law for the momentum density: the change in time of the momentum in a fixed domain can change by a flow of momentum through the boundary, and by a force from the pressure on the boundary.

In the presence of gravity, a volume integral of the local gravity force contributes to the change in integrated momentum:

\[
\partial_t \int_{\mathcal{D}} \rho \mathbf{u} = -\int_{\partial \mathcal{D}} \rho \mathbf{u} \cdot \mathbf{n} - \int_{\partial \mathcal{D}} (pI) \cdot \mathbf{n} - \int_{\mathcal{D}} \rho g e_z.
\]
6.3.4 Energy conservation

For a discrete system with conservative force from a potential energy, the energy can be defined as the sum of kinetic energy and potential energy:

\[ E = \sum_k \frac{1}{2} m_k \dot{q}_k^2 + U(q_1, \ldots, q_N) \]

It is a simple exercise to show that \( E \) is a constant of the motion, i.e., the value doesn’t change during the evolution described by Newton’s equations. For a continuum the generalization is immediate. For a compressible fluid, the local energy density is defined by

\[ E = \frac{1}{2} \rho |\mathbf{u}|^2 + W(\rho) + \rho g z. \]

Using the Euler equations it is readily verified that

\[ \partial_t E + \text{div} \ F_E = 0, \]

where \( F_E \) is the so-called energy flux given by

\[ F_E = \mathbf{u} \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \rho \partial_\rho W(\rho) + \rho g z \right); \]

using the pressure \( p \), it holds \( \rho \partial_\rho W(\rho) = W(\rho) + p \), and the energy flux can be written like

\[ F_E = \mathbf{u} [E + p]. \]

Integrating over a fixed domain \( D \) this result leads to a conservation law for the energy:

\[ \int_D E = - \int_{\partial D} [E + P] \mathbf{u} \cdot \mathbf{n} \]

showing that a change of energy in a domain can only be caused by a flow of energy through the boundary\(^5\).

For incompressible flows the same result holds:

\[ E = \frac{1}{2} \rho |\mathbf{u}|^2 + \rho g z, \quad F_E = \mathbf{u} [E + p] \]

\(^5\)For the isothermal, inviscid fluid flows that we considered, energy conservation is a consequence of the Euler equations and is not an additional dynamic law. For non-isothermal flows, with the temperature as additional state variable, an energy equation (balancing the internal energy that depends on density and temperature with external forces, with local heat production and with heat fluxes through the boundary) an energy law will be an essential additional dynamic equation.

6.4 Exact surface wave equations

Consider a layer of ideal fluid, by which we mean that the fluid is inviscid and incompressible with mass density \( \rho \) normalized to unity. Assume that the fluid is unbounded in the horizontal directions \( x, y \), and that the gravity is pointing in the negative \( z \)-direction. The fluid domain \( \Omega(t) \) above the bottom has a free surface (varying in time in general); we assume that the free surface can at each time be written as the graph of a smooth function \( \eta = \eta(x, y, t) \) (which excludes overturning waves). Then

\[ \Omega(t) = \{ (x, y, z) \mid -h \leq z \leq \eta(x, y, t) \}, \]

where \( h = h(x, y) \) describes the bottom topography. The origin of the coordinate axis is chosen such that \( z = 0 \) coincides with the still water level; the deviations from this level is measured by the surface elevation, or wave height, \( \eta \). Note that for determining the total fluid motion, both the Eulerian velocity in the interior and the surface elevation have to be determined.

In the following we restrict to irrotational flow, i.e., \( \text{curl} \ \mathbf{u} = 0 \). This is a dynamically admissible assumption: it can be shown that if \( \text{curl} \ \mathbf{u} = 0 \) at some initial time, then the Euler equations show that this also holds for \( t > 0 \).

As a consequence, a great simplification becomes possible: one can introduce Stokes’ stream function \( \Phi \) such that the Euler velocity is given by \( \mathbf{u} = \nabla \Phi \). Observe that an arbitrary function of time can be added to \( \Phi \) without changing the physically relevant velocity \( \mathbf{u} \).

The incompressibility condition \( \text{div} \ \mathbf{u} = 0 \) translates to the Laplace equation for \( \Phi \):

\[ \Delta \Phi = 0 \text{ in } \Omega(t). \quad (6.12) \]

The momentum equation \( \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla (p + gz) \) with \( \mathbf{u} = \nabla \Phi \) can be integrated to become

\[ \partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + p + gz = f(t), \quad (6.13) \]
where \( f(t) \) is an arbitrary function of time. Equation (6.13) is called Bernoulli’s equation in the interior of the fluid.\(^6\)

**Dynamic and kinematic surface equations**

Assuming a pressure free atmosphere above the layer, the restriction of Bernoulli’s equation to the free surface becomes

\[
\frac{\partial}{\partial t} \Phi + \frac{1}{2} |\nabla \Phi|^2 + g\eta = 0, \quad \text{at } z = \eta(x, y, t).
\]

This is the *dynamic free surface condition*, a condition that should hold between the surface elevation \( \eta \) and the flow in the interior.

Besides this condition there is a *kinematic boundary condition*, expressing the fact that no fluid particles can pass the free surface. Using the material derivative this is expressed by

\[
D_t z = \partial_x \Phi, \text{ i.e. } \partial_t \eta + \nabla \Phi \cdot \nabla \eta = \partial_x \Phi,
\]

i.e.

\[
\partial_t \eta + \Phi_x \eta_x + \Phi_y \eta_y = \Phi_z.
\]

Observe that \( \mathbf{N} := (-\eta_x, -\eta_y, 1) \) is normal to the free surface (pointing outward the fluid domain), and so

\[
\partial_t \eta = \nabla \Phi \cdot \mathbf{N}
\]

Resuming we have

**Proposition 65** The motion of an inviscid, irrotational layer of fluid is described by the Laplace equation for Stokes’ velocity potential \( \Phi \) in the interior with a non-flux condition at the bottom:

\[
\Delta \Phi = 0 \quad -h < z < \eta(x, y, t),
\]

\[
\nabla \Phi \cdot \mathbf{n} = 0, \quad z = -h(x, y),
\]

and two conditions at the free surface \( z = \eta(x, y, t) \): the kinematic equation

\[
\partial_t \eta = \nabla \Phi \cdot \mathbf{N}, \quad \mathbf{N} := (-\eta_x, -\eta_y, 1),
\]

and Bernoulli’s equation

\[
\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + g\eta = 0.
\]

**Exercise 66** Make a formal linearisation of the free surface equations by assuming that \( |\nabla \Phi| \) and that \( \eta \) and its derivatives are small, along the following lines. (Assume a horizontal bottom, depth \( H \).

1. First observe that

\[
\partial_t \eta = \Phi_z \text{ at the surface}
\]

2. To find \( \Phi_z \) at the surface, show that

\[
\Phi_z = - \int_{-H}^{0} \Delta_2 \Phi \, dz \approx -H \Delta_2 \Phi
\]

where \( \Delta_2 \Phi = (\partial_x^2 + \partial_y^2) \Phi \).

3. Differentiate the linearised Bernoulli equation at the free surface: \( \partial_t \Phi + g\eta = 0 \).

4. Find the resulting equation

\[
\partial_t^2 \eta = gH \Delta \eta
\]

which is a two-dimensional wave equation; what are the characteristic propagation speeds?

\(^6\)Observe that when using the total time derivative, which may be useful in numerical schemes that use Lagrangian grids, Bernoulli’s equation becomes

\[
D_t \Phi - \frac{1}{2} |\nabla \Phi|^2 + p + gz = f(t)
\]
Chapter 7

Asymptotic methods for ode’s

Do small causes have small consequences?
This basic question is now addressed for two classes of problems; both are variants of the simple harmonic oscillator

\[ \ddot{x} + \omega^2 x = 0. \]

The first class concerns ‘adiabatically changing’ problems: some externally given parameter changes slowly in time or space; in the example above the frequency changes adiabatically. For small time/space intervals the solution can well be approximated by taking the value of the parameter at a ‘frozen’ value. But to obtain approximations that are valid on larger time/space intervals (for instance so large that the change in the parameter itself is of order one) such an approximation cannot be reliable, and more sophisticated methods are required. We discuss one such method for linear ode’s in the first section, the well known WKB-method.

Another class of problems are nonlinear, and it is the solution itself that determines through the nonlinearity how a certain ‘parameter’ will vary (not necessarily ‘slowly’ varying); a classical example is a nonlinear oscillator, where the nonlinearity makes the frequency to change: \( \omega = \omega(x) \) in the example above. Periodic oscillations are then disturbed to an extent that is determined by the motion itself. In the second section such self-excited oscillators are studied; notions like mode-degeneration and resonance have to be investigated. All the techniques are illustrated to ode’s, but the methods are equally well applicable, or can be generalized, to study pde’s with the same characteristic properties.

7.1 WKB-method

7.1.1 The general method

Motivation for asymptotic results
Several problems are described by an equation of the form

\[ \frac{\partial^2 u}{\partial z^2} + \omega^2 u = 0. \]

When \( \omega \) is a constant, the solution is a harmonic function with frequency \( \omega \) and arbitrary (complex) amplitude \( A \):

\[ u = Ae^{i\omega z}. \]

The problem becomes much more difficult when \( \omega \) is not a constant but depends on \( z \); in most cases no exact solution can be written down in closed form.

Here we consider the case that \( \omega \) depends slowly varying on \( z \). That is, suppose that \( \omega(z) = \Omega(\epsilon z) \) with \( \epsilon > 0 \) small:

\[ Eqn(u) \equiv \frac{\partial^2 u}{\partial z^2} + \Omega^2(\epsilon z) u = 0. \quad (7.1) \]

A first suggestion to find an approximate solution could be to look for an approximation of the form

\[ v = Ae^{\Omega(\epsilon z)}z. \]

Inserting into the eqn leads to

\[ Eqn(v) = \epsilon[2iA\Omega' - 2A\Omega z\Omega]e^{i\Omega z} + O(\epsilon^2) \]

where here and in the following a prime denotes differentiation with respect to \( \epsilon z \).

This is a reliable, but not very useful result: for \( z \) in bounded intervals, (in which \( \Omega \) changes of the order \( \epsilon \), the proposed solution is valid with residue of order \( \epsilon \). Thus this result is just a continuous dependence result for ode’s.
However, we are interested in a better result, in an approximation that is of the order $\epsilon$ on a longer interval of length $1/\epsilon$! Note that when $z \in [0, O(1/\epsilon)]$, the change in $\Omega$ will be of the order one, and so not small! Then the proposed approximation is not good since the residue is of order one on such intervals.

A major improvement is obtained by looking for an approximation of the form

$$w = Ae^{\theta(z)}$$

with a phase function $\theta$ to be determined. Inserting this Ansatz into the eqn leads to

$$\text{Eqn}(w) = Ae^{\theta}[i\theta'' - \theta'^2 + \Omega^2].$$

From this it follows that if one chooses

$$\partial_z \theta = \pm \Omega(\epsilon z)$$

that then $\theta' = O(\epsilon)$ and the residue is of order $\epsilon$ on the long $z$-interval.

The expression for $\theta = \theta(z)$ (up to an arbitrary phase factor) becomes

$$\theta - \theta_0 = \int_0^z \Omega(\epsilon z')dz' = 1/\epsilon \int_0^{\epsilon z} \Omega(\zeta)d\zeta.$$ 

**Observation**

To compare the two different choices for the phase above, observe that the difference in the phase is given by

$$\theta(z) - \Omega(\epsilon z)z = 1/\epsilon \int_0^{\epsilon z} [\Omega(\zeta) - \Omega(\epsilon z)]d\zeta.$$ 

For small $\epsilon$ and fixed $z$ this difference is small: the integral is approximated by the length of the integration interval times the value of the integrand at $z = 0$. Although $\theta$ resembles $\Omega z$ for bounded $z$:

$$\theta(z) - \Omega(\epsilon z)z = O(\epsilon) \text{ for } z \in [0, O(1)],$$

this phase difference is of essential order on larger intervals:

$$\theta(z) - \Omega(\epsilon z)z = O(1) \text{ for } z \in [0, O(1/\epsilon)]$$

**The WKB-approximation**

In the special case under consideration another improvement is possible by allowing the amplitude to change slowly. Inserting

$$\tilde{u}(z) := A(\epsilon z)e^{\theta(z)}$$

we find as residue

$$\text{Eqn}(\tilde{u}(z)) = e^{\theta}[iA\theta'' + 2i\epsilon A'\theta' + O(\epsilon^2)].$$

With the choice for $\theta$ as above, the order $\epsilon$ term in the right hand side can be made to vanish by letting $A$ satisfy

$$A' = -\frac{\theta''}{2\epsilon A'} A = -\frac{\Omega}{2\epsilon \Omega} A$$

with solution

$$A = \frac{A_0}{\sqrt{\Omega}}.$$

Resuming we have the following result.

**Proposition 67** WKB-approximation (Wentzel-Krammer-Brioullin)

*An asymptotically valid solution of the problem with adiabatically varying coefficient $\Omega = \Omega(\epsilon z)$ is given by the WKB-approximation*

$$w(k) = \frac{1}{\sqrt{\Omega(\epsilon z)}} e^{\theta(z)}, \text{ with } \partial_z \theta(z) = \Omega(\epsilon z);$$

**(7.3)**

*this approximation satisfies the equation to order $O(\epsilon^2)$:*

$$\text{Eqn}(w(k)) = e^{\theta}[i\epsilon^2(\epsilon z) \frac{d^2}{d\epsilon^2} \left( \frac{1}{\sqrt{\Omega(z)}} \right)], \quad \epsilon z = \zeta.$$  

**(7.4)**

*uniformly on z-intervals of order $O(1/\epsilon)$.*

**The WKB-transformation and hierarchy**

Observe the following result for the original equation

$$\text{Eqn}(u) \equiv \partial^2_{zz}u + \Omega^2(\epsilon z)u = 0.$$ 

Define (stimulated by the WKB-approximation found above), a new variable

$$\theta := \int_0^z \Omega(\epsilon z)dz'$$
7.1. WKB-METHOD

and the transformation

\[ B(\theta) := \sqrt{\Omega(\varepsilon)} u(z). \]

Then the following equation for \( B \) is found:

\[ \Omega^{3/2} \left\{ \frac{d^2 B(\theta)}{d\theta^2} + \left[ 1 - \varepsilon^2 \left\{ \frac{\Omega'}{2\Omega^2} - \frac{3\Omega'^2}{4\Omega^2} \right\} \right] B(\theta) \right\} = 0, \]

and so

\[ \frac{d^2 B(\theta)}{d\theta^2} + \left[ 1 - \varepsilon^2 \left\{ \frac{\Omega'}{2\Omega^2} - \frac{3\Omega'^2}{4\Omega^2} \right\} \right] B(\theta) = 0. \]

This leads to some important results.

