ITB

KNAW

UTwente

Lectures on Free Surface Waves

Brenny van Groesen,
Applied Analysis & Mathematical Physics,
University of Twente

Lecture Notes for course

LABMATH: Mathematical support for Hydrodynamic Laboratories

August – September 2001

Course at Pusat Matematika, P4M, Institut Teknologi Bandung, Indonesia,
in bi-national Scientific Cooperation Indonesia-Netherlands,
KNAW - project 95-BTM-29

PREFACE, Acknowledgment

Surface waves are phenomena that are characterised by the dynamic interplay between linear interference and nonlinear interactions between individual waves. For understanding these natural phenomena and for many technological applications, such as in hydrodynamic laboratories, a good understanding and description are important.

Mathematical modelling of the penomena is a challenge that was started by scientific giants like Huygens and Newton, and in the 19th century by Scott Russell, Boussinesq, Korteweg & de Vries. In the 20th century, dynamical system theory and improved methods from mathematical physics made it possible to incorporate nonlinear effects to a better extent.

Investigations of the mathematical models have profited, and are still profiting, from mathematical-physical methods and, on the other hand, stimulated the development of these methods.

This part of the Lecture Notes is written in the spirit of the interplay between

natural phenomena – physics –

– math modelling – analytical methods.

In the above mentioned cooperation programme between the applied mathematics institutes at ITB and UT with the hydrodynamic laboratories MARIN (Wageningen) and IHL (Surabaya), various people have been involved:

senior staff members:

Andonowati, Andi Jamalludin, Edy Soewono, Rene Huijsmans

PhD/S3-students:

Jaap Harm Westhuis, Edi Cahyono, Toto Nusantara, Helena Margaretha, Agus Suryanto

and many participants of the Research WorkShop in 1997 which was the start of this project.

I hope that the enthusiasm with which the problems have been tackled and results have been achieved in the best possible, friendly and stimulating atmosphere, is noticeable in these Lecture Notes.

Specific reference

Many results in Sections 2 and 3 is joint work with Andonowati, Edi Cahyono and Agus Suryanto.

Thanks to Frits van Beckum for critically reading the notes and improving various errors.

'Lectures on Free Surface Waves'

Brenny van Groesen
Lecturers & Tutors:
Edi Cahyono & Helena Margaretha & Frits van Beckum

August 5, 2001

Abstract

The lecture notes consist of three sections, following a general description and overview of the aims and achievements in the LABMATH-project.
In the first section, basic notions of wave propagation are introduced starting at an elementary level: inteference, dispersion, nonlinear effects, and progressing to the description by nonlinear pde's with special mathematical-physical structure.
In section 2 the basic 'laws of nature' are given that describe the actual motion of surface waves. The most complete description is possible, but too difficult to allow a direct investigation, which motivates the design and study of simplified models, in particular of KdV or NLS-type.
In the final section 3 a specific advanced phenomenon is taken: the nonlinear evolution of bi-chromatic waves, the bench-mark problem in LABMATH.

Contents

Schedule – for presentation/exercises

- Overview LABMATH, motivation (sheets)

- Basic notions and phenomena in Wave Propagation (LN section 1)
  - Interference
  - Dispersion
  - Wave groups: modulation (bi-chromatic)
  - Nonlinearity
  - Math-physical structure

- Model Equations for free Surface Waves (LN section 2)
  - Full SW equations
  - Physics of wave evolution: energy, momentum
  - Linearised SW, dispersion
  - KdV model – soliton

- Basic approaches to numerical modelling (lecturer: Frits van Beckum)

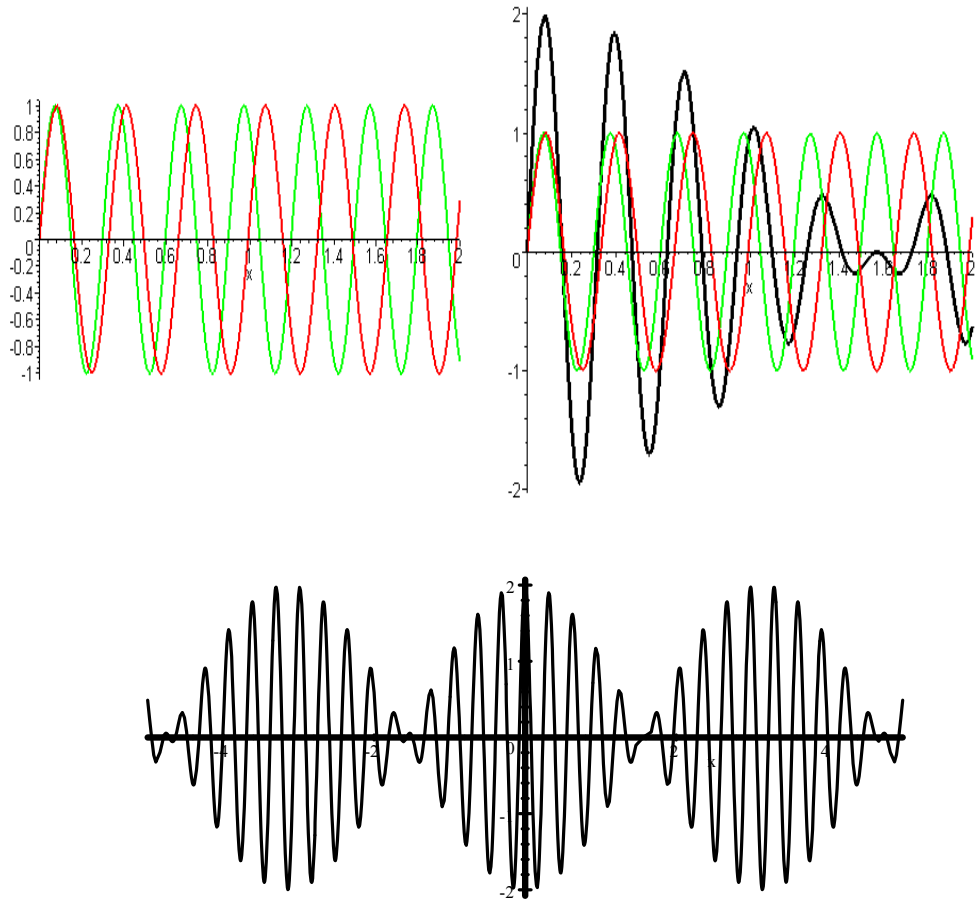
- Nonlinear evolution of Bi-chromatic waves (LN section 3, lecturer: Edi Cahyono)
  - Series expansions
  - Nonlinear Bi-chromatic wave
  - Analytic Wave Code

1 Basic notions Wave Propagation

1.1 Interference

Superposition of 'wiggling' functions can lead to annihilation (cancellation, destructive interference) at some points, and to enlargement (constructive interference) at other points.

For example:



Harmonic function, periodic with period (=wave length)

$$\begin{aligned} x &\rightarrow \cos(kx), \\ k \text{ wavenumber}, \quad \lambda &= 2\pi/k \text{ wavelength} \end{aligned}$$

Superposition of 2 harmonics leads to (harmonic) modulation of harmonic carrier:

$$\begin{aligned} \cos(k_1 x) + \cos(k_2 x) &= 2\underbrace{\cos(\Delta k x)}\underbrace{\cos(\bar{k} x)}, \\ \text{with } \Delta k &= (k_2 - k_1)/2; \bar{k} = (k_1 + k_2)/2 \end{aligned}$$

Intermezzo 1 ***Fourier theory***
Any (square integrable) function f can be expressed as a superposition of possibly infinite number of harmonics. With complex notation for Fourier integral:
Provided the 'energy' is finite, i.e.

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty,$$

f can be written like (all integrals over the wole real axis in the following)

$$f(x) = \int F(k) e^{ikx} dk$$

where F, the Fourier-transform of f, is given by

$$F(k) = \frac{1}{2\pi} \int f(x) e^{-ikx} dx.$$

Some basic properties:

1. $$f \text{ is real, } \Leftrightarrow F(k) = \overline{F(-k)} \quad \text{(overbar is complex conjugate)}$$

2. $$\text{Parceval's indentity: } \int |f(x)|^2 dx = \frac{1}{2\pi} \int |F(k)|^2 dk$$

3. *and more generally*

$$\int f(x)\overline{g(x)} dx = \frac{1}{2\pi} \int \hat{f}(k)\overline{\tilde{g}(k)} dk$$

Exercise 2 Show interference ('MAPLE-plot') for bi-harmonic $\cos(14x) + \cos(16x)$.

1.2 Translation wave

Fixed Profile shifted with constant speed:

$$\eta(x.t) = p(x - ct)$$

For harmonic profile, get a harmonic wave:

$\cos(kx - \omega t)$
k wave number, ω frequency
ω/k phase velocity

Superposition of two harmonic waves with different phase speed:

$$\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t) = 2\underbrace{\cos(\Delta k x - \Delta\omega t)}\underbrace{\cos(\bar{k}x - \bar{\omega}t)},$$

lead to a modulation of a carrier wave; both the carrier wave and the modulated wave propagate at their own speed:

carrier wave	$\cos(\bar{k}x - \bar{\omega}t)$	speed $\bar{\omega}/\bar{k}$
modulation: amplitude	$\cos(\Delta k x - \Delta\omega t)$,	speed $\Delta\omega/\Delta k$ 'group velocity'

Interpretation: superposition of two 'waves', each one propagating with its own phase speed $\frac{\omega}{k}$, leads to 'beat-pattern'.

Animation

Exercise 3 Introduce time dependence and animate ('MAPLE-animate') the bichromatic evolution; verify the propagation speeds of carrier wave and of modulation wave.

1.3 Linear Dispersive wave model

Uni-directional Dispersive wave model: for each wavenumber the physical model assigns a corresponding frequency

$$k -> \Omega(k),$$

the relation being the so-called *dispersion relation*, such that

$$\exp(i\,[kx - \Omega(k)t])$$

is a physical solution, a so-called *harmonic 'mode'*.