1. First, neglecting the terms of order \( \varepsilon^2 \) we find the equation

\[ \frac{d^2 B(\theta)}{d\theta^2} + B(\theta) = 0 \]

with obvious solution

\[ B = A_0 e^{i\theta} \]

and one recovers the WKB-approximation.

2. However, for \( \varepsilon \neq 0 \) but small, the equation for \( B \) with \( \varepsilon \neq 0 \) resembles the original equation, in which \( z \) is replaced by \( \theta \) and \( \Omega^2 \) replaced by

\[ \left[ 1 - \varepsilon^2 \left\{ \frac{\Omega'}{2\Omega^2} - \frac{3\Omega'^2}{4\Omega^2} \right\} \right]. \]

This term is slowly varying (in \( \theta \)) but has as additional property that it is also almost constant: slowly varying and small deviations on an \( z \)- and \( \theta \)-interval of length \( O(1/\varepsilon) \). Hence one can use the WKB-approximation for the \( B \)-equation; this then leads to a residue of order \( O(\varepsilon^4) \).

3. It also possible to apply the same WKB-transformation once more, and repeat the process to obtain inhomogeneities that are slowly varying and have smaller and smaller deviations. In this way a whole hierarchy of WKB-approximations can be obtained, leading to increasingly better results.

Exercise 68 1. Consider the following wave-type equation (for uni-directional surface waves with elevation \( u(x,t) \) running above varying bottom \( h = h(x) \) with \( c(x) = \sqrt{gh(x)} \)):

\[ \partial_t u = -\sqrt{c(x)} \partial_x \sqrt{c(x)} u. \quad (7.5) \]

For constant depth, i.e. \( c \) is constant, the general solution is simple:

\[ u(x,t) = f\left( \frac{x}{c} - t \right), \]

for arbitrary function \( f \).

Now when \( c = c(x) \), show that the general solution can be written like

\[ u(x,t) = A(x) g(\theta(x,t)) \]  \[ (7.6) \]

where \( g \) is arbitrary and the amplitude \( A \) and the phase \( \theta \) are taken appropriately related to \( c(x) \).

Determine the solution of the initial value problem: \( u(x,0) = u_0(x) \) given.

Plot the solution for a typical initial wave consisting of a single hump.

2. Having obtained the result above, or with some more experience or courage, an alternative way to obtain the results is as follows. Introduce the function \( v(x,t) \)

\[ v(x,t) = \sqrt{c(x)} u(x,t) \]

and find the equation for \( v(x,t) \). Then introduce a new independent variable \( \xi \) instead of \( x \) so that the equation for \( w(\xi, t) = v(x(\xi), t) \) becomes the simple translation equation

\[ \partial_\xi w(\xi, t) = -\partial_x w(\xi, t) \]

Perform the inverse transformations to find the general solution.

7.1.2 Pendulum with varying length

We describe the motion of the mass point with mass \( m \) at the end of a massless cord of length \( \ell(t) \) with the angle \( \phi(t) \) that the cord makes with the downward vertical.

**Warning.** You may remember the formula for the pendulum eqn with fixed length which can be written like

\[ \ddot{\phi} + \frac{g}{\ell} \sin \phi = 0; \]

if you substitute in here \( \ell = \ell(t) \), the wrong eqn is obtained, as we shall see.
CHAPTER 7. ASYMPTOTIC METHODS FOR ODE’S

Derivation of governing equation
First look at the kinematics of the motion. In time the position in the plane with axis \( x, z \) is given by
\[
x(t) = \ell(t) \mathbf{n}(t),
\]
with \( n(t) = (\sin \varphi, \cos \varphi) \) the normalized direction vector. Introducing the orthogonal normalized vector \( (t) = (\cos \varphi, -\sin \varphi) \), the velocity and acceleration are easily found to be
\[
\dot{x} = \dot{\ell} \mathbf{n} + \dot{\varphi} \mathbf{r},
\]
\[
\ddot{x} = (\ddot{\ell} - \dot{\varphi}^2) \mathbf{n} + (\dot{\ell} \dot{\varphi} + 2\ddot{\varphi}) \mathbf{r}.
\]

Now, dynamics comes in by applying Newton’s law:
\[
m \ddot{x} = \mathbf{F}
\]
with \( \mathbf{F} \) the force acting on the mass point. Due to gravity, this force is given by
\[
\mathbf{F} = (0, mg) = -mg \sin \varphi \mathbf{r} + mg \cos \varphi \mathbf{n}
\]
Restricting to the tangential component, the governing equation becomes
\[
m(\dot{\varphi}^2 + 2\ddot{\varphi}) = -mg \sin \varphi
\]
i.e.,
\[
\dot{\varphi}^2 + 2\ddot{\varphi} + g \sin \varphi = 0,
\]
which agrees with the eqn above for constant length, but has an additional (damping like-) term:
\[
\dot{\varphi} + \frac{\dot{\ell}}{\ell} \dot{\varphi} + \frac{g}{\ell} \sin \varphi = 0.
\]

Energy consideration
It is not trivial to guess whether the energy will increase or decrease, so let us investigate.

The energy is the sum of kinetic and potential energy
\[
E = \frac{1}{2} m \dot{\varphi}^2 - mg \ell \cos \varphi.
\]
It is constant when \( \ell \) is constant, and in general
\[
\dot{E} = -\frac{\dot{\ell}}{\ell} [m \dot{\varphi}^2 + mg \ell \cos \varphi]
\]
The term in brackets is clearly positive, and hence
\[
\text{sign}(\dot{E}) = \text{sign} \left( -\frac{\dot{\ell}}{\ell} \right)
\]
for increasing length, the energy decreases.

Time transformation and linearization
The eqn can be simplified by introducing a scaling in the time variable: for \( \tau \) such that
\[
\frac{d\tau}{dt} = \frac{1}{\ell^2(t)}
\]
and \( \psi(\tau) \equiv \varphi(t), L(\tau) \equiv \ell(t) \) the equation becomes
\[
\ddot{\psi} + gL^3(\tau) \sin \psi = 0
\]
The transformation is smooth and monotone (and hence invertible) since \( \ell > 0 \). Restricting to oscillations with small deviation from the vertical position, the nonlinear term \( \sin \psi \) can be approximated to \( \psi \), and we get the linear equation
\[
\ddot{\psi} + gL^3(\tau) \psi = 0.
\]

WKB-Analysis for slowly varying length
If we assume that \( \ell(t) \) varies slowly in the original time \( t \), the length \( L = L(\tau) \) varies slowly in the scaled time variable \( \tau \); then the standard WKB-method is applicable to this problem; the result is that (with \( \varepsilon \) a measure for the slow change)
\[
\psi(\tau) = \frac{A}{(gL^3(\tau))^{1/4}} e^{i\theta}
\]
with
\[
\theta = \int_0^\tau \sqrt{gL^3(\tau')} d\tau'
\]
is an \( O(\varepsilon) \) correct solution on an interval \( \tau \in (0, O(1/\varepsilon)) \). Translated to the original variables, the result is that
\[
\varphi(t) = \frac{A}{(gL^3(0/4))^{1/4}} e^{i\theta}
\]
with the phase expressed in the original time variable:
\[
\theta = \int_0^t \sqrt{g} \frac{dt}{\ell(\tau)}
\]
is an \( O(\varepsilon) \) correct solution on an interval \( t \in (0, O(1/\varepsilon)) \).

Exercise 69 1. Derive bounds for the energy based on the observation that \( \frac{d}{dt} E \leq \dot{\ell} E \leq -2\dot{\ell} E \) for decreasing length (similarly \( \frac{d}{dt} E \leq \dot{\ell} E \leq \dot{\ell} E \) for increasing length) using Gronwall’s lemma: if \( \frac{dy}{dt} \leq y \) for integrable functions \( y, g \), then \( y(t) \leq y(0) \exp \int_0^t g(\tau) d\tau \).
2. A mass-pendulum is released without velocity from its initial position at an angle of $\pi/4$ with the vertical. The length of the pendulum is increased slowly until it reaches twice its initial length. Determine the resulting motion after a long time. Calculate the energy of the motion as function of time. Does the final energy depend on the details of the increase in time? Compare the approximation with a numerically calculated solution; make a plot of the energy as function of time.

Remark 70 Observe that the phase factor is the same as would have been obtained if the wrong eqn $\dot{\varphi} + (g/\ell) \varphi = 0$ would have been used. But the amplitude differs essentially: the correct eqn gives an amplitude proportional to $(g/\ell)^{1/4}$, whereas the incorrect eqn would give an amplitude proportional to $(\ell/g)^{1/4}$, and hence an incorrect, opposite, effect for the change of amplitude with changing length!

7.1.3 Reflection of light in slowly varying media using WKB

In a medium of constant index of diffraction (for instance vacuum), a monochromatic light wave propagating in the z-direction with frequency $\omega_0$ is the solution of

$$E_{zz} - c^2 E_{zz} = 0$$

and is of the form

$$E = A e^{ik_0 z - \omega_0 t}$$

where $\omega_0^2 = c^2 k_0^2$.

Fixing the frequency $\omega_0$, write $E(z, t) = u(z) e^{i\omega_0 t}$ then

$$u_{zz} + n^2 u = 0, \text{ where } n^2 : = \frac{\omega_0^2}{c^2}$$

is the effective index of diffraction.

In a slowly varying medium the governing equation is

$$u_{zz} + n^2(\varepsilon z) u = 0$$

where the index of diffraction $n$ is a slowly varying function.

More specifically, we will consider a function $n$ that connects two constant levels, say a smooth transition from level $n_0$ for $z \leq 0$, to a level $n_1$ for $z \geq L$. Assuming $L$ to be large, the relation between $\varepsilon$ and $L$ is given by

$$\varepsilon = \frac{1}{L}.$$  

Assume that in the interval $(0, L)$ the function $n$ is smooth, but that at $z = 0$ and $z = L$ a discontinuity in the first derivative of $n$ may exist.

Exercise 71

1. Write down the WKB-approximation that is correct up to $O(\varepsilon)$ on the whole real line and check the continuity.

2. Calculate the first derivative of this function with respect to $z$ and show that this is only continuous up to $O(1)$.

Scattering problem

A solution of the form $e^{i(k_0 z - \omega_0 t)}$ can be interpreted as a mode travelling to the right. In a non-uniform medium, such a wave will be partly reflected, partly transmitted. The scattering problem is the problem to find a solution consisting of an incoming wave (travelling to the right) of given amplitude, plus a reflected wave (travelling to the left) with unknown amplitude $R$ (‘reflected’) and a transmitted wave (travelling to the right) of unknown amplitude $T$. This means that when the changes in the medium are restricted to the interval $[0, L]$, the solution sought is a function $u$ that satisfies for some values of $R$ and $T$

$$u(z) = e^{i\omega_0 z} + R e^{-i\omega_0 z} \text{ for } z \leq 0, \quad u(z) = T e^{i\omega_1 z} \text{ for } z \geq L,$$

while in the interval $(0, L)$ the WKB-approximation is used for a mode that travels to the right and one that travels to the left, so that the total solution is continuous and once differentiable, also in $z = 0$ and $z = L$.

In doing so it is found that the reflection coefficient depends on the discontinuities as follows:

$$R = \frac{1}{4L} \left\{ \frac{n'(0^+)}{n(0)^2} - e^{2\theta(L)} \frac{n'(L^-)}{n(L)^2} \right\} + O\left(\frac{1}{L^2}\right)$$

where $n'$ denotes the derivative of $n$ with respect to its argument (i.e. the slow variable $\zeta = \varepsilon z$).
Exercise 72 1. Write down the WKB-approximation in the interval \((0,L)\) and determine the amplitudes of the modes.

2. Investigate the formula by dimensional analysis. Observe that a smooth \(C^1\) function \(n\) has higher order reflection: \(R = O(\frac{1}{L^2})\). 

3. To find the solution correct up to \(O(\epsilon^3)\), and so with \(u_\epsilon\) correct up to \(O(\epsilon^2)\), a higher order WKB-approximation is required, e.g. by taking the WKB-approximation of the transformed WKB equation. This improved result looks like

\[
R = \frac{1}{4 L^2} \left( \frac{n''(0)}{n(0)\alpha} - \epsilon^2 \frac{n'(0)}{n(0)^{\alpha+2}} \right) + O(\frac{1}{L^3})
\]

for some exponent \(\alpha\) and phase factor \(\varphi\). Determine \(\alpha\) from dimensional analysis.

7.2 Self-excited oscillators

In this section we consider asymptotic methods for nonlinear equations. We restrict the analysis to ode’s but all ideas and methods are equally useful for the perturbation analysis of pdes (see also Chapter 8). In particular, phenomena as ‘mode generation’ and ‘resonance’ are essential in the infinite dimensional problems as well.

7.2.1 Motivation

As classical examples to which the ideas can be best illustrated we consider the following two nonlinear extensions of the harmonic oscillator (in suitably defined normalized coordinates):

the second order nonlinearity

\[
\ddot{x} + x + x^2 = 0, \tag{7.7}
\]

and the third order nonlinearity (Duffing’s eqn)

\[
\ddot{x} + x + x^3 = 0, \tag{7.8}
\]

both with the initial condition

\[
x(0) = \epsilon, \quad \dot{x}(0) = 0.
\]

Neglecting the nonlinearity, the solution is simply the motion of the harmonic oscillator

\[
x_1(0) = \epsilon \cos t.
\]

For small \(\epsilon\) this may well be a good approximation.

Exercise 73 1. Writing \(y = \epsilon x\) the initial value becomes \(y(0) = 1, \dot{y} = 0\), while \(\epsilon\) appears in the equations:

\[
\ddot{y} + y + \epsilon y^2 = 0, \text{ and } \ddot{y} + y + \epsilon^2 y^3 = 0
\]

In all of the following, do the same calculations for the Duffing eqn with opposite sign of the nonlinearity:

\[
\ddot{y} + y - \epsilon^2 y^3 = 0
\]

2. Show that the problem is a regular perturbation in the parameter \(\epsilon\) at \(\epsilon = 0\). Investigate the linearized problem, and show that \(\cos t\) is the unique solution.

Before discussing perturbation methods, investigate the global properties by phase plane analysis. In most cases this is not possible but here we can exploit the standard analysis for Newtonian systems:

\[
\ddot{x} = -\frac{\partial V}{\partial x}
\]

where \(V\) is the potential energy. For (7.7), the potential \(V\) is given by:

\[
V_2(x) = \frac{1}{2} x^2 + \frac{1}{3} x^3
\]

while for (7.8) the potential is given by

\[
V_3(x) = \frac{1}{2} x^2 + \frac{1}{4} x^4
\]

Sketch the graph of these potentials and investigate the phase portraits of solutions.

We conclude that for (7.7) periodic solutions with amplitude \(a\) smaller than a critical value, while all solutions for (7.8) are periodic.

The smooth distortions of the integral curves near \(a = 0\) compared to the circles of the harmonic oscillator, gives additional confidence in the regularity of the problems.

One phenomenon that is not directly visible from the phase plane analysis is the fact that the periods of the periodic motion depend on the amplitude (on \(\epsilon\)), in contrast to the linear equation for which every solution has period \(2\pi\). This can, however, be expected by observing that both equations can be written like

\[
\ddot{x} + \omega(x)^2 x = 0
\]
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with

$$\omega(x)^2 = 1 + x, \text{ resp. } \omega(x)^2 = 1 + x^2$$

for (7.7) and (7.8) respectively. Here the period depends on the solution (different from the equations from the previous section, WKB-method, where the adiabatic changes in the frequency were described “externally”), and hence is not known in advance. It can therefore be expected that perturbation-methods will also have to be able to vary the frequency with $\epsilon$.

Mode generation from nonlinearity

Differentiation of a ‘mode’, a harmonic function, leads to the ‘same’ mode (in complex notation):

$$\partial_t e^{i \omega t} = i \omega e^{i \omega t},$$

and in real variables, the set $\{ \cos \omega t, \sin \omega t \}$ is invariant:

$$\partial_t \sin \omega t = \omega \cos \omega t, \quad \partial_t \cos \omega t = -\omega \sin \omega t$$

However, nonlinear actions do not leave the mode unchanged: new modes are produced depending on the precise form of the nonlinearity; for instance for a quadratic nonlinearity:

$$x = e^{i \omega t}, \quad x^2 = e^{2i \omega t}:$$

the double harmonic mode is ‘generated’; in real space

$$(\cos \omega t)^2 = \frac{1}{2} (1 + \cos 2\omega t)$$

and

$$(\sin \omega t)^2 = \frac{1}{2} (1 - \cos 2\omega t)$$

Of course, also multiplications of two different modes, say $\omega_1, \omega_2$ produce two (new) modes: $\omega_1 \pm \omega_2$.

Mode generation is the essential cause of many important and difficult phenomena like turbulence: generating effects on small time/space scales from events that start at large time/space scales.

Naive perturbation method

With $\epsilon \cos t$ as a good approximation, we try to find a more accurate solution by substituting a series expansion for the solution:

$$x(\epsilon, t) = \epsilon \cos t + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + \ldots$$

Substituting this series in the equation leads for each $x_k, k \geq 1$ to an equation of the form

$$\ddot{x}_k + x_k = F_k$$

where $F_k$ depends on the functions $x_j, j \leq k - 1$ and is therefore known in a successive treatment. The solution of this linear eqn can be found explicitly; however, even if the forcing $F_k$ is bounded, the solution does not have to be bounded; when the forcing is periodic with the same period as the solution of the homogeneous eqn, the phenomenon of resonance appears.

Resonance

The explicit solution, of

$$\ddot{u} + \omega_0^2 u = a \cos \omega t$$

is given by

$$u(t) = a \frac{\omega_0}{\omega_0^2 - \omega^2} \cos \omega t \text{ for } \omega \neq \omega_0$$

(the solution is forced to be periodic with the period of the force) and by the unbounded solution

$$u(t) = a \frac{\omega_0}{2\omega_0} t \sin \omega_0 t \text{ for } \omega = \omega_0$$

The unboundedness is a consequence of the forcing with the same period as the natural frequency $\omega_0$ of the unforced system and is called resonance. Resonance can be useful to obtain solutions of large amplitude (when pushing a swing) but is often undesirable.

Given a force $f$, how can we investigate if resonance occurs in the equation

$$\ddot{u} + \omega_0^2 u = f(t),$$

or, stated differently, can we give conditions on $f$ that guarantee that the solution is uniformly bounded for all time?

The answer will be clear if $f$ is periodic with period $T_0/N$ for some $N$, where $T_0$ is the fundamental period $T_0 = 2\pi / \omega_0$. (Such functions are called subharmonic). Then, $f$ can be expanded as a Fourier series, and the requirement must be that the coefficients in front of the term with $e^{i \omega_0 t}$ vanish, i.e.

$$\int_0^{T_0} f(t) \cos \omega_0 t \, dt = 0, \quad \int_0^{T_0} f(t) \sin \omega_0 t \, dt = 0.$$
These conditions on \( f \) are known as solvability conditions.

A somewhat different way to derive these conditions is as follows. Since \( \sin \omega_0 t, \cos \omega_0 t \) are solutions of the homogeneous eqn, multiply the equation with \( \sin \omega_0 t \) and \( \cos \omega_0 t \) and integrate over a fundamental period \( T_0 \). Then the left hand sides vanish, demanding the right hand sides to vanish. This leads to the conditions above as being necessary.

**Remark 74** Fredholm alternative (analogy with Linear Algebra)

A more deeper way to interpret the conditions is as follows. Write the equation in operator form by introducing the operator \( L \) such that \( L u = \hat{u} + \omega_0^2 u \). Then the equation can be written like

\[
L u = f
\]

Furthermore, the operator is symmetric with respect to the integral inner-product on the set of \( T_0 \)-periodic functions

\[
\int_0^{T_0} v(t) L u(t) dt = \int_0^{T_0} L v(t) u(t) dt
\]

The kernel of \( L \) is given above:

\[
\ker L = \{ \sin \omega_0 t, \cos \omega_0 t \}
\]

Then the solvability conditions are nothing but the condition that the force \( f \) is perpendicular to the kernel of \( L \). Compare this with the Fredholm alternative in Linear Algebra: for (symmetric) matrices \( A \) the solvability condition for \( Ax = b \) is that \( b \perp \ker(A^*) \).

### 7.2.2 Third-order nonlinearity: Duffing’s eqn

We first consider the oscillator with third order nonlinearity

\[
\text{Eqn} \equiv \ddot{x} + x + x^3 = 0, \quad x(0) = \epsilon, \quad \dot{x}(0) = 0
\]

**Naive perturbation technique**

Substituting the series expansion

\[
x(\epsilon, t) = \epsilon x_1(t) + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + \ldots
\]

in the equation there results

\[
\ddot{x}_1 + x_1 = 0, \quad x_1(0) = 1, \quad \dot{x}_1(0) = 0
\]

\[
\ddot{x}_2 + x_2 = 0, \quad x_2(0) = 0, \quad \dot{x}_2(0) = 0
\]

\[
\ddot{x}_3 + x_3 = -x_1^3, \quad x_3(0) = 0, \quad \dot{x}_3(0) = 0
\]

Clearly the solution for \( x_1 \) is given by \( x_1 = \cos t \).

The solution for \( x_2 \) vanishes identically: \( x_2 \equiv 0 \).

The forcing of the eqn for \( x_3 \) can then be expanded (after some straightforward calculations) to

\[
x_3(t) = (\cos t)^3 = \frac{1}{4} \cos 3t + \frac{3}{4} \cos t
\]

The solution can be written down, but the appearance of the source term \( \frac{3}{4} \cos t \) indicates that resonance will appear:

\[
x_3(t) = -\frac{3}{8} t \sin t + \frac{1}{32} \cos 3t
\]

Hence, up to third order, the solution reads

\[
x(\epsilon, t) = \epsilon \cos t + \epsilon^3 \left[ -\frac{3}{8} t \sin t + \frac{1}{32} \cos 3t \right] + \ldots
\]

**Interpretation.**

For small values of \( t > 0 \), in fact for \( t \) in any interval of prescribed length, \( t \in [0, T] \), this is a good approximation for \( \epsilon \) sufficiently small, since the third order term, although increasing linearly in time, is bounded by constant \( \epsilon^3 T \), and a same result holds for the other terms.

However, in practice we want to go further and have a solution that is also correct for larger values of time. Since the solution is known to be bounded for all time (from phase plane analysis), we would even like to have an approximation that holds for all time, also \( t \to \infty \); an approximation with this property is called a uniformly valid approximation. Clearly the approximation constructed above does not satisfy this criterion, since \( \epsilon^3 t \) will explode for \( t \to \infty \) at fixed \( \epsilon \).

**Remark 75** From symmetry of the equation (if \( x(t) \) is a solution with \( x(0) = \epsilon \), then \( y(t) \equiv -x(t) \) is the solution with \( y(0) = -\epsilon \)) it follows that the solution \( x(\epsilon, t) \) should be odd in \( \epsilon \), and hence all terms \( \epsilon^k x_k(t) \) with \( k \) even should vanish.
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Improved perturbation techniques

The naive method above (series expansion in $\epsilon$) does not give a possibility to prevent the linear growth in time, since this is caused by the resonance phenomenon in the equation for $x_3$. All improved perturbation techniques are based on creating more flexibility in the Ansatz for the solution so that this resonance can be prevented.

One improved technique is as follows; it is often referred to as Lindstedt-Poincaré technique, and is closely related to other methods like ‘uniformization’, ‘renormalization’, ‘averaging technique’(van der Pol, or Krylov-Bogoliubov method), and even ‘multiple time scale’ method (although in this problem no two time scales can be distinguished).

We will demonstrate one technique, which will be called the amplitude-phase method, for reasons to become clear soon.

Neglecting the nonlinear term, the solution is a simple harmonic function:

$$x(t) = \epsilon A \cos(\theta)$$

where the amplitude $A$ is constant, and the phase $\theta$ is simply $\theta = t + \alpha$, where $\alpha$ is a phase constant. In order to choose the constants such that the initial value is satisfied, $A$ and $\alpha$ have to satisfy

$$A \cos(\alpha) = 1.$$  

Of course, $A = 1, \alpha = 0$ is one solution, but not unique.

A straightforward generalization would be to let $A$ and $\theta$ depend on time (and $\epsilon$) in a way to be decided, such that the approximation satisfies the equation as good as possible (variation of ‘parameters’). Substituting the ‘Ansatz’

$$x(\epsilon, t) = \epsilon A(\epsilon, t) \cos(\theta(\epsilon, t))$$

into the equation, will lead to equations for $A$ and $\theta$ which may then be solved approximately. Observe that in this Ansatz, the ‘amplitude’ and ‘phase’ are not uniquely defined, just as above. Also, observe that the WKB-method is based on the same idea.

Let us now investigate the details.

Substitution in the eqn leads to

$$Eqn = \epsilon R_c \cos(\theta) + \epsilon R_s \sin(\theta) - \epsilon^3 \frac{1}{4} A^3 \cos(3\theta)$$

with

$$R_c = \ddot{A} - A \theta^2 + A + \epsilon^2 \frac{3}{4} A^3$$

and

$$R_s = -2\ddot{\theta} - A \dot{\theta}$$

In order to reduce the residue as much as possible, one now chooses $A, \theta$ in such a way that $R_c$ and $R_s$ vanish; this leads to the so-called phase-amplitude equations

$$\ddot{A} - A \dot{\theta}^2 + A + \epsilon^2 \frac{3}{4} A^3 = 0$$

and

$$-2\ddot{\theta} - A \dot{\theta} = 0.$$  

A solution of these eqns can now be found easily: $A$ independent of time and $\theta$ such that

$$\dot{\theta}^2 = 1 + \epsilon^2 \frac{3}{4} A^2$$

which leads to

$$\dot{\theta} = 1 + \epsilon^2 \frac{3}{8} \dot{\theta}^2 + O(\epsilon^4),$$  

i.e. a linear dependence on time for $\theta$. Taking $A=1$, the approximate solution becomes

$$x(\epsilon, t) \approx \epsilon \cos(\omega(\epsilon) t), \text{ with } \omega(\epsilon) = 1 + \frac{3}{8} \epsilon^2.$$  

The term that remains in the residue, $\frac{1}{4} \epsilon^3 \cos(3\theta)$, is of higher order and does not lead to resonance; hence the solution constructed above is uniformly valid:

$$x(\epsilon, t) = \epsilon \cos \left( \left( 1 + \frac{3}{8} \epsilon^2 + O(\epsilon^4) \right) t \right) + O(\epsilon^3).$$

**Remark 76** Compare the solution obtained above with the solution from series expansion, and observe that the series expansion solution is precisely the expansion in $\epsilon$ of the phase-amplitude solution! This explains the nonuniformity of the series expansion solution.

### 7.2.3 Second-order nonlinearity

We consider the oscillator with second order non linearity

$$\ddot{x} + x + x^2 = 0, \ x(0) = \epsilon, \ \dot{x}(0) = 0.$$
Naive perturbation technique

Substituting the series expansion
\[ x(\epsilon, t) = \epsilon x_1(t) + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + \ldots \]
in the equation there results
\[ \ddot{x}_1 + x_1 = 0, \quad x_1(0) = 1, \quad \dot{x}_1(0) = 0 \]
\[ \ddot{x}_2 + x_2 = -\dot{x}_1^2, \quad x_2(0) = 0, \quad \dot{x}_2(0) = 0 \]
\[ \dddot{x}_3 + x_3 = -2x_1 \dot{x}_2, \quad x_3(0) = 0, \quad \dot{x}_3(0) = 0 \]

With \( x_1 = \cos t \), the equation for \( x_2 \) becomes
\[ \ddot{x}_2 + x_2 = -(\cos t)^2 \equiv -\frac{1}{2}(1 + \cos 2t) \]
with solution (satisfying the homogeneous initial data)
\[ x_2(t) = \frac{1}{2} + \frac{1}{6} \cos 2t + \frac{1}{3} \cos t \]

So, in second order the solution is uniformly bounded. Continuing with the third order calculation, the forcing of the eqn for \( x_3 \) can be expanded (after some straightforward calculations) to
\[ -2x_1 \dot{x}_2 = -\frac{1}{3} + \frac{5}{6} \cos t - \frac{1}{3} \cos 2t - \frac{1}{6} \cos 3t \]
The solution can be written down, but the appearance of the source term \( \frac{5}{6} \cos t \) indicates that resonance will appear:
\[ x_3(t) = \frac{5}{12} t \sin t + ( \text{periodic terms} ) \]

Hence, up to third order, the solution reads
\[ x(\epsilon, t) = \epsilon \cos t + \epsilon^2 \left[ -\frac{1}{2} + \frac{1}{6} \cos 2t + \frac{1}{3} \cos t \right] \]
\[ + \epsilon^3 \left[ \frac{5}{12} t \sin t + ( \text{periodic terms} ) \right] \]

Just as above for the Duffing eqn, series expansion leads to resonance and then to a nonuniformly valid approximation.