Example 4 *1. no dispersion: $\Omega(k) = ck$ for some constant c;*

2. surface water waves (see next section) in normalised variables:

$$\textit{SW-dispersion:}\quad \Omega(k) = k\sqrt{\frac{\tanh(k)}{k}}$$

and approximated for 'long waves' (i.e. k small)

$$\Omega(k) \approx k(1 - k^2/6)$$

Then Fourier-theory describes the evolution of an initial wave profile:

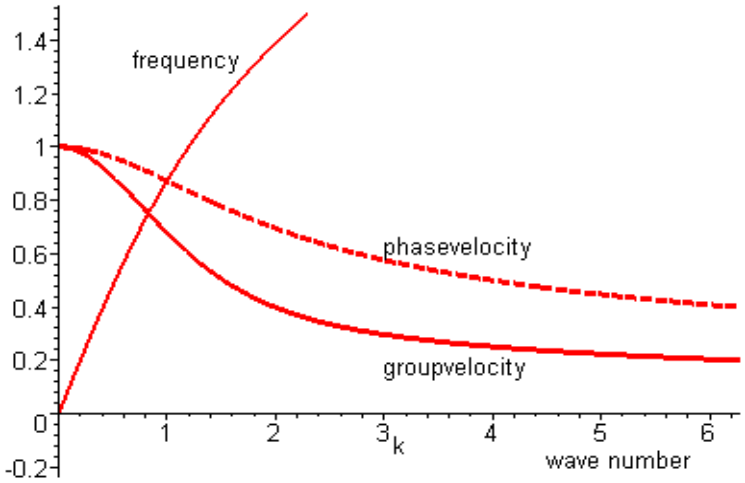
$$\begin{aligned} \text{for initial wave profile} \quad &: \quad \eta(x, t=0) = \int \hat{\eta}(k) e^{ikx} dk, \\ \text{corresponding evolution} \quad &: \quad \eta(x,t) = \int \hat{\eta}(k) e^{i[kx - \Omega(k)t]} dk \end{aligned}$$

Observe/definition:

1. evolution from real initial data is real , provided $\Omega(k) = -\overline{\Omega(-k)}$
2. evolution is *conservative*, meaning $\int \eta(x,t)^2 dx = \int \eta(x,0)^2 dx$, provided $\Omega(k)$ is real.
3. The *group velocity* is defined as

$$V_{gr}(k) = \frac{d\Omega(k)}{dk};$$

   when there is dispersion, this group velocity is not constant but depends on the wave number.
4. Plot of SW-dispersion, the phase- and groupvelocity:



1.4 Wave groups: modulation

When there is no dispersion, $\Omega(k) = ck$ for some constant c, it is found that

$$\eta(x,t) = \int \hat{\eta}(k) e^{i[kx - \Omega(k)t]} dk = \int \hat{\eta}(k) e^{i[k(x-ct]} dk = \eta(x - ct, 0),$$

the case of translating wave.

A *monochromatic wave* (complex notation) is obtained for a spectral function that is a delta-function:

$$\hat{\eta}(k) = \delta(k - k_0) \Rightarrow \eta_{monochrom}(x,t) = e^{i[k_0 x - \Omega(k_0)t]}$$

When there is dispersion, the evolution may be much more difficult; a special example is the *bi-chromatic wave* considered above. This example can be rewritten in complex notation with a spectral function consisting of two delta-functions:

$$\hat{\eta}(k) = \delta(k - k_1) + \delta(k - k_2),$$

from which

$$\eta_{bichrom}(x,t) = e^{i[k_1 x - \Omega(k_1)t]} + e^{i[k_2 x - \Omega(k_2)t]}$$

In general, the solution in the form of the Fourier-integral expression is difficult to grasp. However, when the waves have almost the same wavelenght, one then talks about a *wave packet*, the evolution can be seen (as in the case of the bichromatic wave) as a modulation of a wave with the main wavelenght.

The details of the statement follow from looking at a sharply peaked spectral function, say

$$\hat{\eta}(k) = G(k - k_0)$$

and Taylor expansion of the dispersion relation around $\kappa = k - k_0 \approx 0$:

$$\begin{aligned} \omega(k) &= \Omega_0 + V_0\kappa + \beta\kappa^2 + ..., \\ \Omega_0 &= \omega(k_0), V_0 = \frac{d\Omega}{dk}(k_0), \beta = \frac{d^2\Omega}{dk^2}(k_0) \end{aligned}$$

Then, the solution can be written as a harmonic wave at the central wavenumber modulated by an amplitude

$$\begin{aligned} \eta(x,t) &= \int G(k-k_0)e^{i[kx-\Omega(k)t]}dk = A(x,t)e^{i[k_0 x = \Omega_0 t]}, \\ A(x,t) &= \int G(\kappa)e^{i\left[\kappa(x-V_0 t)-\beta\kappa^2 t\right]}dk \end{aligned}$$

Observe that upon neglecting the quadratic terms in the dispersion relation, the amplitude has fixed profile and travels at constant speed V_0, which is the group velocity at the central wavenumber. For instance, for a Gaussian wave packet,

$$G(\kappa) = \frac{1}{\sigma\pi}\exp\left[-\kappa^2/(4\sigma^2)\right]$$

the amplitude is a Gaussian itself:

$$A(x,t) = \exp\left[-\sigma^2(x - V_0 t)^2\right]$$

In the more general case, the quadratic and higher order terms in the dispersion will lead to some distortion (broadening) of the amplitude's profile. Below we will investigate this deviation by deriving the equation for the amplitude.

1.5 Nonlinearity: mode-interaction and -generation

Most systems, in particular the evolution of surface waves, are non-linear. For instance, in case of Quadratic Nonlinearity, a term appears for which the quantity is squared, say

$$N(\eta) = \eta^2$$

The Fourier transform of a product is not a simple multiplication, but a convolution.

Intermezzo 5 *For functions f, g the Fourier transform of the product fg is the convolution of the Fourier-transforms:*

$$\widehat{fg}(k) = (\hat{f} * \hat{g})(k) \equiv \int \hat{f}(k-\ell)\hat{g}(\ell)d\ell$$

In the simplest case (verify)

$$e^{i\kappa_1 x} \cdot e^{ik_2 x} = e^{i(k_1+k_2)x}$$

which expresses that two Fourier components produce, 'generate', a third one. Clearly, if such nonlinearity is present, modes will be mixed up. For instance, products of long waves produce short waves, and conversely We will see the drastic consequence below.

1.6 Wave Equations: pde's

All the topics above are mathetical methods that are useful to describe phenomena that show characteristic wave properties. But so far no physics (the 'LAWS OF NATURE') has been made explicit. We will discuss this in the next section, but here we just prepare for the general frame work, which is the description by *partial differential equations*. The physical laws, say for the wave elevation $\eta(x,t)$, will be described by relations between the elevation, η, and its derivatives with respect to time $\partial_t\eta$ and space $\partial_x\eta$, and possibly higher order derivatives. (The fact that partial derivatives are involved, explains the name *partial* de, in contrast to *ordinary* de's for functions of only one variable.)

We will now show some examples. As will be relevant for the wave equations, mosttimes we will emphasize the time evolution, that is express $\partial_t\eta$ in terms of the spatial derivatives. This has a fundamental reason: if the equation is of first order in time (meaning that the highest order time derivative is 1, possibly achieved by increasing the number of variables), and some smoothness conditions are satisfied, the IVP (Initial Value Problem) is well posed: for any given initial 'state' (at initial time), the successive evolution (positive time) is uniquely defined.

Translation wave equation

For any profile function p the translation wave: $u(x,t) = p(x-ct)$ satisfies the simple pde

$$\partial_t u = -c\partial_x u$$

and conversely[1]!!

Linear Dispersive Wave equation

[1] To show the converse, introduce new variables $\xi = x - ct, \upsilon = x + ct$ and observe that the pde becomes in new variables: $\partial_\upsilon u = 0$, which is satisfied for any function $u = p(\xi)$, and hence the result.

A single harmonic function $u = \exp(i(kx - \omega t))$ is a special solution (not the only one) of the pde

$$\partial_t u = -\frac{\omega}{k}\partial_x u$$

However, for a dispersive wave model, for which $\omega = \Omega(k)$ is prescribed, this formulation should be generalised as follows. First observe that $\exp(i(kx - \Omega(k)t))$ can be written as a time dependent coefficient multiplying $\exp(ikx)$; writing $a_k(t) = \exp -i\Omega(k)t$, this is for each k a solution of the ode

$$\partial_t a_k = -i\Omega(k)a_k, \quad a_k(0) = 1$$

This motivates to find the equation for the general solution

$$\eta(x,t) = \int \hat{\eta}(k)e^{i[kx-\Omega(k)t]}dk = \int \hat{\eta}(k)e^{-i\Omega(k)t}e^{i[kx]}dk = \int A_k(t)e^{ikx}dk$$

Since

$$\partial_t A_k = -i\Omega(k)A_k, \quad A_k(0) = \hat{\eta}(k)$$

upon multiplying by e^{ikx} and integrating over k there results

$$\partial_t \eta(x.t) = -i\int \Omega(k)A_k(t)e^{ikx}dk$$

Intermezzo 6 ***Pseudo-differential operators***
The right hand side is an operator action on the function $\eta(x,t)$ by multiplying each Fourier coefficient of η by $\Omega(k)$ and then inverse Fourier transformation. This so-called Fourier-integral operator, or pseudo-differential operator, *will be denoted symbolically as $\check{\Omega}$; it is defined as*

$$\textit{for } \; f(x) = \int \hat{f}(k)e^{ikx}dk \quad \textit{by definition } \; \check{\Omega}f(x) = \int \Omega(k)\hat{f}(k)e^{ikx}dk$$

and the function Ω is called the symbol *of the operator $\check{\Omega}$.*

Example 7 *When Ω is a polynomial, the operator $\check{\Omega}$ is a simple differential operator, since*

$$\Omega(k) = k, \check{\Omega} = i\partial_x, \quad \textit{and hence } \; \Omega(k) = k^n, \check{\Omega} = (i\partial_x)^n$$

However, $\check{\Omega}$ is also well-defined when it is not a polynomial as in te case of surface waves.

Using this notation, the equation $\partial_t \eta(x.t) = -i\int \Omega(k)A_k(t)e^{ikx}dk$ can be written like

$$\partial_t \eta(x.t) = -i\check{\Omega}\eta(x,t).$$

When $\Omega(k)$ is real (for real k) and odd, this is the general form of a *linear dispersive wave equation.*

Proposition 8 *When $\Omega(k)$ is real (for real k) and odd, $\Omega(k) = -\Omega(-k)$, the corresponding pseudo-differential operator is* skew-symmetric *in the sense that for any two real functions f, g*

$$\int f(x)\check{\Omega}g(x)dx = -\int g(x)\check{\Omega}f(x)dx.$$

Proposition 9 *For a linear dispersive wave equation the square energy is a constant of the motion:*

$$\partial_t \int \eta^2(x.t)dx = 0$$

Exercise 10 Verify (prove) these two propositions.