**Exercise 77  Uniform approximation**

Find a uniformly valid approximation up to order \( O(\epsilon^3) \); take as ‘Ansatz’ (why??)
\[ x(\epsilon, t) = \epsilon A \cos(\theta) + \epsilon^2 \left( B + C_1 \cos(2\theta) + C_2 \sin(2\theta) \right) \]

Give the interpretation of the solution.
Chapter 8

Asymptotic models

This chapter starts with some general remarks about the core of mathematical modelling in science. Starting point are well defined sets of equations which are, however, too difficult in general to be analysed directly. Therefore approximations are sought: approximations of the equations, by restricting to special phenomena and thereby losing part of the validity of the original model, or approximations of special classes of solutions.

In the second section we illustrate the process of finding approximate model equations for the problem of surface waves. Starting with the exact set of equations, simplified models are derived by restricting to rather small, rather long waves; the famous model eqns of Boussinesq and of Korteweg–de Vries will be derived. The section ends with a short outlook for an even better approach to the modelling process: the consistent modelling retaining the Hamiltonian structure of the basic eqn.

In the third section we deal directly with special solutions. Taking the KdV eqn as basic eqn (for simplicity), the dynamics of wave groups (already encountered in ‘Basics 1’) is considered in more detail. The analysis is restricted to the linearized case, and the amplitude eqn for the envelope of the periodic train is derived in a way that prepares for the derivation of the amplitude eqn for nonlinear wave groups.

8.1 General aspects of modelling

Consider a basic set of equations that describe a large number of phenomena; to be specific we will consider the example of surface waves in this section. Let us write this basic model as follows: \( u \) will denote the (collection of) state variables, and the set of equations (ode’s, pde’s, etc) is symbolically written like

\[
\mathcal{E}(u) = 0.
\]

The state space will be denoted by \( \mathcal{U} \). We assume that there are trajectories in \( \mathcal{U} \) that describe various type of phenomena; for instance the full set of surface waves allow the description of waves of different type: long waves, short waves, small amplitude waves, steep waves, very regular, and irregular waves.

If we want to consider only one specific type of phenomena (for instance long waves), one may try to find a simpler description: an approximation of the equation that describes in a good approximation the desired phenomena. In general, this approximation may not reveal the other phenomena that are described by the basic model; this is the price one has to pay for dealing with a simpler problem. On the other hand, the approximate model may have ‘spurious solutions’, solutions of the approximate model that are not relevant for the basic model since they do not satisfy the assumptions put forward in the derivation of the approximation. Many mathematical problems may arise and in specific situations one may easily loose track of what is going on. Therefore let’s try to schematize the procedure.

Let the phenomena we are specifically interested in be characterized by a part of the state space, say a subset \( \mathcal{U} \subset \mathcal{U} \); the appearance of \( \epsilon \) symbolically emphasizes that in general the specific phenomena are limiting cases. For instance, when looking for small waves, \( \epsilon \) (small) may indicate the amplitude; for long waves, \( 1/\epsilon \) may be the wave length, etc.
Then, for functions from \( \mathcal{U} \) we approximate the original set of equations, say we derive an approximation \( \mathcal{E}_\epsilon \) such that
\[
\left| \mathcal{E}(u) - \mathcal{E}_\epsilon(u) \right| = o(1) \quad \text{for } u \in \mathcal{U}:
\]
for the functions describing the desired phenomenon \( (u \in \mathcal{U}) \), the set of equations \( \mathcal{E}_\epsilon \) approximates the original set well.

Then we expect that the approximate model equations
\[
\mathcal{E}_\epsilon(v) = 0
\]
can be investigated (more easily than the basic model equations). Then, \( \textit{provided } v \textit{ is a solution within } U_\epsilon \), it holds that
\[
\mathcal{E}(v) = o(1).
\]
That is, \( v \) satisfies ‘almost’ the original basic equations; the quantity \( \mathcal{E}(v) \) is called the \textit{residue}: it is the measure of how much \( v \) does not satisfy the basic set of equations. Intuitively it is expected (hoped) that then \( v \) is also close to a solution of the exact equation:
\[
\left| V - v \right| = o(1) \quad \text{for some solution } V \text{ of } \mathcal{E}(V) = 0.
\]

Unfortunately, most of the steps outlined above are often very difficult to justify in general, and even in specific cases. Moreover, in trying to prove the results, the general norms used above, \( \left| \cdot \right| \), have to specified, while different choices may lead to positive or negative results, or to results of a rather different nature. This was shown already in the chapter on Perturbation methods; there equations were considered which contained some small parameter \( \epsilon \), and very different behaviour was observed, depending on the specific perturbation (where the \( \epsilon \)-term appeared) and on the norms used. This will only be more profound when dealing with pde’s.

In view of the mathematical problems with formal verifications, it is essential for at least a partial justification, to make the steps and underlying assumptions as explicit as possible, and to view the process of modelling and the task of justification as two intimately related topics.

In this chapter we will illustrate several aspects to the surface wave problem; various models will be constructed for different types of wave motions. Specifically we will deal with

- linear theory for small waves, with full and approximate dispersion,
- theory for waves on a shallow layer, nonlinear, without dispersion,
- Boussinesq equations, describing waves with a specified relation between (long) wave length and (small) amplitude for which nonlinearity and dispersion appear in a balanced way,
- \textit{KdV} (Korteweg de Vries) equation, for Boussinesq types of waves that travel in one direction.

We will also investigate in more detail than in ‘Basics 1’ the propagation of wave groups.

### 8.2 Surface wave models

#### 8.2.1 Full set of equations

We consider the motion of a layer of fluid under the following simplifying assumptions:

- the fluid is \textit{inviscid, incompressible} (density normalized to unity), and no surface tension;
- the bottom is flat, at depth \( z = -H \);
- the fluid motion is \textit{irrotational}, assumed to be uniform in the (horizontal) \( y \)-direction and unbounded in the \( x \)-direction; if the horizontal and vertical velocities are denoted by \( U = U(x, z, t) \) and \( W = W(x, z, t) \) respectively, irrotational motion means
  \[
  U_z - W_x = 0
  \]
  and allows the introduction of the fluid potential \( \Phi \) such that
  \[
  (U, W) = \nabla \Phi, \quad U = \Phi_x, \quad W = \Phi_z;
  \]
  incompressibility implies
  \[
  U_x + W_z = 0, \quad \Delta \Phi = 0;
  \]
the surface elevation is the graph of a function
(no overturning waves) \( \eta = \eta(x, t) \).

Then the governing equations are (see ‘Motion of
Continua’)

\[
\Delta \Phi \equiv \Phi_{xx} + \Phi_{zz} = 0, 
- H < z < \eta(x, t) \tag{8.1}
\]

\[
\Phi_z = 0 \text{ at } z = -H, \tag{8.2}
\]

\[
\partial_t \eta = -\eta_x \Phi_x + \Phi_z \text{ at } z = \eta(x, t), \tag{8.3}
\]

\[
\partial_t \Phi + \frac{1}{2} (\Phi^2_x + \Phi^2_z) + g\eta = 0 \text{ at } z = \eta(x, t). \tag{8.4}
\]

**Alternative descriptions**

For various reasons it is convenient to introduce
the potential at the free surface

\[
\varphi(x, t) = \Phi(x, \eta(x, t), t)
\]

and its \( x \)-derivative (which is a velocity type of
quantity)

\[
u(x, t) = \partial_x \varphi = \Phi_x + \Phi_z \eta_x
\]

Then \( \partial_x \varphi = \Phi_t + \Phi_{xx} \eta_t \), and the free surface
conditions become

\[
\partial_t \eta = -\eta_x \nu_x + w(1 + \eta_x^2), \tag{8.5}
\]

\[
\partial_t \nu = -\partial_x [g\eta + \frac{1}{2} w^2 - \frac{1}{2} w^2(1 + \eta_x^2)] \tag{8.6}
\]

where \( w \) is the vertical velocity at the free surface:

\[
w(x, t) = W(x, \eta(x, t), t).
\]

This is a set of equations for \( \eta, \nu \) in which, however, \( w \) still has to be expressed as a function of
\( \eta, \nu \).

Another way to rewrite the kinematic surface condition
(the continuity equation) is as a local con

\[
\partial_t \eta = -\partial_x \int_{-H}^{\eta} \Phi_x \,dz \tag{8.7}
\]

which shows that the flux equals the integrated
horizontal velocity. Stated differently, this is nothing
but mass-conservation.

**8.2.2 Approximations**

The equations above are the description of the ‘ba
sic model’, describing large classes of various wave
profiles. In the equations (8.5,8.6) the quantity \( w \)
can only approximately be expressed in term of
\( \eta, \nu \).

Looking for approximations, observe that the ge
ometry has only one characteristic length: the
depth of the layer \( H \). Limiting cases, \( H \rightarrow 0 \) or
\( H \rightarrow \infty \), may lead to simplified equations, pro
vided the characteristic length scales of the waves
are restricted appropriately. By restricting to spe
cific classes of waves we will construct simplified
sets of equations: approximate models.

To prepare for the various approximations, we start
to introduce the potential at the still water level as an auxiliary variable:

\[
\phi_0(x, t) := \Phi(x, \eta = 0, t).
\]

Then the solution of the Laplace problem \( \Delta \Phi = 0 \)
with \( \Phi_z = 0 \) at \( z = -H \) will depend on the poten
tial at the still water level and can be found with
Fourier (integral) transformation with respect to
\( x \); the result is

\[
\Phi(x, z, t) = \int \hat{\phi}_0(k, t) e^{ikx} \frac{\cosh k(H + z)}{\cosh kH} \, dk \tag{8.8}
\]

where \( \hat{\phi}_0(k, t) \) is the Fourier transform of \( \phi_0(x, t) \).

An expression for \( w \) has now to be found by elim
inating \( \phi_0 \) from the two relations

\[
\varphi(x, t) = \int \hat{\phi}_0(k, t) e^{ikx} \frac{\cosh k(H + \eta)}{\cosh kH} \, dk \tag{8.9}
\]

\[
w(x, t) = \int \hat{\phi}_0(k, t) k \sinh k(H + \eta) \frac{\cosh kH}{\cosh kH} \, dk \tag{8.10}
\]

In doing so the following, it is convenient to in
troduce the pseudo differential operator \( \hat{R} \) that
has symbol

\[
\hat{R}(k) = \frac{\tanh kH}{k};
\]

recall that this means that for any function \( v \) with
Fourier transform \( \hat{v} \) it holds that

\[
\hat{R}(v)(k) = \hat{R}(k) \hat{v}(k).
\]
8.2.3 Small amplitude waves: exact dispersion

The simplest approximation is obtained by looking for infinitesimal small amplitude waves. Taking \( \eta \) to be small, the free surface eqns show that also \( u \) is small, and the equations become (the linearized eqns)

\[
\begin{align*}
\partial_t \eta &= w, \\
\partial_t u &= -\partial_x g \eta
\end{align*}
\]

while the relations between \( \varphi \) and \( w \) follow simply by substituting \( \eta = 0 \) in (8.9,8.10), leading to

\[
\begin{align*}
\varphi &= \phi_0 \\
w &= -R \partial_x^2 \phi_0
\end{align*}
\]

and so \( w = -R \varphi_{xx} = -R u_x \).

Hence, the linearized equations become

\[
\begin{align*}
\partial_t \eta &= -\partial_x u, \\
\partial_t u &= -\partial_x g \eta
\end{align*}
\]

This is a closed set of two (linear) equations in the variables \( u, \eta \), a set of dispersive wave equations. Upon eliminating \( u \), there results an equation of second order in time for \( \eta \):

\[
\partial_t^2 \eta = g \partial_x^2 R \eta \tag{8.11}
\]

The dispersion relation for this equation is found by substituting mono-chromatic modes \( \eta = e^{i(kx + \omega t)} \), showing that

\[
\omega^2 = gk^2 \hat{R}(k) = gk \tanh kH
\]

and so

\[
\Omega_\pm = \pm \sqrt{gk \tanh kH}.
\]

Approximate dispersion for long waves

Having found the dispersion relation, the solution of the initial value problem can be written down in Fourier integrals.

Let us investigate the specific properties of the dispersion as found here.

First note that the phase velocity and the group velocity, given by

\[
\begin{align*}
V_{ph}(k) &= \frac{\omega}{k}, \\
V_{gr}(k) &= \frac{\partial \Omega}{\partial k}
\end{align*}
\]

are qualitatively as depicted (for \( gH = 1 \)):

The wave number \( k \) is inversely proportional to the wave length; hence the limit \( k \to 0 \) is the long wave limit. For long wave lengths we have the approximations:

\[
\begin{align*}
\Omega &= k\sqrt{gH(1 - \frac{1}{6}(kH)^2)} + O(k^5), \\
V_{ph}(k) &= \sqrt{gH(1 - \frac{1}{6}(kH)^2)} + O(k^4), \\
V_{gr}(k) &= \sqrt{gH(1 - \frac{1}{2}(kH)^2)} + O(k^4);
\end{align*}
\]

in particular, for long waves, the group and phase velocity equal the characteristic velocity \( \sqrt{gH} \).

In this approximation, the symbol of the operator \( R \) becomes \( k^2H(1 - \frac{1}{3}(kH)^2) \), which corresponds to the differential operator

\[
R_{kr} = H(1 + \frac{1}{3}H^2\partial_x^2).
\]

With this long wave length approximation, the equation for the surface elevation becomes

\[
\partial_t^2 \eta = gH(\partial_x^2 \eta + \frac{1}{3}H^2\partial_x^4 \eta) \tag{8.12}
\]

and incorporates the dispersive effects for long waves.

Relevance

Under the assumption of small amplitude waves (and correspondingly small fluid velocities) we derived an equation for the waves, eqn (8.11), which in the long wave limit, simplified to (8.12).

The relevance of these equations should now be investigated.

First the relevance of (8.12) for (8.11). Since the equation is linear, no mode interaction takes place:
all mode numbers present in the initial profile remain there for later times, but no other wave numbers are created by (8.12). Hence, starting with initial data with small wave numbers, this property remains true for all time, and for these initial data (8.12) is a valid approximation of (8.11).

Initial data that are short wave profiles evolve smoothly as long as $\omega(k)^2$ remains positive, but may not represent solutions of (8.11) anymore when $k$ becomes too large. Note, however, that when (8.12) is discretized (finite difference, or finite elements) for numerical purposes, short wave errors will be introduced in general which will become unstable ($\omega^2$ negative). This makes this long-wave length approximation less useful. (Modifications are possible to avoid this undesired property).