1.7 Distortion of linear wavegroups: linear envelope equation

Let's now investigate once again the linear wave group, i.e. the solution with a sharply peaked spectral function, say

$$\hat{\eta}(k) = G(k - k_0).$$

From Taylor expansion of the dispersion relation around $\kappa = k - k_0 \approx 0$:

$$\begin{aligned} \omega(k) &= \Omega_0 + V_0\kappa + \beta\kappa^2 + ..., \\ \Omega_0 &= \omega(k_0), V_0 = \frac{d\Omega}{dk}(k_0), \beta = \frac{d^2\Omega}{dk^2}(k_0) \end{aligned}$$

we found that the solution can be written as a harmonic wave at the central wavenumber modulated by an amplitude

$$\begin{aligned} \eta(x,t) &= A(x,t)e^{i[k_0x=\Omega_0t]}, \\ A(x,t) &= \int G(\kappa)e^{i\left[\kappa(x-V_0t)-\beta\kappa^2t\right]}dk \end{aligned}$$

where qubic and higher order terms in the dispersion relation have been neglected.

Using $\xi = x - V_0t$ and $\tau = t$ as the basic variables (which means introducing a frame of reference moving with the group velocity),

$$A(\xi,\tau) = \int G(\kappa)e^{i\left[\kappa\xi-\beta\kappa^2\tau\right]}dk$$

Observe that $A(\xi,\tau)$ is the solution of a dispersive wave equation that has dispersion relation $\kappa \to -\beta\kappa^2$, i.e. A satisfies

$$\partial_\tau A = -i\beta\partial_\xi^2 A.$$

Hence, the distortion of the amplitude of the wavegroup is described by this *envelope equation.*

Observe that for $\beta = 0$, the amplitude is fixed, meaning travelling at constant speed V_0 in the laboratory frame of reference.

Exercise 11 Study the 'slow' deformation of the envelope in a wave packet due to dispersion. (Hint: make plots in Maple, subtract the undeformed evolution at the group velocity; approximate the integral by a Riemann sum.)

1.8 Nonlinear wave equation: "Breaking" waves

Consider the following prototype of a nonlinear equation:

$$\partial_t u(x,\, t) + u(x,t)\, \partial_x\, u(x,\, t)) = 0$$

The solution with initial value f can be written analytically in an implicit way as follows. First observe that on a curve $t \to x(t)$ the total derivative of u is given by

$$\frac{d}{d\,t}\, u(x(t),\, t) = \partial_x u(x(t),\, t)\, \frac{d}{d\,t}\, x(t) + \partial_t u(x(t),\, t).$$

From this it follows that

$$\frac{d}{d\,t}\, u(x(t),\, t) = 0 \text{ on } \frac{d}{d\,t}\, x(t) = u,$$

and hence

$$u(x,t) = \text{constant on } \frac{d}{d\,t}\, x(t) = u.$$

Therefore, the 'curve' with $\frac{d}{d\,t}\, x(t) = u$, which is called a *characteristic curve*, is in this case a straight line with slope u. From this the construction becomes clear by using the information of the initial value. In detail:
Let y be a point at the x-axis of the x, t plane; at this point u has the value $f(y)$. The characteristic through this point is the straight line with tangent $f(y)$:

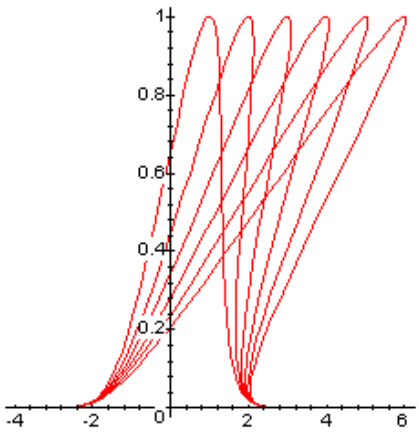
$$x = y + f(y)\, t$$

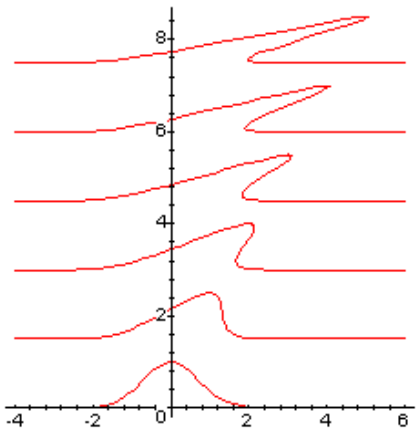
and on this line the function u has the value $f(y)$, and so

$$u(y + f(y)\, t,\, t) = f(y)$$

is the solution in implicit form.

Example 12 For a Gaussian function as example of an initial profile, the solution can be plotted easily using the parametric representation; a succession of profiles at different times is given below, together with a different presentation that is obtained by shifting the profiles vertically with increasing time.





Exercise 13 Consider the nonlinear "breaking" equation $\partial_t u + u\partial_x u = 0$.

1. Draw the profiles at two successive (positive) times of the solution with initial condition the positive step function:

$$step(x) = \begin{cases} = 0 & for \;\; x < 0, \\ = 1 & for \;\; x > 0. \end{cases}$$

2. Do the same for the reversed step function, i.e. the initial condition $step(-x)$.
3. Draw in each case characteristic curves in the x, t- plane.
4. Determine the first time t^* at which "breaking" occurs: $\partial_x u(x, t^*) = \infty$.
   [[Hint: Note that

$$\frac{d}{dy} u(y + f(y)\, t, \, t) = \frac{\partial u}{\partial x}(x, \, t) \left(1 + t \frac{d}{dy} f(y)\right)$$

   and show that from this it follows that on the characteristic through y this is the case for the time (if it exists) such that ...., etc; complete the argument.]]

1.9 Math-physical structure

'Laws of Nature' express basic properties and describe special phenomena. Mathematical equations that express these LofN will/should contain these properties and phenomena in the description. That is, such properties will lead to a special structure of the math equations; sometimes this is not obvious, but it is worthwhile to try to *recognise basic properties of physics from the mathematical structure.*

A special example of a basic property is conservation of 'energy'.

Exercise 14 Show that for the following ode's in time the given quantity E is conserved (constant in time for solutions)

$$\begin{array}{ll} \ddot{x} + \omega^2 x = 0, & E = \frac{1}{2}\left(\dot{x}^2 + \omega^2 x^2\right) \\ \dot{z} + i\omega z = 0, & E = \frac{1}{2}|z|^2 \\ \left\{ \begin{array}{c} \dot{q} = p \\ \dot{p} = -\omega^2 q \end{array} \right. & E = \frac{1}{2}\left(p^2 + \omega^2 q^2\right) \end{array}$$

The *phase plane* for these equations is the plane $(\dot{x}, x), (\mathrm{Re}\, z, \mathrm{Im}\, z), (q, p)$ respecively. Level sets of the function E give information about the dynamics (why?). Draw th so-called *phase portrait,* i.e. various levellines in the phase-plane.

These examples are special cases of certain classes of equations, each of which has 'energy conservation':

1. Newton's equation with conservative force from potential energy $V = V(x)$:
$$\ddot{x} + \frac{\partial V}{\partial x} = 0, \quad E = \frac{1}{2}\dot{x}^2 + V(x)$$
2. In complex notation
$$\dot{z} = -i\frac{\partial H}{\partial z}, \quad E = H(z)$$
3. Classical Hamilton's equations in so-called *canonical- variables* (q, p):
$$\partial_t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix}, \quad E = H(q, p)$$

Actually, all these classes of equations themselves are just representatives of one larger class of equations, namely all these are examples of ode's with a Hamiltonian (Poisson) structure.

Definition 15 A dynamical system in R^n (or C^n) with state variable $u(t)$ is called a *Hamiltonian (Poisson) system* if the system of ode's has the following form:

$$\partial_t u = \Gamma \partial_u H(u),$$

where

$$\Gamma \textit{ is a skew-symmetric operator, the } \text{'structure map'}$$

meaning that in a suitable innerproduct

$$< \Gamma u, v > = - < u, \Gamma v >,$$

and where

$$u \to H(u) \textit{ is a given function, called the } \text{Hamiltonian}$$

Exercise 16 Show that the given equations above are indeed Hamiltonian systems by specifying the structure map and the Hamiltonian.

Exercise 17 Show that for a Hamiltonian system the Hamiltonian is a constant of the motion itself:

$$\partial_t H(u) = 0 \textit{ for solutions.}$$

Remark 18 This is an example of the opening sentences of this subsection: the mathematical structure of Hamilton's equations implies that there is a constant of the motion (H), i.e. a Hamiltonian structure contains a conservation property.
In Classical Mechanics Hamilton's equations are studied in detail; in most cases the Hamiltonian is (related to) the *total physical energy, sum of kinetic and potential energy*, just as in the examples above.

For systems described by partial differential equations, it is possible to generalise the notion of Hamiltonian system. For instance, all wave equations, when viscous effects which cause dissipation are neglected, can be recognised as such. As a simple example, the translation equation at constant speed

$$\partial_t u = -\partial_x (cu)$$

is of this form:

- the state variable $t \to u(t)$ now depends on x, i.e. $u(t)(x) = u(x,t)$, and hence $u(t)$ now belongs to an infinite dimensional state space
- the symplectic structure is now

  $$\Gamma = \partial_x$$

  and is indeed skew-symmetric for the usual L_2-innerproduct of functions of x that decay at infinity::

  $$< u, v >= \int u(x)v(x)dx;$$

- the Hamiltonian is now a function on the function space, called functional, in this case given by

  $$H(u) = \int \frac{1}{2} cu^2(x)dx;$$

  it can be shown that the 'derivative' of such a functional $H(u)$, called the 'variational derivative' and denoted by $\delta H(u)$, can be defined just as for functions on R^n with the directional derivative as a generalisation of the gradient:

  $$< \delta H(u), v >= \frac{d}{d\varepsilon} H(u + \varepsilon v)\Big|_{\varepsilon=0};$$

for the given Hamiltonian

$$H(u) = \int \frac{1}{2} c u^2(x) dx \Longrightarrow \delta H(u) = cu$$

from which the Hamiltonian structure of the translation equation follows.

Exercise 19 Verify by direct calculation that this Hamiltonian is conserved for solutions of the translation equation (restrict to functions that vanish at infinity). Give also a more direct argument for this result, based on the observation of the change of this functional when the function is translated over an arbitrary distance.

2 Model Equations for free Surface Waves (SW)

The evolution of waves on the surface of a layer of fluid (such as water) remains a challenging task. Assuming the fluid to be incompressible and inviscid, and the flow to be irrotational, the full surface wave equations (full SWE) are well known. The combination of dispersive and non-linear effects present major problems in the numerical simulation as well as in the theoretical analysis of the resulting interesting phenomena.
Although the full SWE's are well known and describe the complete physics, this set of equations is too complicated for a direct investigation. Therefore, to gain insight in interesting characteristic phenomena, except from detailed experiments and numerical simulations, simplified models are desired that are amenable for theoretical investigations, while, at the same time, should be accurate enough to capture the phenomenon of interest.
In this section we present a unified view on results from the literature and describe models of KdV- and NLS-type of equations and their relation. However, since this is essential for the phenomena to be expected, we will carefully model the dispersive properties. That is, we distinguish between long waves (shallow water) and short waves (deep water). In the first case, the dispersion for long waves leads to the classical KdV-equation, and to the defocusing NLS-equation for wave packets. For waves with small wave lengths, comparable to the depth of the fluid, the full dispersive properties should be dealt with, which leads to a non-local version of the KdV-equation and the focusing NLS-equation. In the latter case, initially smooth wave packets will show large deformations (caused by side band instabilities, though different from the well known Benjamin-Feir instability). Using the available results from experiments and simulations, we investigate the performance of the simplified NLS-model.
The mathematical models to be presented, describe, to varying degree of accuracy, the surface wave problem. In all cases, the basic assumptions are that the fluid is incompressible and inviscid (such as water in a good approximation), and that the flow is irrotational; furthermore we mainly restrict to wave propagation in one horizontal direction. The full set of equations (SWE's) is given first. These equations describe in detail the evolution of the surface elevation, and the interior fluid velocity. For designing simpler models, it is important to identify some fundamental properties, which are physically most relevant, and then keep these properties in simplified models. Mass conservation and energy conservation are the most important properties to be retained. Mathematically, the variational structure present in the equations, is just as important to be retained; this is a Hamiltoinian structure as briefly discussed in the previous section. The simplified models will be checked for such properties.
A class of simplified models are KdV type of equations for uni-directional wave propagation; these will be considered both for shallow and deep water waves. Finally, for wave packets with narrow banded spectra, NLS-type of equations are described, and the derivation is recalled as far as necessary for the appreciation of the two different cases (focusing and defocusing).

2.1 Full surface wave equations

We consider the motion of a layer of fluid under the following simplifying assumptions:

- the fluid is *inviscid*, *incompressible* (density normalized to unity), and no surface tension;

- the bottom is flat, at depth $z = -H$;
- the fluid motion is *irrotational*, assumed to be uniform in the (horizontal) y-direction and unbounded in the x-direction; if the horizontal and vertical velocities are denoted by $U = U(x, z, t)$ and $W = W(x, z, t)$ respectively, irrotational motion means

  $$U_z - W_x = 0$$

  and allows the introduction of the fluid potential Φ such that

  $$(U, W) = \nabla\Phi, \quad U = \Phi_x, \quad W = \Phi_z.$$

  Then incompressibility implies

  $$U_x + W_z = 0, \quad \Delta\Phi = 0;$$

- the surface elevation is the graph of a function (no overturning waves) $\eta = \eta(x, t)$.

Then the governing equations are

$$\Delta\Phi \equiv \Phi_{xx} + \Phi_{zz} = 0, \qquad -H < z < \eta(x, t) \tag{1}$$

$$\Phi_z = 0 \quad \text{at} \quad z = -H, \tag{2}$$

$$\partial_t\eta = -\eta_x\Phi_x + \Phi_z \quad \text{at} \quad z = \eta(x, t), \tag{3}$$

$$\partial_t\Phi + \frac{1}{2}\left(\Phi_x^2 + \Phi_z^2\right) + g\eta = 0 \quad \text{at } z = \eta(x, t). \tag{4}$$

Equation (3) is a kinematic condition; equation (4) is a dynamic condition, resulting from Bernoulli's equation restricted to the free surface.

Remark 20 *Alternative descriptions*

1. For various reasons it is convenient to introduce the potential at the free surface

   $$\varphi(x, t) = \Phi(x, \eta(x, t), t)$$

   and its x-derivative (which is a velocity type of quantity)

   $$u(x, t) = \partial_x\varphi = \Phi_x + \Phi_z\eta_x$$

   Then $\partial_t\varphi = \Phi_t + \Phi_z\eta_t$, and the free surface conditions become

   $$\partial_t\eta = -u\eta_x + w(1 + \eta_x^2), \tag{5}$$

   $$\partial_t u = -\partial_x[g\eta + \frac{1}{2}u^2 - \frac{1}{2}w^2(1 + \eta_x^2)] \tag{6}$$

   where w is the vertical velocity at the free surface:

   $$w(x, t) = W(x, \eta(x, t), t).$$

   This is a set of equations for η, u in which, however, w still has to be expressed as a function of η, u.

2. Another way to rewrite the kinematic surface condition (the continuity equation) is as a local conservation law:

$$\partial_t \eta = -\partial_x \int_{-H}^{\eta} \Phi_x \, dz \tag{7}$$

which shows that the flux equals the integrated horizontal velocity. Stated differently, this is nothing but mass-conservation.

3. This also shows that the boundary conditions are in 'conservation form': (7) and (6).

4. Yet another interpretation of the kinematic surface condition is that the vertical fluid velocity should equal the vertical displacement of the fluid surface:

$$\Phi_z = \frac{d}{dt}\eta(x(t),t) \equiv \partial_t \eta + \partial_x \eta \frac{dx}{dt}$$

For *multi-directional waves*, depending on both horizontal variables, a simple extension is as follows.
Due to the irrotationality of the velocity field $\vec{v}$ a scalar potential function Φ can be defined as $\nabla\Phi = \vec{v}$. Mass conservation within the fluid domain $\mathcal{D}$ can now be expressed as Laplace's equation for the potential

$$\Delta\Phi \equiv \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0 \qquad \text{in } \mathcal{D}. \tag{8}$$

Now the two conditions that govern the free surface dynamics are

$$\Phi_t = -\frac{1}{2}|\nabla\Phi|^2 - g\eta \quad \text{on} \quad z = \eta(x,t), \tag{9}$$

$$\eta_t = \Phi_z - \eta_x \Phi_x - \eta_y \Phi_y \quad \text{on} \quad z = \eta(x,t). \tag{10}$$

Additional initial and boundary conditions are required to make the problem well-posed. For the applications in hydrodynamic laboratories, these are an impermeable bottom condition, lateral boundary conditions, absorption at the down stream end of the domain, and some prescribed condition at the upstream position to define a suitable signalling problem. Alternatively, on a spatially unbounded domain, an initial surface elevation and horizontal fluid velocity may be prescribed.
For a simple interpretation of the SWE's, most important is to observe how the interior Laplace problem is linked to the dynamic equations. The algorithm for one timestep can be described as follows:

- start with a given surface elevation and given potential at the surface;
- then solve the Laplace equation, and calculate the velocity ($\nabla\Phi$) at the free surface;
- use this information in the right hand side of the dynamic equations, and evolve these for one time-step;
- this then provides updated fluid surface and potential,

- start the next timestep.

The main problem in this algorithm is the calculation of the velocity at the free surface, actually the solution of the *Dirichlet to Neumann (DtN) problem*: given Φ at the surface, find $\nabla\Phi$ at the surface. Numerically this can be done nowadays quite efficiently and accurately. In finding simplified models for theoretical investigations, various different approximations for the solution of the DtN problem are used to arrive at equations on the free surface, thereby having eliminated the problem to calculate the interior flow. We will describe this in the next subsection.

2.1.1 Variational structure

Another way to interpret the dynamical structure is using a variational description. In fact, it has been observed (independently) by Zakharov (1968) and Broer (1974) and Miles (1977) that SWE can be described as a Hamiltonian system. Summarising, this can be described by using as variables the fluid potential at the free surface and the surface elevation. Then the full surface wave equations can be described as a Hamiltonian system (see for full details [4])

$$\partial_t \eta = -\delta_\phi H(\phi, \eta), \quad \partial_t \phi = \delta_\eta H \tag{11}$$

where H is the Hamiltonian functional and $\phi = \Phi|_{z=\eta}$ is the free surface potential. Just as for Hamiltonian systems from Classical Mechanics, the Hamiltonian is the sum of kinetic and potential energy

$$H(\phi, \eta) = K(\phi, \eta) + \int \frac{1}{2} g\eta^2 dx$$

where the kinetic energy is given for solutions of the Dirichlet problem for the Laplace problem in the fluid domain: $K(\phi, \eta) = \int\int \frac{1}{2}|\nabla\Phi|^2 dxdz = \int \phi \left[\partial_n \Phi\right]_{z=\eta} dx$. Since this functional cannot be expressed explicitly in terms of ϕ, η, which would require to solve the DtN problem, simplified models can be obtained by constructing approximations for this functional. In doing so, and taking as governing equations the system (11) with the approximated Hamiltonian, leads to a model that has retained the basic variational structure, which is not guaranteed in a direct approach.

2.2 Physics of wave evolution: energy, momentum

2.2.1 Intermezzo: Conservation laws and conserved quantities

We start with some general ideas about conservation in pde's, which will after that be applicable for the wave problems.
Let the state of a dynamic system be described by a function $u = u(x,t)$ of the spatial variable x and time t. Let $e = e[u]$ be some local density, i.e. an expression that depends on u in such a way that the value of e at (x,t) is determined by $u(x,t)$ and a finite number of x- derivatives of u at (x,t). We write $e = e[u](x,t)$ for its value at (x,t).

The integral of e along a given spatial interval $[a, b]$ defines the integrated quantity $E_{[a,b]}$ that will depend on u and time:

$$E_{[a,b]}[u](t) \equiv \int_a^b e[u](x,t)\,dx$$

In specific cases, depending on the evolution equation and the density, the time derivative may depend only on the value of a (local) flux density Q at the end points of the interval:

$$\frac{d}{dt}E_{[a,b]}[u](t) = -Q[u](b,t) + Q[u](a,t). \tag{12}$$

This is then called a *global conservation law* for the density e, and Q is called the *flux density* corresponding to e.
Rewriting the right hand side

$$-Q[u](b,t) + Q[u](a,t) \equiv -\int_a^b \partial_x\, Q[u](x,t)\,dx$$

and (assuming differentiability) the left hand side

$$\frac{d}{dt}E_{[a,b]}[u](t) \equiv \int_a^b \partial_t\, e[u](x,t)\,dx,$$

the global law can be rewritten like

$$\int_a^b \{\, \partial_t\, e[u](x,t) + \partial_x\, Q[u](x,t)\,\}\,dx = 0.$$

If this holds for any interval of integration $[a, b]$ (within some given interval), it follows[2] that

$$\partial_t\ e[u](x,t) + \partial_x\ Q[u](x,t) = 0. \tag{13}$$

This expression is called the *local conservation law* for the density e.

Remark 21 *1.* As described above, the local conservation law is derived from the global conservation law; the reverse is equally true (and simpler): integrating (13) along an interval $[a, b]$, the result will be (12).