Now the relevance of (8.11), which is equivalent to the linearized set of equations for the boundary value problem for $\Phi$: how well do the solutions of (8.11) satisfy the full set of equations? To satisfy the a priori assumptions to derive the equations, the amplitude $\eta$ should be bounded, such that $c\eta$ is small. Since (8.11) has all solutions bounded for all time, with bound determined by the initial condition, the smallness can be assured for the correct set of initial data. Then the terms that were neglected in the derivation remain small, provided also all derivatives (that appear in the neglected terms) remain bounded. Again, this is true for solutions of this linear equation (8.11), showing that the residue is small.

The final step in the justification now boils down to showing that from the fact that the residue is small, the difference with some exact solution of the basic set of equations is small (maybe on a limited scale of time). Unfortunately, this requires a difficult investigation of the full set of (nonlinear) equations, which cannot be done here.

8.2.4 Shallow water waves

Now we want to look at another limiting case: for finite amplitude waves, i.e. for which $\eta/H = O(1)$, we consider the case of small depth. More precisely, looking at the expressions derived for $\varphi$ and $w$, we will have to assume that

$$kH = O(\alpha)$$

with $\alpha$ a small parameter. This means that only waves can be considered that are “long” compared to the depth of the fluid, wave length $\lambda$ such that

$$\frac{H}{\lambda} = O(\alpha);$$

this situation is usually referred to as shallow water limit.

In this situation we have, for instance,

$$\frac{\sinh k(H + \eta(x, t))}{\cosh kH} = kH + O(\alpha),$$

and, to the same order of approximation,

$$\varphi = \phi_0 + O(\alpha^2),$$

$$w = \int \frac{k \sinh k(H + \eta)}{\cosh kH} e^{ikx} \, dk$$

$$= \int \hat{\phi}_0 k^2 (H + \eta) e^{ikx} + O(\alpha^2) \, dk$$

and so

$$w = -(H + \eta) u_x + O(\alpha^2).$$

Hence, (8.5) becomes

$$\partial_t h = \partial_x [h \partial_x \varphi] = -\partial_x [hu],$$

where $h = H + \eta$

while (8.6) becomes

$$\partial_t u + \partial_x [g\eta + \frac{1}{2}u^2] = 0.$$

These equations describe in the shallow water theory finite amplitude waves with small depth/wavelength ratio $H/\lambda = O(\alpha)$.

Observe that although $w = O(\alpha)$, i.e. vertical fluid velocities are small, the derivative $W_x$ is of order unity.

Interpretation and analysis of the equations

The two equations,

$$\partial_t h = -\partial_x [hu]$$

$$\partial_t u = -\partial_x [gh + \frac{1}{2}u^2].$$

have the form of a set of conservation laws, denoting the continuity and momentum equation respectively.
Let’s investigate the vertically integrated form of these equations, using the fact that in the required approximation the horizontal velocity is depth independent.

- Equation (8.13) is mass conservation (density \( \rho \equiv 1 \)): on an interval \( x \in [a, b] \), the change of mass

\[
\partial_t \int_a^b dx \int_{-H}^{\eta(x,t)} \rho dz = \partial_t \int_a^b \rho h(x,t) \, dx
\]

is a consequence of mass flux through the boundaries:

\[
- \int_{-H}^{\eta(x,t)} \rho u(x,t) \, dz \bigg|_{x=a}^{x=b} = - \rho h(x,t) u(x,t) \bigg|_{x=a}^{x=b}
\]

- Equation (8.14) is the momentum law: the change of momentum

\[
\partial_t \int_a^b dx \int_{-H}^{\eta(x,t)} \rho u(x,t) \, dz = \partial_t \int_a^b \rho h(x,t) u(x,t) \, dx
\]

is caused by momentum flux through the boundary:

\[
- \int_{-H}^{\eta(x,t)} \rho u^2 \, dz \bigg|_{x=a}^{x=b} = - \rho uu^2 \bigg|_{x=a}^{x=b}
\]

and change in potential energy

\[
- \int_{-H}^{\eta(x,t)} \rho g z \, dz \bigg|_{x=a}^{x=b} = - \frac{1}{2} \rho g [\eta^2 - H^2] \bigg|_{x=a}^{x=b}
\]

Together this leads to

\[
\partial_t (hu) = -\partial_x \left[ \frac{1}{2} g \eta^2 + hu^2 \right].
\]

which, with the mass conservation, leads to (8.14).

The shallow water equations are nonlinear, and of a special mathematical structure: the two equations (8.13,8.14) are of the form of a set of first order hyperbolic equations:

\[
\partial_t \begin{pmatrix} h \\ u \end{pmatrix} = -\partial_x \begin{pmatrix} hu \\ gh + \frac{1}{2} u^2 \end{pmatrix}.
\]

Equations with such a structure can be analyzed with the method of characteristics. We can summarize the results as follows.

The characteristic velocities are given by \( u \pm \sqrt{gh} \) and the corresponding Riemann invariants are \( u \pm 2\sqrt{gh} \). This means that

\[
D_t (u \pm 2\sqrt{gh}) \equiv \partial_t (u \pm 2\sqrt{gh}) + \frac{dx}{dt} \partial_x (u \pm 2\sqrt{gh}) = 0
\]

on the characteristics

\[
\frac{dx}{dt} = u \pm \sqrt{gh}.
\]

One simple wave solution, the one moving to the right, is the solution for which \( u - 2\sqrt{gh} = 0 \); then \( D_t \sqrt{gh} = 0 \) on \( \frac{dx}{dt} = 3\sqrt{gh} \); explicitly it is given for an initial elevation \( h(x,0) = h_0(x) \) by

\[
h(x,t) = h_0(\xi) \text{ for } x = \xi + 3t \sqrt{gh_0(\xi)}.
\]

The characteristics show that every initial condition with increasing elevation will eventually “break”; in fact, this simple wave solution is described by the ‘nonlinear breaking equation’ investigated in ‘Basics 2’.

Relevance

When looking at solutions of the shallow wave model, two observations should be made. First, when small amplitude solutions are considered, linearization of the equations, the equations are the simple second order wave equation:

\[
\partial_t^2 \eta = g H \partial_x^2 \eta;
\]

hence the linear limit of the shallow wave model differs from the small amplitude model derived in the previous section in the fact that no dispersion is incorporated.

Secondly, for finite amplitudes, the solutions of this model equation are very different from the solutions of the linear model: now solutions ‘break’, i.e. existence is assured only on a limited time interval. In fact, starting with initial conditions that satisfy the assumptions of the derivation, i.e. \( kH = O(\epsilon) \) at \( t = 0 \) for all relevant wave numbers, for increasing time, during steepening of the solution, this condition will become violated and the
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Solutions of these equations will loose their relevance as approximate solutions of the original set of free surface equations. Note that, contrary to the linear equations of the small amplitude model, the nonlinearity generates small waves through mode interaction.

8.2.5 Boussinesq approximation

It turns out that for a more restricted class of wave forms, a more relevant approximation is possible than the linear or the shallow water approximation. This model incorporates nonlinear effects of the shallow wave model, as well as some of the dispersive effects that are present in the linear model; taken together in an appropriate way, the dispersive effects in some sense balance the nonlinear steepening in such a way that smooth solutions exist that approximate the free surface equations in a good way. Therefore, let \( \eta = O(\epsilon) \) and observe

\[
\varphi = \int \hat{\phi}_0 \left[ 1 + k^2 (\eta \hat{R}(k) + \frac{1}{2} \eta^2) + O(\epsilon^3) \right] e^{ikx} \, dk
\]

while

\[
w = \int \hat{\phi}_0 \left[ k^2 \hat{R}(k) + k^2 \eta + k^4 \hat{R}(k) \eta^2 + O(\epsilon^3) \right] e^{ikx} \, dk
\]

and so

\[
\varphi = \phi_0 - (\eta R + \frac{1}{2} \eta^2)(\phi_0)_{xx}
\]

\[
w = -(R + \eta^2 R \partial_x^2)(\phi_0)_{xx}
\]

From this it follows

\[
\phi_0 = \varphi + (\eta R + \frac{1}{2} \eta^2) \varphi_{xx}
\]

Inserting this in the expression for \( w \) one arrives at

\[
w = -(R + \eta^2 R \partial_x^2) \left[ \varphi + (\eta R + \frac{1}{2} \eta^2) \varphi_{xx} \right]_{xx}
\]

Now, in addition to \( \eta = O(\epsilon) \) we further restrict to waves with sufficiently large wave length: \( kH = O(\alpha) \). Then the operator \( R \) has expansion

\[
\hat{R} = 1 - \frac{1}{3} k^2 H^2 + O(\alpha^4).
\]

Using this in the expression for \( w \), the various terms are of the form \( O(\alpha^n \epsilon^m) \).

For different classes of waves the relation between \( \alpha \) and \( \epsilon \) will be different. For instance, for the linear theory \( \alpha = 1 \) and \( \epsilon \) is small; for shallow water theory, \( \epsilon = 1 \) and \( \alpha \) is small.

A well known intermediate model is based on the so-called Boussinesq assumption, which relates the amplitude of the waves and their wavelength according to

\[
\alpha = O(\sqrt{\epsilon});
\]

this means that \( \partial_x^2 = O(\epsilon) \) is of the order of the amplitude, i.e. this assumption refers to rather long, rather low waves.

With this relation between \( \epsilon \) and \( \alpha \), the expression for \( w \) becomes

\[
w = -h u_x + \partial_x \left[ \frac{1}{3} H^3 u_{xx} + \eta u \right] + O(\epsilon^{5/2}).
\]

Substituting this in the free surface conditions, and omitting higher order terms, the set of two Boussinesq equations are found.

The final result can be summarized as follows. Consider the scaling of independent and dependent variables

\[
\eta = \epsilon \hat{\eta}, \quad u = \epsilon \hat{u}, \quad x = \frac{1}{\sqrt{\epsilon}} \hat{x}, \quad t = \frac{1}{\sqrt{\epsilon}} \hat{t}
\]

Then upon omitting hats, and terms of order \( \epsilon^2 \) and higher, the Boussinesq equations are given by

\[
\begin{align*}
\partial_t \eta &= -\partial_x \left[ Hu + \epsilon \left( \frac{1}{3} H^3 u_{xx} + \eta u \right) \right], \\
\partial_t u &= -\partial_x \left[ \eta \eta + \frac{1}{2} \epsilon u \right]
\end{align*}
\]

(8.15)

Observe that the dynamic eqn for \( \eta \) can in the same order of accuracy be written (in nonscaled variables) with \( h = H + \eta \) like

\[
\partial_t \eta = -\partial_x \left[ hu + \frac{1}{3} H^3 u_{xx} \right];
\]

this is an improvement (including dispersive effects) of the expression for shallow waves

\[
\partial_t \eta = -\partial_x \left[ hu \right];
\]

and is in the required order an approximation of the exact mass balance:

\[
\partial_t \eta = -\partial_x \int_H \Phi_x \, dz.
\]
8.2.6 Uni-directionalization to KdV-eqn

Bi-directional character of B-eqn’s

The B-eqn’s derived above are two scalar pde’s of first order in time for the two scalar variables η, the surface elevation, and u, a velocity-type variable. Roughly speaking, B-eqn’s describe waves in two directions, the positive and negative x-axis: it is a bi-directional wave eqn. In more precise terms this can be described in two ways.

1. Observe that B-eqn’s are invariant for the transformation

\[ (\eta, u, x, t) \mapsto (\eta, -u, -x, t) \]

(there is also the symmetry, \( (\eta, u, x, t) \mapsto (\eta, -u, x, -t) \) and \( (\eta, u, x, t) \mapsto (\eta, u, x, -t) \)). This means that if some solution can be interpreted as a wave running to the right, the same surface elevation can also be found running to the left (with opposite sign of the velocity).

2. Consider the limit problem with \( \epsilon = 0 \):

\[ \partial_t \eta = -\partial_x Hu, \quad \partial_t u = -\partial_x g\eta. \]

This can be written as a second order eqn for \( \eta \), and the same for \( u \):

\[ \partial^2_t \eta = c^2 \partial^2_x \eta, \quad \text{with } c = \sqrt{gH} \]

The general solution is given by

\[ \eta = F(x - ct) + G(x + ct) \]

for arbitrary functions \( F \) and \( G \); \( F(x - ct) \) represent waves running to the right, and \( G(x + ct) \) waves to the left. The corresponding velocity reads

\[ u = \frac{c}{H}[F(x - ct) - G(x + ct)]. \]

The waves running to the right, and also those to the left, can be seen as subclasses of solutions of the second order wave equation; this is obvious if we rewrite this eqn as

\[ (\partial_t + c\partial_x)(\partial_t - c\partial_x)\eta = 0. \]

In particular, waves to the right are the solutions of

\[ (\partial_t + c\partial_x)\eta = 0, \quad (8.16) \]

and for these waves the elevation and velocity are related:

\[ \eta = F(x - ct), \quad u = \frac{c}{H}F(x - ct), \]

and so

\[ u = \frac{c}{H}\eta. \quad (8.17) \]

This can be interpreted as follows: for the 2-nd order wave eqn \( \eta \) and \( u \) can be prescribed arbitrary, and in general a combination of right and left travelling waves will emerge. However, when we restrict the initial data to belong to the subset (8.17), the evolution remains in this set, and the evolution is a wave running to the right; this wave is in fact described by the first order wave eqn (8.16).

For the full B-eqn’s with \( \epsilon \neq 0 \) such a strict splitting into right and left travelling waves is not possible, but we can aim at an approximate splitting. This turns out to be possible, and the result is the KdV-eqn.

The KdV-eqn

In physical variables the Korteweg-de Vries eqn for waves (satisfying the Boussinesq assumptions) running to the right (waves to the left) are obtained for backward time) reads as follows:

\[ \partial_t \eta = -c\partial_x \eta + \epsilon \left( \frac{H^2}{6} \eta_{xx} + \frac{3}{4H} \eta^2 \right) \]

(8.18)

For \( \epsilon = 0 \) in KdV, the simple first order eqn for waves running to the right is obtained. The effect on the direction of propagation of the order \( \epsilon \)-terms is not so simple. One is a dispersive term leading to spatial spreading of a superposition of waves with different wave numbers, the other one is a nonlinear term which has effect of steepening/breaking of waves; together they are able to counterbalance so that travelling waves, travelling with undisturbed profile exist, as was shown in ‘Basics 2’.

Remark 78 1. Applying scalings of dependent and independent variables, and transforming to a coordinate frame moving with velocity \( c \),
KdV can be casted in the so-called normalized form of KdV
\[ \partial_t f = -\partial_x [f_{xx} + \frac{1}{2}f^2], \]
or
\[ \partial_t f + f_{xxx} + ff_x = 0. \]
(Actually, any value of coefficients before the various terms can be obtained, and sometimes normal forms with other coefficients are considered.)