2. When for a given solution, the net flux through the boundary vanishes, i.e. when

$$Q[u](a,t) = Q[u](b,t)$$

then

$$\partial_t \int_a^b e[u](x,t)\,dx = 0$$

and we have a conserved quantity or *constant of the motion:* $\int_a^b e[u](x,t)\,dx$.

[2] Intuitively clear; strictly speaking use Lagrange's Lemma and assume continuity of the expression in brackets in the integral.

3. In the special case that the evolution equation for the state variable itself is of the form of a local conservation law

$$\partial_t\, u(x,t) = -\partial_x\, Q[u](x,t)$$

it is custom to talk about an *evolution equation in conservation form.*

4. For a given evolution equation, and a local density e, the existence of a local/global conservation law is special; in general the density will satisfy a *local balance law* of the form:

$$\partial_t\, e[u] + \partial_x\, \Phi[u] = S[u]$$

where S is the so-called local *source density.*

For a system in *more spatial dimensions*, write $\mathbf{x} \in \Omega \subset R^n$ and let Ω be a bounded or unbounded domain with $\mathbf{n}$ the outward pointing normal at the boundary $\partial\Omega$. The flux of a (scalar) density e is an n-vector $\mathbf{Q}$ and the flux through the boundary is determined by the normal component:

$$\text{flux through boundary:} \quad \int_{\partial\Omega} \mathbf{Q} \cdot \mathbf{n}\, dA$$

With a source density S, the global balance law reads

$$\frac{d}{dt} \int_\Omega e(\mathbf{x},t)\ d\mathbf{x} = -\int_{\partial\Omega} \mathbf{Q}\cdot\, \mathbf{n}\, dA + \int_\Omega S(\mathbf{x},t)\, d\mathbf{x}$$

Using Gauss' theorem:

$$\int_{\partial\Omega} \mathbf{Q} \cdot \mathbf{n}\, dA \equiv \int_\Omega \operatorname{div} \mathbf{Q}\, d\mathbf{x},$$

and assuming smoothness, a local balance law is obtained in the form

$$\partial_t\ e + \operatorname{div} \mathbf{Q} = S.$$

If $S = 0$, the balance law becomes a conservation law.
All these notions become particularly relevant when for $e(u)$ a physically motivated density is taken.

2.2.2 Conservation properties for full SW equations

1. *Conservation of mass (fluid):* The 'continuity equation' in the description of continua with a mass density ρ and Eulerian velocity v reads

$$\partial_t \rho + div(\rho v) = 0,$$

expressing mass conservation: in a fixed domain Ω with boundary $\partial\Omega$ with normal n; the total amount of mass can only change by flux though the boundary as is expressed as follows:

$$\partial_t \int_\Omega \rho dx = -\int_\Omega div(\rho v) dx = -\int_{\partial\Omega} \rho v.n$$

For water, we assume ρ =constant, corresponding to

$$div(v) = 0,$$

which leads to Laplace equation with the expression for the Eulerian velocity (from irrotationallity assumption) $v = \nabla\Phi$.

2. For the depth integration of mass-conservation the kinematic condition follows as we have seen already:

$$\partial_t \int_{-H}^{\eta} dz = \partial_t \eta = -\partial_x \int_{-H}^{\eta} \Phi_x \, dz$$

The flux density is the horizontal momentum, to be denoted by $I(x,t)$:

$$I = \int_{-H}^{\eta} \Phi_x \, dz$$

3. Total Energy

The local kinetic energy density is given by $\frac{1}{2}\rho|\nabla\Phi|^2$. The potential energy of mass ρ at height z equals ρz, and therefore the total energy integrated over depth is found to be:

$$E(x,t) = \int_{-H}^{\eta} \left(\frac{1}{2}\rho|\nabla\Phi|^2 + \rho z\right) dz = \int_{-H}^{\eta} \left(\frac{1}{2}\rho|\nabla\Phi|^2\right) dz + \frac{1}{2}\rho\eta^2 + \left(\frac{1}{2}\rho H^2\right)$$

We can and will neglect the inessential constant $\frac{1}{2}\rho H^2$ in the following.

2.3 Linearised SW, dispersion

As stated above, any simplifying model is based on an approximation of the DtN problem. Without surface elevation and bottom variations, the problem is on a straight strip and can be solved in closed form by Fourier techniques. This explains the various approximations, which are based (given a horizontal straight bottom) on specifying simplifying properties of the surface elevation. In particular, for infinitesimally small amplitude waves, the linearised problem can be solved and leads to the basic (linear) dispersion relation between frequency and wavenumber, given by (using normalised variables)

$$\omega = \pm\Omega(k), \text{ with } \Omega(k) = k\sqrt{\tanh(k)/k}. \tag{14}$$

In more dimensions, for a plane wave with wavevector k, the same dispersion relation is found with $k = |k|$.

We now present the details of the derivation.
First we have to consider the linearised equations. This means, we suppose that the wave elevation η is small, and just as well the fluid velocity $\nabla\Phi$. Then consistently in lowest order, we retain only linear terms in the surface equations, and, moreover, take the boundary in the Laplace problem to be the undisturbed level $z = 0$. Then the full set becomes
Laplace problem on strip:

$$\Delta\Phi \equiv \Phi_{xx} + +\Phi_{yy} + \Phi_{zz} = 0, \quad -H < z < 0 \tag{15}$$

with boundary condition at bottom:

$$\Phi_z = 0 \quad \text{at} \quad z = -H, \tag{16}$$

and linearised free surface conditions: kinematic surface condition

$$\partial_t\eta = \Phi_z \quad \text{at} \quad z = 0, \tag{17}$$

and dynamic linearised surface condition

$$\partial_t\Phi + g\eta = 0 \; z = 0. \tag{18}$$

To find the dispersion relation, we look for time harmonic solutions, periodic in the horizontal directions, i.e. both η and Φ of the form $e^{i(k_1x+k_2y-\omega t)}$. The general solution for Φ then has the form

$$\Phi(x, y, z, t) = a(z)e^{i(k_1x+k_2y-\omega t)}$$

and inserting in the equation there results:

$$a_{zz} - |k|^2a = 0.$$

With the boundary condition at bottom (16), which requires $a(z, t) = 0$ at $z = -H$, the solution becomes:

$$a(z) = A\sinh(|k|(z + H))$$

with A arbitrary. Hence $\Phi = A\sinh(|k|(z + H))e^{i(k_1x+k_2y-\omega t)}$.
For the kinematic surface condition (17) we need to calculate Φ_z at $z = 0$, which is $\Phi_z|_{z=0} = A|k|\cosh(|k|H)e^{i(k_1x+k_2y-\omega t)}$ Writing

$$\eta(x, y, t) = Be^{i(k_1x+k_2y-\omega t)}$$

the kinematic boundary condition requires:

$$-i\omega B = A|k|\cosh(|k|H) \tag{19}$$

while the dynamic boundary condition gives

$$-i\omega A \sinh(|k|H) + gB = 0. \tag{20}$$

These two conditions for the coefficients A, B only can give a nontrivial result provided

$$\omega^2 = g|k| \tanh(|k|H).$$

which is the desired dispersion relation.
In one space dimension, allowing the wave number to have different signs, the odd solution should be taken and there results the dispersion relation

$$\omega = k\sqrt{\frac{g \tanh(kH)}{k}}$$

Note the limiting behaviour, for small wave numbers (long waves) and large wave numbers (short waves)

$$\begin{aligned} \omega &\sim k\sqrt{gH} \text{ for } k \to 0, \\ \omega &\sim \sqrt{gk} \text{ for } k \to \infty. \end{aligned}$$

In more dimensions, given the wave vector, the positive value of ω is taken for waves travelling in the wave vector direction, and the negative sign for waves in the opposite direction.

2.4 KdV type of equations

One step further than the linear approximation, is to take for small amplitude solutions a first order nonlinear effect into account. At the same time, assumptions on the characteristic wave length are commonly made; mostly the restriction is to 'long' waves. A characteristic, often used approximation is the so-called Boussinesq approximation, which corresponds to the specific relation between the wave amplitude ε and wave length λ given by $1/\lambda^2 \sim \varepsilon$, the case of 'rather small, rather long waves'. This is the basic assumption to arrive at what are called Boussinesq-type of equations. These equations describe both waves running to the right and the left. Further restricting to waves running mainly in one direction, then leads to KdV-type of equations[3]. The general form of these equations is a first order in time equation of the form

$$\partial_t u = -\partial_x(Ru + \frac{3}{4}u^2) \tag{21}$$

[3] *Korteweg-de Vries equation (1895)*
Korteweg and de Vries derived in 1895 a model equation for the motion of waves on the surface of a layer of fluid above a flat bottom. Restricting to rather low, rather long waves, they derived the equation (21) that now bears their name with $R_{KdV}(k)$. This equation became well known in the sixties since it turned out that from a mathematical point of view it was the first partial differential equation shown to be completely integrable, leading to a huge extension of the theory of nonlinear pde's. It also became clear that many problems in physics and technics are modelled by this equation.
Being an evolution equation, first order in time, the *initial value problem* requires to find the evolution of the surface profile from a given initial profile. This initial value problem for KdV is not easy to solve; for arbitrary initial profiles, numerical calculations have to be used to find the subsequent wave profiles; the complete integrability makes it possible in principle to write down the time-asymptotic profile.

for the normalised surface elevation u. Here R is a linear operator that determines the specific dispersive properties, since the linearised equation has dispersion relation $\omega = kR(k)$. For the classical KdV equation, in the Boussinesq approximation, the operator is given by

$$R_{KdV}(k) = 1 + \frac{1}{6}\partial_x^2 \simeq 1 - \frac{1}{6}k^2$$

where the symbol $\simeq$ means the action of the operator in Fourier space. With this choice, the dispersion is correct for long waves, since $\Omega(k) = k(1 - \frac{1}{6}k^2 + ..)$ for $k \to 0$. Then the operator is a local operator, and KdV is a partial differential equation. An improvement of the dispersive properties can be achieved by taking for R the exact dispersion operator for linear waves, i.e.

$$R \simeq \sqrt{\tanh(k)/k}$$

Then the operator is no longer an ordinary differential operator and the equation (21) is no longer a standard partial differential equation. For modelling waves of shorter wave lengths, as in hydrodynamic laboratories, this improvement turns out to be essential. Observe, for instance, that for R_{KdV} the phase and group velocity become negative for shorter waves, very different from the exact relation for which both tend to zero for shorter waves. In finite difference approximations of (21) this non-local behaviour is a serious problem, and various improvements have been proposed and studied (for instance Pade-approximations). In the next subsection we will see explicitly the importance of the choice of the dispersion relation. The quadratic nonlinearity in (21) is the second order term from the quadratic character of the original problem. However, the specific form is just a simplification and higher order terms could be added. This is actually done in an alternative model, the so-called Stokes expansion, in which a series expansion in the steepness of the wave, essentially ka, is taken as parameter. Then the equation has also a dispersive operator in the quadratic term. In view of the rather good performance of the model (21), almost up to the point of breaking waves, as will be shown in the next sections, we restrict to the non-dispersive quadratic nonlinearity for simplicity.
Finally, as stated above, a model should retain the Hamiltonian character of the full SWE's. Indeed, the KdV-type of equations have a Hamiltonian structure too, given by

$$\partial_t u = -\partial_x \delta_u H(u)$$

where the symplectic operator is the differential operator ∂_x and the Hamiltonian is now given by

$$H(u) = \int \left[\frac{1}{2}uRu + \frac{1}{4}u^3\right] dx$$

In [4] a derivation of the KdV-equation is described using the variational structure as the guiding principle to obtain consistent simplified models.