2. KdV is of the form of a local conservation law. Actually it is an infinite dimensional Hamiltonian system, and more particularly completely integrable. For one thing this means that there are infinitely many conserved quantities. The most simple ones, which have a physical meaning, are: mass conservation
\[ M(f) = \int f \, dx, \]
momentum conservation
\[ I(f) = \int f^2 \, dx, \]
energy conservation
\[ E(f) = \int \left( f_x^2 - \frac{1}{6}f^3 \right) \, dx. \]

3. In its normalized form, the Boussinesq-class of waves for which KdV has been derived clearly shows itself by the presence of the following symmetry: if \( f(x, t) \) is a solution, then so is \( \epsilon f(x/\sqrt{\epsilon}, t/\sqrt{\epsilon}) \) (two \( x \)-derivatives balance against one amplitude).

4. KdV has been derived by Korteweg and De Vries in 1895 as an approximate eqn to describe waves of small amplitude and long wave length (the Boussinesq class) that run mainly in one direction. Their aim was to settle a scientific dispute that lasted for a long time in the nineteenth century about the existence of waves of permanent form. With the eqn they derived, they were able to show that indeed solitary wave solutions and periodic (cnoïdal) wave solutions exist that propagate without change of form.

**Derivation of KdV from B-eqn’s**
Motivated by (8.17) we look for the approximate manifold of right-travelling waves in the following way:
\[ u = c \frac{H}{\ell} (\bar{\eta} + a) \quad (8.19) \]
where \( \bar{\eta} \) and the intermediate variable \( a \) are functions in a frame moving with velocity \( c \):
\[ \eta(x, t) = \bar{\eta}(\xi, t), \quad a = a(\xi, t), \]
with \( \xi = x - ct \). This manifold will describe right travelling waves if for \( \eta = O(1) \) the variable \( a = O(\epsilon) \). Inserting the transformation into B-eqn’s, there results
\[ \partial_t \bar{\eta} = -c \partial_\xi (\bar{\eta} + a + \epsilon \frac{H^2}{3} (\bar{\eta}_{\xi\xi} + a_{\xi\xi} + \epsilon \frac{1}{H} (\bar{\eta}^2 + a \bar{\eta})), \]
\[ \partial_\xi a = -c \partial_\xi (2a - \epsilon \frac{H^2}{3} (\bar{\eta}_{\xi\xi} + a_{\xi\xi} - \epsilon \frac{1}{H} (\eta^2 - a^2)). \]

This last equation shows that for any \( \bar{\eta} \) a corresponding \( a = O(\epsilon) \) can be found, with \( \partial_\xi a = 0 \), given by
\[ a = -\frac{1}{2\epsilon} (\bar{\eta}_{\xi\xi} + \frac{1}{2H} \bar{\eta}^2). \]

Inserting this in the equation for \( \bar{\eta} \) there results up to \( O(\epsilon^2) \)
\[ \partial_t \bar{\eta} = -c \partial_\xi \epsilon \frac{H^2}{6} (\bar{\eta}_{\xi\xi} + \frac{3}{4H} \bar{\eta}^2). \]

Translating to the original \( x, t \) variables, this eqn is the KdV eqn given above.

**Remark 79** The equation obtained for \( a \) is precisely the condition that, upon substituting \( u \) given by (8.19) the two B-eqn’s for \( \eta \) and \( u \) are the same up to and including terms of order \( O(\epsilon) \).

### 8.3 Linear wave groups

For an equation with dispersion relation given by \( \omega = \Omega(k) \) the solution with initial spectral function \( \tilde{f} \) is given by
\[ u(x, t) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{i(kx - \Omega(k)t)} \, dk \]
It was argued in Basics 1 that when $\hat{f}$ is peaked around a wave number $k_0$, the solution is a wave group that approximately travels as a whole with the group velocity $V_{gr}(k_0)$:

$$u(x, t) \approx e^{-i\alpha(k_0) t} \int_{-\infty}^{\infty} \hat{f}(k)e^{i(kx-V_{gr}(k_0)t)} dk$$

with \(\alpha(k_0) = \Omega(k_0) - k_0 V_{gr}(k_0)\)

and hence

$$u(x, t) \approx e^{-i\alpha(k_0) t} u(x - V_{gr}(k_0)t, 0).$$

This shows that the initial profile $u(x, 0)$ is approximately translated with the group velocity $V_{gr}(k_0)$, at the same time being subject to a time periodic modulation.

**Exercise 80** Take as example a Gaussian spectrum profile,

$$\frac{1}{\sigma \pi} e^{-\frac{(k-k_0)^2}{2\sigma^2}}$$

centered around $k_0 = 2.5$, with standard deviation $\sigma = .2$.

For $\Omega(k) = k - .2 k^3$ we calculate (with Maple) the profiles with a discrete version of the integral

$$v(x, t) = \frac{1}{10} \sum_{m=-25}^{25} f(2.5 + \frac{1}{10} m) \cos((2.5 + \frac{1}{10} m) x - \Omega(2.5 + \frac{1}{10} m) t))$$

and show the initial profile, and the profiles shifted vertically with increasing time:

Observe that each constituent (monochromatic) wave is translated with phase velocity (almost vertically), while the package as a whole translates with the group velocity; in this example the group velocity is approximately 10 times the phase velocity.

**Exercise 81** When a Gaussian function is taken as initial spectral function, centered at $k_0 > 0$, and with standard deviation $\sigma$, the initial profile $u(x, 0)$ is of the form

$$u(x, 0) = e^{ik_0 x} a(x, 0),$$

where the amplitude function $a(x, 0)$ has a width proportional to $\frac{1}{\sigma}$. (Plot figures for various values of $\sigma$).

Take in the following $\sigma = .2$, which implies that the initial profile has an envelope of much larger extension than the wave length of $e^{ik_0 x}$, which is $\frac{2\pi}{k_0} \approx 2.5$ for $k_0 = 2.5$; there are approximately 10 waves in the interval of modulation.

This motivates to write the solution as a modulation of the carrier monochromatic wave:

$$u(x, t) = a(x, t) e^{(k_0 x - \Omega(k_0)t)}$$

(8.20)

with $a = a(x,t)$ the amplitude function, given by

$$a(x, t) = u(x, t) e^{-i(k_0 x - \Omega(k_0)t)}$$

The approximation of the solution given above is the same as stating that the amplitude develops like

$$a(x, t) \approx a(x - V_{gr}(k_0)t, 0)$$

i.e. a translation of the amplitude with the group velocity.
8.3. LINEAR WAVE GROUPS

8.3.1 Deviations from uniform translation

Let us investigate this approximation in more detail as follows.

At \( t = 0 \) the amplitude function is the envelope as shown below:

The approximation is of the form:

\[
app(x, t) = e^{(-i\alpha(k_0)t)} u(x - V_{gr}(k_0) t, 0)
\]

the initial value translated with group velocity, time-modulated.

Making various plots (of the real part), it can be observed that this approximation looks quite good, at least for times that are not too large. In fact, investigating the difference of the approximation with the exact solution gives the following results at 11 successive time steps, \( t = 0, t = 1/4, \ldots t = 2.5 \):

For another measure of the quality of the approximation, we plot the difference of the envelope of the exact solution and the approximation (back-shifted with the group velocity). At \( t = 2 \) the difference is shown below.

At a much later time, \( t = 10 \) for instance, the difference becomes larger: (at \( t = 10 \) the profile has shifted a distance \(-27.5\).

This shows that the profile of \( a(x, t) \) becomes smaller, broader and develops an oscillation.

For a better description one can either investigate the exact integral in more detail (asymptotic analysis of the Fourier integral), or derive an equation for \( a(x, t) \), a so-called amplitude equation. This last approach is possible by the special origin of the Fourier integral, i.e. the possibility to relate it to the solution of a given dynamic equation, namely that which has dispersion relation \( \omega = \Omega(k) \).

We follow the second possibility in order to get experience to extend the analysis to nonlinear equations.

8.3.2 Derivation of amplitude equation

Consider the linear KdV-equation:

\[
\partial_t u(x, t) + c \partial_x u(x, t) + \frac{1}{5} \partial_x^3 u(x, t) = 0
\]
Substituting $u(x, t) = a(x, t)e^{(ik_0 x - \Omega(k_0) t)}$ the following eqn for $a(x, t)$ is found:

$$\begin{align*}
\partial_t a(x, t) + c \partial_x a(x, t) - \frac{3}{5} k_0^2 \partial_x a(x, t) \\
+ \frac{3}{5} i k_0 \partial_x^2 a(x, t) + \frac{1}{5} \partial_x^2 a(x, t) &= 0
\end{align*}$$

This is the amplitude equation, (for complex-valued solutions) to be investigated.

Observe that the equation can be rewritten like:

$$\begin{align*}
\partial_t a(x, t) + V_{gr}(k_0) \partial_x a(x, t) \\
+ \frac{3}{5} i k_0 \partial_x^2 a(x, t) + \frac{1}{5} \partial_x^2 a(x, t) &= 0
\end{align*}$$

e.g. a translation with the group velocity, a dispersion and imaginary diffusion (also dispersion). Introducing a frame of reference moving with the group velocity

$$X = x - V_{gr}(k_0) t$$

the equation becomes

$$\begin{align*}
\partial_t a(X, t) + \frac{3}{5} i k_0 \partial_X^2 a(X, t) + \frac{1}{5} \partial_X^2 a(X, t) &= 0
\end{align*}$$

The approximation in the previous section did not take both dispersive terms into account, i.e. the approximation took $a(X, t) = a(X, 0)$. Clearly, this is not an exact result; however, the (good) quality of the approximation can be understood by realizing that the spatial variations in the amplitude take place over a long interval, and hence higher order spatial derivatives become smaller with the increase of the order. This can be made more precise by introducing a small parameter $\varepsilon$, and a scaled spatial variable

$$\xi = \varepsilon X \equiv \varepsilon (x - V_{gr}(k_0) t).$$

This small parameter $\varepsilon$ has a clear meaning: it is a measure of the width of the spectral function, something like $\frac{\Delta k}{k_0}$.

Writing $a(X, t) = b(\xi, t)$ the amplitude eqn becomes

$$\begin{align*}
\partial_t b(\xi, t) + \frac{3}{5} i k_0 \partial_\xi^2 b(\xi, t) + \frac{1}{5} \partial_\xi^2 b(\xi, t) &= 0
\end{align*}$$

This means that the change of the amplitude in time is of the order $O(\varepsilon^2)$. Stated differently, the amplitude changes in order one on long time intervals of the order $O(\frac{1}{\varepsilon^2})$. This leads to the introduction of a slow time variable $\tau = \varepsilon^2 t$. Collecting the transformations,

$$a(x, t) = A(\xi, \tau), \quad \xi = \varepsilon (x - V_{gr}(k_0) t), \quad \tau = \varepsilon^2 t,$$

the amplitude equation becomes

$$\begin{align*}
\partial_\tau A(\xi, \tau) + \frac{3}{5} i k_0 \partial_\xi^2 A(\xi, \tau) + \frac{1}{5} \partial_\xi^2 A(\xi, \tau) &= 0
\end{align*}$$

This is the final form of the amplitude eqn; observe that on time scales $\tau \in (0, 1)$, i.e. $t \in (0, \frac{1}{\varepsilon^2})$, the term of order $\varepsilon$ in the equation, produces an error of order $\varepsilon$. Therefore, on these intervals one may neglect this term to find a solution correct up to order $\varepsilon$:

$$\begin{align*}
\partial_\tau A(\xi, \tau) + \frac{3}{5} i k_0 \partial_\xi^2 A(\xi, \tau) &= 0
\end{align*}$$

For the linear KdV eqn considered, it is no restriction to work with complex valued solutions, even when the interest is in real solutions, since with every complex valued solution, the real and imaginary part is also a solution.

For nonlinear eqns this is no longer true, and one restricts from the onset to investigate real valued solutions. For the case considered here, that would mean to take the combination of a complex function plus its complex conjugate ($c.c.$):

$$r(x, t) := A(\xi, \tau) e^{i\theta} + c.c.$$

i.e. $r(x, t) := A(\xi, \tau) e^{i\theta} + \overline{A(\xi, \tau)} e^{-i\theta}$ with $\theta = k_0 x - \Omega k_0 t$ and where $\overline{A}$ denotes the complex conjugate of $A$.

Substituting this in the original eqn, the result is of the form:

$$\begin{align*}
\text{Eqn } e^{i\theta} + \overline{\text{Eqn } e^{-i\theta}} &= 0
\end{align*}$$

with Eqn given by the eqn (8.21). Hence, the same result is found in this linear case.

Now consider a linear wave eqn with general dispersion relation $\Omega(k)$:

$$\partial_t u(x, t) + i \Omega(-i \partial_x) u = 0$$

For a derivation of the amplitude eqn for this case, consider a wavegroup with slowly varying amplitude:

$$\begin{align*}
u(x, t) &= a(\zeta, t) e^{i\theta_0}, \quad \zeta = \varepsilon x, \\
\theta_0 &= k_0 x - \Omega(k_0) t.
\end{align*}$$
8.3. LINEAR WAVE GROUPS

Then a spatial derivative $\partial_x$ acts on the amplitude and the phase in a different way:

$$\partial_x = ih_0 + \varepsilon \partial_k.$$ 

Hence, for the amplitude it holds

$$\partial_t a(\zeta, t) + i \left[ \Omega(k_0 - i \varepsilon \partial_k) - \Omega(k_0) \right] a(\zeta, t) = 0.$$ 

Expanding $\Omega(k_0 - i \varepsilon \partial_k)$ with respect to $\varepsilon$, the group-velocity appears in the first order term. In a moving and scaled coordinate frame as introduced above, $A = A(\xi, \tau)$, with

$$\xi = \varepsilon(x - V_{gr}(k_0)t), \quad \tau = \varepsilon^2 t,$$

and we find

$$\begin{cases} 
\partial_x A - i \beta \partial^2_x A = O(\varepsilon) \\
\text{with } \beta = \frac{1}{2} \partial^2_k \Omega(k_0). 
\end{cases}$$

The special linear case considered above is in agreement with this.

This more general result shows that the interesting coefficient in the amplitude equation is the second derivative $\partial^2_k \Omega(k_0)$, which measures the dispersion of the group velocity. In particular, when this coefficient vanishes, the next order term has to be investigated; in that case the change of the amplitude takes place on an even longer time interval of order $O(1/\varepsilon^3)$.

**Exercise 82.** 1. Observe that the amplitude equation is again a linear dispersive eqn, with dispersion relation $\omega = \beta k^2$. In the exact integral expression in the previous subsection, show that the term that has been neglected is of the form $e^{i \Omega(k_0 - \Omega(k_0) - V_{gr}(k_0)(k-k_0)t)}$, so approximately $e^{i \beta (\Delta k)^2 t}$.

2. Specialise the foregoing for the case of the BBM-equation

$$(1 - \partial^2_x) \partial_t u = -\partial_x (u + u^2)$$

and also derive the result in a direct way.
Chapter 9

Structures in dynamical systems

Consider a set of first order ode’s, i.e. for a vector function $t \mapsto \mathbf{x}(t)$ and vector field $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

We restrict to autonomous systems in the following. The initial value problem (ivp) is the problem to find for given initial value $\mathbf{x}_0$ the solution on an interval around $t = 0$ for which $\mathbf{x}(0) = \mathbf{x}_0$. Under mild conditions for $\mathbf{f}$, Lipschitz continuity, the ivp is well posed, meaning that there exists a unique solution. Then it is natural to call $\mathbf{x}$ a state variable; it collects that information that is necessary to predict its futures values from the values at a certain initial time. If the solution exists for all time, it is called a global solution. The solution can be depicted in the state space as a trajectory: the curve $t \mapsto \mathbf{x}(t)$. This can be done for solutions belonging to different initial values. Then:

- through each point in state space there goes a trajectory;
- in fact, from uniqueness of ivp it follows that through each point passes precisely one trajectory (i.e. trajectories cannot intersect).

Only in special cases a solution can be found in closed form. Special solutions are equilibrium solutions, which are the vectors $\mathbf{x}$ such that $\mathbf{f}(\mathbf{x}) = 0$. Alternatively it can be said that the equilibria are the fixed points of the phase flow. Especially when the solution cannot be found in closed form, it is important to try to understand some qualitative properties, such as: do solutions exist globally, are solutions bounded, do they tend to a special (equilibrium) solution for increasing time, etc. In such investigations the dependence of the solution on the initial value is particularly important; this information is precisely described by the phase flow. When dealing with nonlinear equations, often the equilibrium points are the only solutions that can be found explicitly. In that case, linearization near an equilibrium solution will give some indication of the phase flow near this equilibrium, since linear (autonomous) equations can be solved explicitly.

9.1 Indicator functions

As stated, solutions in closed form cannot be found in general; therefore, any information that will give some qualitative information is useful. When dealing with an ode of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

it is in fact quite difficult to classify the possible phase flows in terms of $\mathbf{f}$, i.e. to find ‘classes’ of vectorfields $\mathbf{f}$ such that for each member the phase flow has some clearly recognizable and characteristic property. Below we briefly describe one way of partially analyzing an equation by using a scalar function that gives some indication of the phase flow; we will call it an indicator function. Based on what information such an indicator function gives, we will identify two classes of equations.

The idea of an indicator function is instead of looking at the state of the system, which completely determines all details, to associate with each state $\mathbf{x}$ in the state space $S \equiv \mathbb{R}^n$ a real number: the value of the indicator function. This means that the indicator function, say $E$, maps the state space to the real numbers

$$E : S \mapsto \mathbb{R}.$$
Clearly, in general there will be many states that have the same value: in general (when \( \nabla E \neq 0 \)) the levelset will be \( n - 1 \)-dimensional. This indicates that the value of \( E \) describes only partly the state. Given the indicator value at the initial time, which is \( E(x_0) \) when \( x_0 \) is the initial state, consider its value at subsequent times: the indicator at subsequent times. When \( x(t) \) denotes the evolution of the state from \( x_0 \), this provides a scalar function of time:

\[
t \mapsto E(x(t))
\]

How the indicator changes in time depends of course on the change of the state:

\[
\frac{d}{dt} E(x(t)) = \nabla E(x(t)) \cdot \dot{x}.
\]

Using the eqn for \( x \), there results

\[
\frac{d}{dt} E(x(t)) = \nabla E(x(t)) \cdot f(x).
\]

This is nothing but the directional derivative of \( E \) in the direction of the vector field. The change in \( E \) vanishes if \( f \) is tangent to the level set, while the change is maximal when \( f \) is perpendicular to the level set.

For a given ode and a specified function \( E \) its change is well defined. An arbitrary function \( E \) will not be very useful, in general. Only special functions, that are in some way related to the vector field, may provide useful information. Hence, in practice, it is important to try to find suitable, relevant functions \( E \). Maybe there are none, or maybe there are more than one. We will give examples below.

**Remark 83** The idea above is simple and can be applied to any difficult evolutionary system. Suppose we have a system with state variable \( u \) evolving in time \( t \mapsto u(t) \). In applications, the state \( u \) may describe a rather complicated situation and involve many different aspects. The evolution may be equally difficult, or even not known in all details. An example is the ‘economy’ of a country, or its ‘productivity’. These terms are poorly defined, but any practical definition will involve many factors; their dependence is equally poorly understood, and so is their evolution. Suppose we can define to each possible state one number, that in some sense should be ‘characteristic’; this can then be interpreted as an indicator; often the change of such an indicator is more important than its actual value.

**Example 84**

1. A barometer, which measures the air pressure, can predict the change of the weather rather well; changes are more indicative than the value itself. The body temperature is an indication of illness when it outside a small range around 37° Centigrade. Within this range it is no indicator of health at all.

2. The economy of a country is difficult to define; some global parameters may be unemployment, interest rates, national productivity, etc. An indicator function is usually a weighted average of such (and many other) factors. Changes in the indicator function may give an indication that has to be interpreted depending on the purpose: decrease of unemployment is good if you are looking for a job, but generally interpreted to be negative for the national industries.

3. The change in value of stock at stock market is often more indicative than its actual value; unless you want to buy or sell.

### 9.2 Gradient and Hamiltonian systems

For ode’s we consider two classes of eqns (that are not disjoint) for which at least one indicator function is easy to detect.

**Definition 85** Consider a dynamical system for state variable \( u \in \mathbb{R}^n \).

1. The system is called a gradient system if there exists a scalar function \( E : \mathbb{R}^n \to \mathbb{R} \) such that the dynamics is given by

\[
\frac{\partial}{\partial t} u = -\nabla E(u). \tag{9.1}
\]

2. The system is called a (generalized) Hamiltonian system if there exists a scalar function \( E : \mathbb{R}^n \to \mathbb{R} \) and a skew symmetric matrix \( A : \mathbb{R}^n \to \mathbb{R}^n \) such that the dynamics is given by

\[
\frac{\partial}{\partial t} u = A \nabla E(u). \tag{9.2}
\]
9.3. VARIATIONAL CALCULUS

The motivation to distinguish equations of this form from more general equations is the fact that for both the explicit function $E$ has a special role, as is expressed in the following propositions.

Proposition 86  
1. A critical point of the function $E$, i.e. $U$ such that $\nabla E(U) = 0$, is an equilibrium for the (generalized) gradient system.

2. For any solution of the gradient system the value of $E(u)$ is decreasing (or constant for an equilibrium):

$$\partial_t E(u(t)) = -\|
abla E(u(t))\|_2;$$

for a generalized gradient system the same holds when the matrix $S$ is positive definite:

$$\partial_t E(u(t)) = -S \nabla E(u) \cdot \nabla E(u).$$

3. As a consequence, any solution will evolve towards a critical point of $E$ (i.e. towards an equilibrium); for (almost) all initial data, the solution tends to a local minimizer of $E$ as $t \to \infty$

Proposition 87  
1. For the Hamiltonian system (9.2), equilibrium solutions are the points for which $\nabla E(\dot{u}) \in \ker(A)$, including all critical points of $E$.

2. For any solution of the Hamiltonian system the value of $E$ is constant during the evolution:

$$\partial_t E(u(t)) = \nabla E(u) \cdot A \nabla E(u) \equiv 0;$$

this means that the motion takes place on the levelset of $E$ determined by the initial condition:

$$E(u(t)) = E(u(0)) \text{ for all } t.$$

$E$ is called a constant of the motion, or first integral.

3. As a consequence, the state space is foliated by levelsets of $E$ that are invariant sets for the evolution.

Sketch of proofs
The main ingredient is the resulting change in $E$:

$$\partial E(u(t)) = \nabla E(u(t)) \cdot \partial_t u(t)$$

For gradient systems this becomes

$$\partial E(u(t)) = -\|
abla E(u(t))\|_2^2$$

which is nonnegative (and implies that $E$ is non-increasing) and only vanishes at critical points. For Hamiltonian systems there results

$$\partial E(u(t)) = \nabla E(u(t)) \cdot A \nabla E(u(t))$$

which vanishes identically since for any skew-symmetric matrix $A$ it holds that $A \xi \cdot \xi = 0$ for all vectors $\xi$.

Remark 88 It turns out that a Hamiltonian or gradient structure is important in the study of models described by equations with such structure. In fact, also evolution equations, i.e. partial differential equations, can have the same structures. To be able to describe this, we need to know the generalisation of the gradient of a scalar function, like the Hamiltonian above. We will first briefly consider this generalisation, which is 'variational calculus', and then show various examples.

9.3 Variational Calculus

The basic calculational aspects for functions can be generalized for functionals (functions on infinite dimensional spaces, mostly function spaces). In this text we deal with the first variation (directional derivative) and the variational derivative (the “gradient” of a functional), mainly of so-called density functionals: integrals over a certain interval of expressions of functions defined on that interval. We also provide the maple-procedures to perform the algebraic manipulations.

9.3.1 First variation

For a given functional $F$ the first variation is defined to be the directional derivative:

Definition 89 The first variation of the functional $F$ at a point $u$ in the direction $v$ is denoted by $\delta F(u,v)$ and is the expression (when defined)

$$\delta F(u,v) := \frac{d}{d\varepsilon} F(u + \varepsilon v) \big|_{\varepsilon = 0}.$$
Maple procedure

In Maple-notation we write \( \text{fvar}(F, u, v) \) for \( \delta F(u, v) \). Note that \( \varepsilon \to u + \varepsilon v \) should be viewed as a line through \( u \) in the direction \( v \) in the function space. In Maple we have to describe this line as a parameterized function:

\[
\text{line} := \varepsilon \to \text{unapply}(u(t) + \varepsilon v(t), t)
\]

and the functional evaluated on this line

\[
F(t \to u(t) + \varepsilon v(t))
\]

has to be differentiated with respect to \( \varepsilon \). In one procedure, with some precautions for general use,

**Procedure fvar**

\[
\begin{align*}
\text{fvar} & := \text{proc}(F, u, v) \\
local \ f v, \ \varepsilon; \\
f v & := \text{unapply}(\text{simpify}(\text{subs}(\varepsilon = 0,
\text{diff}(F(\text{unapply}(U(\tau + \varepsilon V(\tau, \tau), \varepsilon)), U, V)), U, V)); \\
f v(u, v)
\end{align*}
\]

**Example 90**

1. L2-norm of functions

\[
L := u \to \int_0^1 u(x)^2 \, dx
\]

\[
\text{fvar}(L, u, v) = 2 \int_0^1 v(x) \, u(x) \, dx
\]

2. Second order derivative, non-quadratic integrand

\[
G := u \to \int \left( \frac{\partial^2}{\partial x^2} u(x) \right)^2 + u(x)^4 \, dx
\]

\[
\text{fvar}(G, u, v) = 2 \int \left( \frac{\partial^2}{\partial x^2} u(x) \right) \left( \frac{\partial^2}{\partial x^2} v(x) \right) + 2 u(x)^3 v(x) \, dx
\]

**Derivative of a functional along a vectorfield (Lie-derivative)**

Assuming the function \( u(x) \) to depend on an additional parameter (in many applications, \( t = \text{‘time’} \)), substituting in a functional \( F \) reduces the value to a function of \( t \): \( F(u(x, t)) = f(t) \).

The derivative of \( f \) with respect to \( t \) is related to the first variation of the functional:

\[
\frac{d}{dt} f(t) = \frac{d}{dt} F(u(x, t)) = \delta F(u, \partial_t u)
\]

This can be seen as a special case of the chain-rule for functionals. If the evolution \( t \to u(t) \) satisfies a first order equation

\[
\frac{\partial}{\partial t} u(x, t) - K(u(x, t)) = 0
\]

with vector field \( K \), the time derivative of the functional is known as the “Lie-derivative”:

**Example 91** The restriction of the functional \( F \)

\[
F := v \to \int v(x)^3 \, dx
\]

to an evolution \( u := t \to \text{unapply}(w(x, t), x) \) is given by

\[
F(u(t)) = \int w(x, t)^3 \, dx
\]

and a direct differentiation with respect to \( t \) provides (the chain-rule!!)

\[
\partial_t F(u(t)) = 3 \int w(x, t)^2 \left( \frac{\partial}{\partial t} w(x, t) \right) \, dx
\]

For an evolution equation given by:

\[
\text{Eq} := \frac{\partial}{\partial t} w(x, t) - K(w(x, t))
\]

the result is

\[
\partial_t F(u(t)) = 3 \int w(x, t)^2 K(w(x, t)) \, dx
\]

This is the same as a direct calculation of the Lie-derivative of \( F \) along vector field \( K \):

\[
\text{fvar}(F, u(t), \text{unapply}(K(w(x, t)), x));
\]

\[
3 \int w(x, t)^2 K(w(x, t)) \, dx
\]

**9.3.2 Variational derivative**

The first variation of the functional \( F \) at the point \( u \) in the direction \( v \) is a functional that is linear in \( v \).

For a density functional (with density that depends on \( u \) and a finite number \( N \) of its derivatives), this linear functional when \( N = 1 \) is of the form

\[
\int a_0(x) v(x) + a_1(x) \partial_x v(x) \, dx
\]
and more generally
\[
\int \sum_{k=0}^{N} a_k(x) \frac{\partial^k}{\partial x^k} v(x) \, dx
\]
where the coefficients \(a_0(x), \ldots, a_N(x)\) are expressions in \(u\); the precise form is of no interest at this moment.