Exercise 22 The KdV-eqn in normalized form:

$$\partial_t u + u_{xxx} + uu_x = 0,$$

can be written as a conservation law:

$$\partial_t u = -\partial_x \left[u_{xx} + \frac{1}{2}u^2\right],$$

and so u is a conserved density. In fact there are more conserved densities; three relevant conserved densities are given below. Verify the given corresponding fluxes:

$$\begin{aligned} e(u) = u, \quad & \textit{mass} \\ & f(u) = u_{xx} + \tfrac{1}{2}u^2 \\ e(u) = u^2 \quad & \textit{momentum} \\ & f(u) = 2uu_{xx} - u_x^2 + \tfrac{1}{3}u^3 \end{aligned}$$

$$e(u) = \frac{1}{2}u_x^2 - \frac{1}{6}u^3, \quad \textit{energy}$$

$$f(u) = -\frac{1}{2}u_{xx}^2 + u_x u_{xxx} + u_x^2 u - \frac{1}{2}u^2 u_{xx} - \frac{1}{8}u^4$$

For solutions decaying (with all derivatives) at infinity (or for periodic solutions), the global conserved quantities are the integrals over the whole real line (or with the integrals over one period, respectively). Determine these conserved quantities.

Exercise 23 Consider the socalled BBM eqn. (Benjamin, Bona & Mahony, 1972):

$$(1 - \partial_x^2)\partial_t u = -\partial_x u - u\partial_x u \tag{22}$$

This model equation is a variant of the KdV eqn (normalised variables).

1. Determine the dispersion relation of the linearised equation. What is the relation with the dispersion relation of the full SW eqn's, and with that of the standard-KdV-equation.

2. Looking for travelling waves, $u(x,t) = f(x - V\,t)$, write down the equation for the profile function f; do you recognise this (form of the) equation? Find the solution explicitly.

2.5 Focusing/defocusing NLS-models

In this subsection we derive the equation for the envelope of a wave group. In view of the applications of propagation in wave tanks, we will sketch the derivation of the spatial NLS-equation, in which the evolution parameter is in the down stream direction (instead of in time). Then the spatial evolution is expressed in terms of temporal derivatives; this is the convenient description for a signalling problem: sending waves from a specified point (the wave maker) and investigating the down-stream evolution. This equation can be derived in a certain approximation as follows. Without the nonlinearity, the next equations are equivalent

$$\partial_t u + i\Omega(i\partial_x)u = 0 \Leftrightarrow \partial_x u - iK(-i\partial_t)u = 0$$

where the function K is the inverse of Ω, the wavenumber now taken to depend on the frequency:

$$k = K(\omega) \Leftrightarrow \omega = \Omega(k).$$

Now, with $i\Omega = \partial_x R$ as in (21), assume in lowest order that $R \approx 1$, so that approximately $\partial_t u + \partial_x u = 0$ Then, in this approximation we can replace $\partial_x(u^2)$ in (21) by $-\partial_t(u^2)$, leading to

$$\partial_x u - iK(-i\partial_t)u - \partial_t(u^2) = 0, \tag{23}$$

where we removed the factor $\frac{3}{4}$ for convenience (by rescaling u).
Starting point is a given time signal at the initial position, centerd at a central frequency $\bar{\omega}$. The result will be an NLS-equation with coefficients that are determined by the specific dispersion relation. The derivation given here follows the description in [6] (apart from some change of notation and corresponding changes in signs).

2.5.1 Linear dispersive envelope equation

First we deal with dispersive effects upon neglecting the quadratic nonlinearity. Then we include effects of the nonlinearity. We will do this in some detail, since the precise relation with the actual wave height u contains some intricate phase effects which are important.
The general solution of the linearised equation can be written down as

$$u(x,t) = \int \alpha(\omega) e^{i[K(\omega)x - \omega t]} d\omega$$

where α is the spectral function of the field at $x = 0$. Taking an initial spectrum sharply centred at $\bar{\omega}$, the resulting wave group is a modulation of a harmonic mode, given by

$$u(x,t) = A(x,t)e^{i\Theta(\bar{\omega})} + cc,$$

where here and in the following we use the notation $\Theta(\omega)$ to denote the phase of a mode with frequency ω satisfying the dispersion relation

$$\Theta(\bar{\omega}) = K(\bar{\omega})x - \bar{\omega}t,$$

where K is the inverse of Ω : $K(\omega) = \Omega^{-1}(\omega)$. The complex amplitude that describes the modulation is given by

$$A(x,t) = \int_0^\infty \alpha(\bar{\omega} + \nu) e^{i[(K(\bar{\omega}+\nu) - K(\bar{\omega}))x - \nu t]} d\nu$$

and satisfies the linear dispersive equation

$$\partial_x A - i\left[K(\bar{\omega} + i\partial_t) - K(\bar{\omega})\right] A = 0.$$

To eliminate the first order term in the dispersion, a frame moving with the group velocity $1/K'(\bar{\omega})$ is introduced, i.e.

$$\tau = t - K'(\bar{\omega})x, \zeta = x.$$

Writing

$$K_2(\nu) = K(\bar{\omega} + \nu) - K(\bar{\omega}) - K'(\bar{\omega})\nu$$

the equation can then be rewritten like:

$$\partial_\zeta A - iK_2(i\partial_\tau)A = 0. \tag{24}$$

Exercise 24 *Linear envelope equation*

1. Show that K_2 is second order in ν :

   $$K_2(\nu) = \beta\nu^2 + O(\nu^3);$$

   determine β and show it is positive for surface wave dispersion. This coefficient is called the *group-velocity dispersion coefficient*; can you explain why?

2. Taking only the second-order approximation, the linear envelope equation (24) becomes

   $$\partial_\zeta A + i\beta\partial_t^2 A = 0. \tag{25}$$

   What is the relevance of the sign of β in this equation? This is a simple dispersive wave equation; what is te dispersion relation? Is there a conservation property (local conservation law, constant of the motion)?

3. Show that, in contrast to the so-called *diffusion equation* (when $\beta > 0$) for the real function u

   $$\partial_\zeta u = \beta\partial_t^2 u$$

   (a parabolic pde) is dissipative: Calculate for instance $\partial_\zeta \int (\partial_t u)^2 dt$ and show that it has damping solutions. So, very different from the equation (25) which is not dissipative but dispersive.

2.5.2 Nonlinear contributions

To incorporate nonlinearity in the deformation of the envelope amplitude, the generation of a second order double harmonic bound wave and a variation of the equilibrium level is anticipated:

$$u = A_0 e^{i\Theta(\bar{\omega})} + Be^{2i\Theta(\bar{\omega})} + C + cc. \tag{26}$$

The amplitudes A_0, B and C are allowed to vary slowly in the frame of reference, and B and C are of second order in the amplitude. It is natural to restrict to solutions u which are, either, square integrable in time (decaying at infinity) or are periodic with some period T. In the latter case $\int u dt = 0$, and the variations in the equilibrium level should satisfy at each position x

$$\int C dt = 0.$$

Inserting the Ansatz in the equation gives a residue that has to be made as small as possible. This defines the second order coefficients

$$B = 2\sigma_2 A_0^2, \text{ and } C = \sigma_0 \left[|A_0|^2 - I_0\right] \text{ with } I_0 = \int |A_0|^2 d\tau$$

where

$$\sigma_0 = \frac{1}{K'(\bar{\omega}) - K'(0)}, \sigma_2 = \frac{\bar{\omega}}{2K(\bar{\omega}) - K(2\bar{\omega})} \tag{27}$$

are (transfer) coefficients from the generation of the second harmonic and non-harmonic term in the solution. Then, to avoid resonant terms in third order, for A_0 there results the equation

$$\partial_\zeta A_0 + 4i\sigma_0 I_0 \bar{\omega} A_0 - iK_2(i\partial_\tau)A_0 + i\gamma|A_0|^2 A_0 = 0$$

with

$$\gamma = 4\bar{\omega}\left(\sigma_0 + \sigma_2\right). \tag{28}$$

One can simplify this result by getting rid of the term $4i\sigma_0 I_0 \bar{\omega} A_0$ in the equation by introducing an additional phase in A_0 :

$$A_0 = Ae^{-4i\bar{\omega}\sigma_0 I_0 \zeta}$$

leading to the generalised NLS-equation, denoted by gNLS:

$$\partial_\zeta A - iK_2(i\partial_\tau)A + i\gamma|A|^2 A = 0. \tag{29}$$

The phase shift indicates that the physical solution should actually be seen in a frame of reference moving at a slightly adjusted speed; this effect is part of the nonlinear dispersion relation.
From a solution of this equation, the actual physical solution is then up to and including second order given by

$$u = A\exp i\left[\Theta(\bar{\omega}) - 4\bar{\omega}\sigma_0 I_0 \zeta\right] + \sigma_0\left[|A|^2 - I_0\right] + 2\sigma_2 A^2 \exp i2\left[\Theta(\bar{\omega}) - 4\bar{\omega}\sigma_0 I_0 \zeta\right] + cc \tag{30}$$

This can be rewritten in a more attractive way by writing the complex amplitude A with real amplitude a and phase ϕ like $A = ae^{i\phi}$; then

$$u = 2a\cos(\Theta(\bar{\omega}) + \phi - 4\bar{\omega}\sigma_0 I_0 x) + 2\sigma_0[a^2 - I_0] + 4a^2\sigma_2\cos(2(\Theta(\bar{\omega}) + \phi - 4\bar{\omega}\sigma_0 I_0 x)). \tag{31}$$

The non-harmonic term with σ_0 modifies locally the equilibrium level of the field (keeping the averaged value at zero), while the double harmonic term with σ_2 deforms the first-order harmonic profile to a cnoidal-type of profile.
In the envelope equation above, we have retained the full dispersive properties of the problem, just as in the KdV-type of equation in the previous subsection. This may be less relevant for the envelope equation since then, from the start on, the attention is to waves with a sharply peaked spectrum, while (21) is valid for waves with a broad spectrum. However, retaining the

full dispersion makes it possible to study the influence of truncating the dispersive properties. Indeed, it is custom to expand the dispersion operator to second or third order, i.e.

$$K_2(\nu) \approx \beta_2\nu^2 + \beta_3\nu^3$$

where $\beta_2 = \frac{1}{2}K''(\bar{\omega}), \beta_3 = \frac{1}{6}K'''(\bar{\omega})$, and then A satisfies

$$\partial_\zeta A + i\beta_2\partial_\tau^2 A - \beta_3\partial_\tau^3 A + i\gamma|A|^2 A = 0. \tag{32}$$

Taking $\beta_3 = 0$ and performing a simple scaling transforms this equation to the standard form of the NLS-equation

$$\partial_\zeta A + i\partial_\tau^2 A + i\text{sign}(\beta_2\gamma)|A|^2 A = 0,$$

a well known equation that has been studied extensively (e.g. [1]). For this standard NLS, the quadratic function K_2 is even; with the third order dispersive term β_3 included, the equation is known as the Dysthe equation ([3]).
The sign of the coefficients, or better sign$(\beta_2\gamma)$, determines the character of the NLS equation:

- defocusing NLS if sign$(\beta_2\gamma) < 0$, and
- focusing NLS if sign$(\beta_2\gamma) > 0$.

The focussing NLS has soliton-type of solutions and more 'confined' solutions, as we shall see in the next section. In this case, the dispersive and the nonlinear effects counterbalance eachother, while in the defocussing NLS the waves will spread.
For the problem under consideration it is important to realise that the sign depends on the value of the central frequency. In fact, for the surface wave dispersion, and also for the classical KdV long wave dispersion, the sign of β_2 is positive for each $\bar{\omega}$. However, as can be easily verified, for the full dispersion relation, the value of γ changes sign, being negative for $\bar{\omega} < \omega_{crit}$ and positive for $\bar{\omega} > \omega_{crit}$. The specific value of ω_{crit} is related to the critical wave number known as the Davey-Stewartson value, $k_{crit} \approx 1.363$. In contrast, when the long-wave dispersion as in the classical KdV equation is used, the value of γ is negative for all $\bar{\omega}$, leading to the defocusing NLS. This means that essentially different behaviour is found for these cases. This has as physical interpretation that when the wave length of the carrier wave is sufficiently large, the waves constituting a wave group have too little interaction to remain confined in a group, while for sufficiently short waves the interaction length covers several waves which leads to grouping behaviour.

Summary 25 For the spatial NLS as considered above, in laboratory coordinates the NLS equation reads

$$NLS - spat: \ (\partial_x + \frac{1}{V_0}\partial_t)A + i\beta\partial_t^2 A + i\gamma|A|^2 A = 0$$

In the same way we can derive the temporal NLS:

$$\begin{aligned} NLS - temp \quad &: \quad (\partial_t + V_0\partial_x)A + i\beta'\partial_x^2 A + i\gamma'|A|^2 A = 0 \\ \beta' \quad &= \quad V_0^3\beta; \gamma' = V_0\gamma \end{aligned}$$

2.5.3 Phase-amplitude equations for NLS

Writing

$$u(x,t) = 2a(x,t) * \cos(\Phi(x,t)) + hot$$

all of the above can also be written in terms of the

- real **amplitude** $a(x,t) = \sqrt{|A|^2}$, (defining the envelope) ; in many cases one uses the 'energy' $E = |A|^2$;

and

- quantities that define the carrier wave, namely the

phase	$\Phi(x,t) = k_0 x - \omega_0 t + \phi(x,t)$
local wave number	$k := \partial_x \Phi$
local frequency	$\omega := -\partial_t \Phi$

Notice that here we define the wavenumber and frequency as local quantites, to be determined from the phase.
Then NLS can be found from the follwoing basic relations, the phase-amplitude relations:

amplitude (energy)	$\partial_t E + \partial_x (V(k)E) = 0$
phase equation	$\omega - \Omega(k) = \gamma\ E + \beta\ \frac{\partial_x^2\ a}{a}$
kinematic relation	$\partial_t k + \partial_x \omega = 0$

where Ω is the standard linear dispersion relation, and

energy	$E = a^{\ 2}$
groupvelocity	$V(k) = \Omega'(k)$
gr.vel. dispersion	$\beta = -\frac{1}{2}\Omega''(k) > 0$
transfer-coefficient	$\gamma > 0$

.

Remark 26 These equations have a variational structure. The Variational formulation is found from the Lagrangian functional

$$L = \int\int [(\omega - \Omega(k))a^2 + \beta a_x^2 - \frac{1}{2}\gamma a^4] dx dt.$$

Observe, in particular, the expression for the *Phase equation*, which has been written above like

$$\omega - \Omega(k) = \gamma\ E + \beta\ \frac{\partial_x^2\ a}{a}$$

This is to be interpreted as an extension of '*nonlinear dispersion relation*' (NDR) which reads for one mode with amplitude q

$$\omega - \Omega(k) = \gamma q^2$$

The present modification results from the degeneracy caused by the 'arbitrary' choice of k_0, ω_0,but, more important, NDR now includes an 'envelope-profile'-dependence $\beta \frac{\partial_x^2 a}{a}$ from dispersion ($\beta \neq 0$), independent from magnitude but locally defined and dependent on the whole envelope.

Remark 27 The NDR can be written in an alternative form like

$$\beta \partial_x^2 a - \nu a + \gamma a^3 = 0, \ \textit{with} \ \ \nu = \omega - \Omega(k).$$

This form is particularly interesting when looking for steady state solutions, for which $\nu = \omega - \Omega(k)$ constant. As we shall see later, this equation has as simple mechanical analogy the dynamic (now 'time' is x) evolution of a particle with mass β in a potential force field as described by Newton's equation. When ν is constant, the solutions are easy to find; when non-constant, the solutions may be very complicated, showing periodic, pseudo-periodic and even chaotic behaviour. This is at the heart of the complicated dynamics of surface waves!!

3 Advanced Surface Wave Phenomena

In this section we derive increasingly more complex mathematical methods to deal with more and more detailed and complicated wave phenomena.
We start with the well known soliton-solutions, which can be written down explicitly for the standard KdV-equation, the equation that was derived under the assumption of 'rather small, rather long waves' in the Boussinesq approximation. Then some explicit solutions will be derived for the NLS equation, i.e. the equation that describes the evolution of the amplitude of a wave group. Both for KdV and NLS there are opposing effects: the *dispersive effect* that tends to inrease the spreading of waves (different phase velocities for different wave lengths) and the *nonlinear effect* that tends to steepen the waves. The combined effect, when in a delicate balance as described by these equations, makes it possible that they counterbalance, with the result that waves (or wave groups) of 'permanent' form (only for a special form) become possible.
Although these examples may easily give the impression that explicit solutions are common, the opposite is true and solutions in closed form are usually impossible to obtain. That should be kept in mind in the follwoing. We show that, even for the KdV-model, it is quite difficult to obtain the simplest but one solution, i.e. the nonlinear analogue of the linear bi-chromatic solution that is the result of linear interference of two mono-chromatic waves. The nonlinear effects cause slow, but clearly noticeable, deviations: an (almost periodic) amplitude increase and wave profile distortions that are large on long time or space intervals. To study such kind of problems, series expansions are a common tool and are based on an expansion in the 'order' of some quantity that is assumed to be 'small' (a quantity measuring the amplitude is a standard choice). A naive perturbation expansion will, however, lead to resonant terms which will ruin the approximation: the deviation from the exact solution increases linearly with time or space. In a preparatory subsection, this is illustrated for simple ODE's, the Duffing equation. Also the method to remedy this artefact of the expansion method is illustrated and motivated: an expansion not only in the amplitude, but just as well in the frequency. This is the basis of the so-called Lindstedt-Poincaré method, and leads to uniformly valid (that is, for al time, or space) approximations. The same method will be used to obtain uniformly valid approximations for the nonlinear evolution of bi-chromatic waves; then this remedy leads to the so-called nonlinear dispersion relation.
The bi-chromatic wave results from interaction of two mono-chromatic waves, i.e. only two frequencies are involved initially. More realistic waves have a continuous spectrum, and the interaction takes place for each pair of frequencies (an infinite number). The total solution is therefore quite complicated, but can nevertheless be tackled. Computer-algebraic tools to deal with the Fourier-manipulations can be designed, leading to an Analytical Wave Code (AWC) that can predict correctly the evolution of an irregular wave (with many initial frequencies) over the large distances that are relevant for hydrodynamic laboratories.

3.1 Solitons of KdV

The motivation for Korteweg and de Vries to study the problem of surface waves, was to settle a dispute that continued throughout the nineteenth century about the existence of *travelling*

waves: is it possible that a wave exists that doesn't change in time, but is merely travells at a fixed speed?
They showed, by deriving their (KdV-) equation and analyzing it, that the answer is affirmative. More so, it is possible to write down the wave shapes and speeds explicitly. This is quite unexpected at first sight, since KdV combines nonlinearity (leading to "breaking"-phenomenon) and dispersion ("spreading" of initial profile). The remarkable property is that these combined effects make it possible that there exist *travelling waves*, waves with a specific profile, say f, that will neither break nor spread (an exact balance between the counteraction of breaking and spreding), and that travel undisturbed in shape at a specific speed, say V. That is, a solution of the form

$$\eta(x,\, t) = f(x - V\, t)$$

just as in the translation equation, but now only for specific profiles f and specific velocity V.

To find the wave profile f and the velocity V, we substitute this form in the KdV-equation, in normalised variables

$$\partial_t \eta + \partial_x \left[\partial_x^2 \eta + \frac{1}{2} \eta^2 \right] = 0.$$

Then the pde becomes an ode for the function f in which V enters as a parameter to be determined together with the profile. We shall see that, in fact, there is a whole family of such waves; the higher the amplitude, the larger the velocity.
Writing $\xi := x - V\, t$, the equation becomes

$$-V\, \partial_\xi f(\xi) + f(\xi)\, \partial_\xi\, f(\xi) + \partial_\xi^3 f(\xi) = 0$$

A solution of this equation, for certain V, produces the wave profile f of the wave that travels undisturbed in shape at speed V.