**Definition 92** The variational derivative of \(F\) at the point \(u\) is defined to be that function, denoted by \(\delta F(u)\), such that for all functions \(v\) that vanish (together with its derivatives) near the boundary points of the integral, it holds that:
\[
\delta F(u, v) = \int \delta F(u) \, v(x) \, dx
\]
By partial integration the variational derivative can be obtained from the first variation:
\[
\delta F(u, v) = \int \sum_{k=0}^{N} a_k(x) \frac{\partial^k}{\partial x^k} v(x) \, dx
\]
then (partial integrations, neglecting possible boundary contributions):
\[
\delta F(u, v) = \int \sum_{k=0}^{N} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial x^k} a_k(x) \times v(x) \, dx
\]
and hence the variational derivative is given by
\[
\delta F(u) = \sum_{k=0}^{N} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial x^k} a_k(x)
\]

**Maple procedure**

Formulated as done above, there is a clear procedure to obtain the variational derivative from the first variation. However, starting with a given functional, the coefficients \(a_k\) in ‘\text{fvar}’ are calculated (by Maple), and first have to be detected before we can write the expression for \text{vander}. The procedure to do so is an iterative one on the density; the coefficients are recursively obtained by substitution of powers of \(x\) for the function \(v\). For instance, taking \(v \equiv 1\) there results
\[
\delta F(u, 1) = \int a_0(x) \, dx
\]
and hence the integrand gives the first coefficient \(a_0\) immediately. Taking \(v(x) = x\) the integrand becomes \(a_1(x) + x a_0(x)\) from which \(a_1\) can then be found, etc. The ‘\text{student}’ package allows one to detect the integrand from a given integral (provided the whole expression is under the integral sign); we modify the procedure ‘\text{fvar}’ slightly for that aim. Although that could be avoided, for computational simplicity (and without too much restriction) the procedure asks as input the order \((N)\) of the highest derivative in the integrand of the functional, and the variable that is used in the description of the functional with ‘int’. In Maple notation we write \text{vander}(F, u) for \(\delta F(u)\).

**Procedure \text{vander}**

\[
\text{vander} := \text{proc}(F, u, N, x) \rightarrow \text{local } v, t, vd, a, k, n, m, \varepsilon, \text{den, locf fvar};
\]
\[
\text{locf fvar} := \text{unapply(subs(} \varepsilon = 0, \text{diff}(F(\text{unapply}(u(t + \varepsilon v(t), t)), \varepsilon)), v));
\]
\[
\text{den} := \text{unapply(}\text{integrand(locf fvar(v)), v)};
\]
\[
a[0] := \text{den(}\text{unapply}(1, t)\text{)};
\]
\[
vd := a[0];
\]
for \(n\) from 1 to \(N\) do
\[
a[n] := \text{simplify(}\text{den(}\text{unapply}(t^n/n!, t)) \rightarrow \text{sum(}a[k] \times x^n - k)/n, k = 0..n - 1\text{)};
\]
\[
vd := vd + (-1)^n \times \text{Diff}(a[n], x(n));
\]
end;

**Example 93**

1. \(L(u) := \int_0^1 u(x)^2 + (\frac{\partial^2}{\partial x^2} u(x))^2 \, dx\)

\[
\text{vander}(L, u, 1, x) = 2 x u(x) - 2 \left( \frac{\partial^2}{\partial x^2} u(x) \right)
\]

2. \(L(u) := \int_0^1 \sin(x) u(x)^2 + x^3 (\frac{\partial^2}{\partial x^2} u(x))^2 \, dx\)

3. \(L(u) := \int u(x)^2 + (\frac{\partial}{\partial x} u(x))^2 \, dx\)

\[
\text{vander}(L, u, 1, x) = 2 u(x) - \left( \frac{\partial}{\partial x} \left( 2 \left( \frac{\partial}{\partial x} u(x) \right) \right) \right)
\]

4. \(H(u) := \int_0^1 \sin(u(x)) + (\frac{\partial^2}{\partial x^2} u(x))^2 \, dx\)

\[
\text{vander}(H, \psi, 2, x) = \cos(\psi(x)) - \left( \frac{\partial}{\partial x} 0 \right) + \left( \frac{\partial^2}{\partial x^2} (2 \left( \frac{\partial^2}{\partial x^2} \psi(x) \right)) \right)
\]
5. $K(u) := \int_0^1 u(x)^4 + \left(\frac{\partial}{\partial x} u(x)\right)^7 \, dx$

6. $J(u) := \int_0^1 n(x) \sqrt{1 + \left(\frac{\partial}{\partial x} u(x)\right)^2} \, dx$

$$\text{var} \left(\text{der}(J, u, 1, x)\right) =
- \left(\frac{\partial}{\partial x} n(x) \left(\frac{\partial}{\partial x} u(x)\right)\right)
\sqrt{1 + \left(\frac{\partial}{\partial x} u(x)\right)^2}$$

### 9.4 Structures in evolution equations

The generalization of a gradient structure or Hamiltonian structure to evolution equations that are described by partial differential equations, infinite dimensional evolution eqns, is formally rather straightforward.

Using the notation $\delta E(u)$ for the variational derivative of a functional $E$, a gradient system is of the form

$$\partial_t u = -S \delta E(u)$$

where now $S$ is a symmetric operator.

A Hamiltonian system is generalized analogously:

$$\partial_t u = A \delta E(u),$$

where $A$ is a skew-symmetric operator.

**Example 94** 1. KdV as Hamiltonian system.

The KdV eqn has the following Hamiltonian structure

$$\partial_t u = \partial_x \delta H(u),$$

where now $\partial_x$ is the skew-symmetric operator $A$, and the Hamiltonian is given by

$$H(u) = \int \frac{1}{2} u_x^2 - \frac{1}{6} u^3$$

for which $\delta H(u) = -u_{xx} - \frac{1}{2} u_x^2$.

2. The linear diffusion eqn

$$\partial_t u = u_{xx}$$

can be written as a gradient system in two ways:

$$\partial_t u = -\delta E(u), \quad \text{with } E(u) = \int \frac{1}{2} u_x^2,$$

and

$$\partial_t u = \partial_x^2 \delta E(u), \quad \text{with } E(u) = \int \frac{1}{2} u^2;$$

in the last case, the operator $\partial_x^2$ is the symmetric operator $S$ from the generalized gradient system.

3. Burgers’ eqn can be written as a mixed system:

$$\partial_t u = \partial_x^2 \delta D(u) + \partial_x \delta E(u),$$

for

$$D(u) = \int \frac{1}{2} u_x^2, \quad E(u) = \int \frac{1}{3} u_x^3.$$

**Example 95** Hamiltonian structure of full surface wave equations

It is possible to describe the equations that describe surface wave equations as a set of Hamiltonian eqns. Using the formulae as in Chapter 6, Section 6.4, we will simplify the presentation by restricting to the case that the flow is uniform in the $y$-direction. The basic variables (canonical variables) in this description are the surface elevation $\eta(x, t)$, and the potential at the free surface:

$$\phi(x, t) := \Phi(x, \eta(x, t), t).$$

Furthermore, the total energy (sum of potential and kinetic energy) plays the role as Hamiltonian $H$, where it is understood that $H$ is considered as functional of $\phi$ and $\eta$:

$$H(\phi, \eta) := K(\phi, \eta) + \int \frac{1}{2} g \eta^2 \, dx,$$

with $K$ the kinetic energy. Standard variational techniques show that the kinetic energy can

---

^1However, care should be taken with boundary conditions that may arise in calculating the variational derivatives.

^2Symmetry of $S$ is understood here with respect to the same inner product as that used to define the variational derivative:

$$\langle Su, v \rangle = \langle u, Sv \rangle;$$

in all cases we will deal with, the $L_2$-innerproduct is used.

^3In the course 'Applied Analytical Methods', the following will be treated in more detail.
equally be defined as the value function of a minimization problem:

\[ K(\phi, \eta) = \text{Min}_\Phi \int dx \int_{-H}^{\eta} \frac{1}{2} |\nabla \Phi|^2 dz, \]

with

\[ \Phi = \phi \text{ at } z = \eta(x,t) \]

which can equally be formulated as

\[ K(\phi, \eta) = \int_{\Omega(t)} \frac{1}{2} |\nabla \Phi|^2, \]

where \( \Phi \) is the solution of

\[
\begin{align*}
-\Delta \Phi &= 0, & & \text{for } -H < z < \eta(x,t) \\
\nabla \Phi \cdot n &= 0, & & \text{at } z = -H \\
\Phi &= \phi, & & \text{at } z = \eta(x,t)
\end{align*}
\]

Then it can be shown that the two surface equations are equivalent to the set of Hamiltonian eqns

\[
\begin{align*}
\partial_t \eta &= \delta_\phi H(\phi, \eta) \\
\partial_t \phi &= -\delta_\eta H(\phi, \eta)
\end{align*}
\]

Here, \( \delta_\phi H, \delta_\eta H \) denote the variational derivative of \( H \) with respect to \( \phi \) and \( \eta \) respectively. From the Hamiltonian character, it follows immediately that the Hamiltonian is a conserved quantity (remains constant during the evolution):

\[ \partial_t H(\phi, \eta) = 0. \]

An equivalent formulation can be obtained by using the variables \( u, \eta \) where \( u \) is a velocity-type variable defined as \( u = \partial_x \phi(x,t) \). Then, with \( H \) the Hamiltonian expressed in \( u \) and \( \eta \), the eqns transform to

\[
\begin{align*}
\partial_t \eta &= -\partial_x \delta_u H(u, \eta), \\
\partial_t u &= -\partial_x \delta_\eta H(u, \eta)
\end{align*}
\]

This formulation, as a ‘generalized’ Hamiltonian system (a so-called *Poisson system*), clearly shows the conservative nature of the equations since it is a set of conservation laws. Except from the ‘masses’ \( \int u \), \( \int \eta \) and the energy \( H(u, \eta) \), there is another conserved quantity. This is a quantity that is a consequence of translation symmetry and is therefore called (horizontal) momentum; explicitly it is given by

\[ I(u, \eta) = \int u \eta dx. \]

The alternative formulations of the equations as a (generalized) Hamiltonian system as given above are useful for at least two aspects:

- approximations (that may be more convenient for analytical investigations) can be found by looking for approximations of the Hamiltonian;
- for numerical purposes the special form of the eqns can be exploited: finite volume techniques are motivated from the description as conservation laws, while consistent discretizations can guarantee that the expressions \( \delta_u H(u, \eta), \delta_\eta H(u, \eta) \) are derivatives of (an approximate) Hamiltonian \( H \).

**Example 96 Consistent Hamiltonian structure of surface wave approximations**

The exact surface wave equations have a Hamiltonian structure: with the variables \( u, \eta \) and the total energy \( H \) expressed in \( u \) and \( \eta \), the eqns are given by (9.3). The structure of these equations, a generalized Hamiltonian system (a so-called Poisson system), is very specific and reflects many of the properties that underlie the physical phenomenon. The approximations of the exact surface wave eqns in Chapter 8, Section 8.2, did not take this special structure into account. However, it turns out that the Boussinesq eqns, and also KdV, are approximations that have the same structure as the exact equations: these approximations have retained the Hamiltonian structure. In fact, it can be shown that we could have obtained these approximations by approximating the exact Hamiltonian, i.e. by approximating the exact total energy into an explicit functional of \( u, \eta \). We will not give all the details, just present the final results. The Boussinesq eqns are given by the same eqns (9.3) but now with the following expression for the Hamiltonian:

\[ H_B(\eta, u) = \int \frac{1}{2} g \eta^2 + \frac{1}{2} Hu^2 + \epsilon \left[ \frac{1}{2} \eta u^2 - \frac{H^3}{6} u_x^2 \right] \]

The KdV eqn as given by (8.18) can be written with the Hamiltonian

\[ KdV(\eta) = \int \frac{1}{2} \eta^2 + \epsilon \left[ \frac{1}{4} H^3 \eta^3 - \frac{H^2}{12} \eta_x^2 \right]. \]
in the following form of a Hamiltonian system:
\[
\partial_t \eta = -c \partial_x \delta H_{K,dv}(\eta).
\] (9.5)

### 9.5 Exercises

1. The action functional of the harmonic oscillator is the integral of kinetic minus potential energy:
\[
L := q \to \int \frac{1}{2} (\partial_t q)^2 - \frac{1}{2} q^2 \, dt.
\]
The equation of motion is obtained from the so-called action principle: the dynamics is governed by
\[
\delta L(q) = 0.
\]
Derive this equation explicitly.
The total energy \( E \) is given by
\[
E := \frac{1}{2} \partial_t q^2 + \frac{1}{2} q^2.
\]
Determine the evolution of \( E \) during the evolution.

2. Show that the bi-directional wave equation for the function \( u(x,t) \) can also be obtained from an action principle for the action functional (which is now an integration over the spatial variable \( x \) and the time):
\[
L(u) = \int \int \left( \frac{1}{2} u_t^2 - \frac{1}{2} u_x^2 \right) \, dx \, dt
\]
Determine the total energy in this case, and verify that it is constant during the evolution.

3. Show that the KdV-eqn can be written as (a Hamiltonian system)
\[
\partial_t u = -\partial_x \delta H(u)
\]
where \( H \) is the functional given by
\[
H(u) = \int \frac{1}{2} u_x^2 + \frac{1}{3} u^3
\]
Show that the following functionals are constants of the motion for solutions of KdV that decay, together with all their derivatives, to zero at infinity:
- mass \( \int u(x) \, dx \),
- momentum \( \int u(x)^2 \, dx \),
- energy \( H(u) \).

4. Show that Burgers’ eqn can be written as (a combination of a conservative and a dissipative structure):
\[
\partial_t u = -\partial_x \delta H(u) - \delta D(u),
\]
with
\[
H(u) = \int \frac{1}{3} u^3, \quad D(u) = \int \frac{1}{2} u_x^2
\]
Can you find an alternative way to describe the dissipative part? Determine the evolution of the functional \( \int u(x,t)^2 \, dx \).

5. Second variation
The second variation of a functional \( F \) at a point \( u \) in a given direction \( v \), denoted by \( \delta^2 F(u,v) \) is nothing but the second derivative of the functional when restricted to the line in the given direction:
\[
\delta^2 F(u,v) := \frac{d^2}{d\varepsilon^2} F(u + \varepsilon v) \big|_{\varepsilon = 0}
\]
This leads to a quadratic functional in \( v \). In Maple-notation we write \( \text{Svar}(F,u,v) \) for \( \delta^2 F(u,v) \). In a different way, the second variation can be obtained by repeated differentiation; this leads to a bilinear functional of two directions, that will be denoted by \( \text{svar}(F,u,v,w) \).

**Procedures** \( \text{Svar}(F,u,v,w) \) and \( \text{svar}(F,u,v,w) \)

\[
\text{Svar} := \begin{align*}
(G, u, v) &\rightarrow \text{simplify(subs(} \varepsilon = 0, \\
\text{diff}(G(\text{unapply}(u(t) + \varepsilon * v(t), t), \varepsilon))));
\end{align*}
\]
\[
\text{svar} := \begin{align*}
(G, u, v1, v2) &\rightarrow \text{simplify(subs(} \varepsilon = 0, \rho = 0, \\
\text{diff}(G(\text{unapply}(u(t) + \varepsilon * v1(t) + \rho * v2(t), t), \varepsilon, \rho))));
\end{align*}
\]
Verify these procedures (take examples of functionals from above), and show the relation that is expected from repeated differentiation:
\[
\text{Svar}(F, u, v) = \text{svar}(F, u, v, v).
\]
Chapter 10

Previous exams and written tests
Bibliography