3.1.1 Analysis of solitary wave profiles

To find the solution we have to distinguish two cases:

- *space-(and time-) periodic solutions*, for which f is a periodic function of ξ, the socalled cnoidal waves (since the profile is expressed with the elliptic cnoidal function), and
- *solitary wave solutions*: wave profiles of a single hump that decay, together with all derivatives, sufficiently fast at infinity ("almost confined", exponentially small outside a certain interval).

We will concentrate on the solitary wave profiles.
Then by integrating the equation above once, noticing that the constant of integration has to vanish as a consequence of the decay at infinity, leads to the second order ode for the profile:

$$-V\, f(\xi) + \frac{1}{2}\, f(\xi)^2 + \partial_\xi^2\, f(\xi) = 0$$

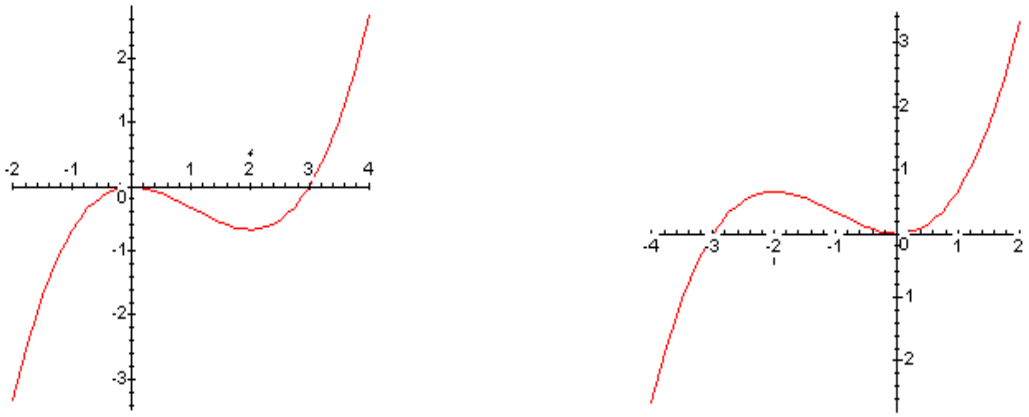
This equation can be solved in a standard way by observing the *mechanical analogue:*
when ξ is interpreted as the time, and f as the position, the equation describes the motion of a particle of unit mass subject to a potential force with potential energy U according to Newtons law:

$$\partial_\xi^2 f(\xi) + \frac{dU}{df} = 0 \tag{33}$$

with potential energy

$$U(f) = -\frac{1}{2} V f^2 + \frac{1}{6} f^3.$$

The plot of U is qualitatively as shown below, at the left for positive values of V, at the right for negative values:



Looking for a solitary wave profile f that decays to zero for ξ tending to $-\infty$, ∞, we look for the solution that is nontrivial and connects the origin with itself: a *homoclinic orbit*. Clearly, this can only be achieved for positive values of V.
In more detail, for the profile equation *mechanical-energy conservation* holds. Multiplying (33) with $\partial_\xi f$ and integrating the equation again, there results:

$$\frac{1}{2} [\partial_\xi f]^2 + U(f(\xi)) = E.$$

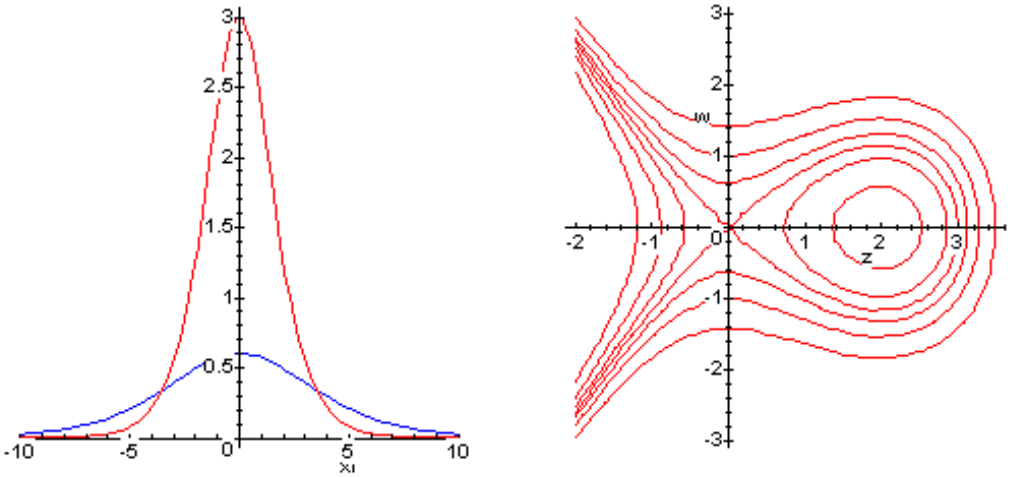
Since E should be zero for a solitary wave profile, the equation becomes

$$\frac{1}{2} [\partial_\xi f(\xi)]^2 - \frac{1}{2} V f(\xi)^2 + \frac{1}{6} f(\xi)^3 = 0.$$

This is a first order equation for the profile function and its solution can be given explicitly. This solution is a *solitary wave profile*: for each V, with $V > 0$, it is given by

$$f(\xi, V) = \frac{3V}{\cosh(\frac{1}{2}\sqrt{V}\,\xi)^2}$$

Two profiles, for $V = .2$ and $V = 1$, are shown below:



Observe: *with V the velocity, the amplitude is proportional to V, and the width proportional to $\frac{1}{\sqrt{V}}$: the larger the amplitude, the more confined the wave, and the larger its speed.*

The solution above can be found in the *phase plane* $z = f$, $w = \partial_\xi f(\xi)$ in the following way. The curves of constant energy, given by

$$\frac{1}{2}w^2 + U(z) = E$$

are sketched in the phase-plane (z, w), and in this phase portrait the solitary wave corresponds to the homoclinic orbit which is the level curve through the origin (for which $E = 0$).

3.2 Special solutions of NLS

Referring to the previous section, we use the following form of the NLS equation for the amplitude A depending on ζ, τ which may be time-space variables or conversely:

$$\text{NLS:}\quad \partial_\zeta A + i\beta\partial_\tau^2 A + i\gamma|A|^2 A = 0.$$

A simple scaling of ζ, τ shows that the parameters β, γ can be scaled away to arrive at an equation with no parameters; the signs in this equation only depend on the sign of the product $\beta\gamma$. We will see the relevance in a moment.
We restrict the investigation of special solutions to solutions of form

$$A(\tau, \zeta) = a(\tau)e^{-i\mu\zeta}.$$

Such solutions are a pure harmonic modulation $e^{-i\mu\zeta}$ of the fixed profile $a(\tau)$, and can therefore be interpreted as 'standing waves'. (Note: in laboratory-variables this corresponds to a travelling wave, since NLS is formulated in a moving frame of reference.)

In order to be a solution, the parameter μ and the profile a should satisfy the equation

$$\beta a_{\tau\tau} - \mu a + \gamma a^3 = 0.$$

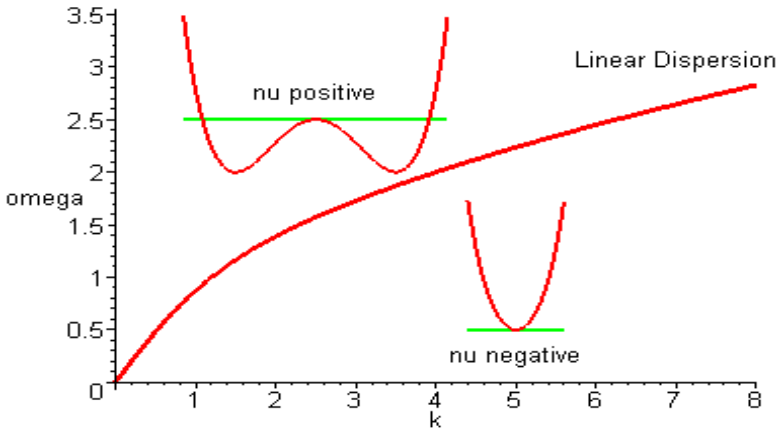
The study of this equation is, just as for the investigation of the KdV-soliton, simplified by looking at its *Mechanical analog:* the equation is Newtons Law for a mass particle of mass $\beta(> 0)$ in a field with potential P:

$$\beta \partial_\tau^2 a = -\frac{\partial}{\partial a} P(a), \quad \text{potential } P(a) = -\frac{1}{2}\mu a^2 + \frac{1}{4}\gamma a^4$$

The sign of μ and γ determine the behaviour of the potential. Having chosen $\beta > 0$, two essentially different cases are given by $\gamma > 0$ and $\gamma < 0$. We will restrict to the case $\gamma > 0$, which is known as the *'focussing' NLS*, since then 'soliton'-type of solutions exist. The defocussing case, $\gamma < 0$ does not have such solutions.
In the plot below we sketch the potential profiles for the focussing case $\gamma > 0$. For the water-wave problem, for sufficiently large k (larger than the Davey-Stewardson value, approx. 1.3), the coefficient γ is indeed positive but the value of μ in the NLS-equation depends on whether the local wave number and frequency are above or below the curve of the linear dispersion relation. This plot gets its meaning from the interpretation of the equation as the phase-equation, as described in section 2, where μ defines the difference from the linear dispersion plot:

$$\mu = \omega - \Omega(k)$$

.

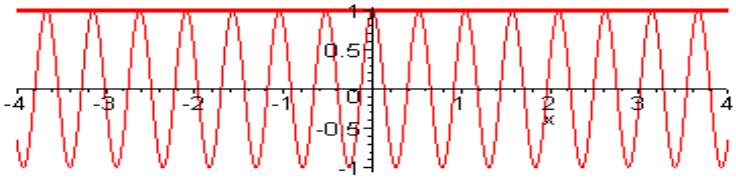


3.2.1 Nonlinear harmonic

The simplest solution is a constant amplitude $a(\tau) = q$, which is a solution if $\mu > 0$ and $q = \sqrt{\frac{\mu}{\gamma}}$. Then

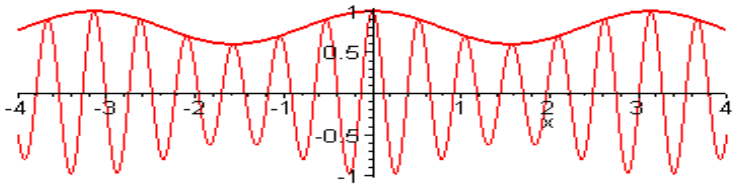
$$A = qe^{-i\gamma q^2 \zeta}.$$

The constant amplitude corresponds to the point of minimal potential. In a plot of the real part as function of ζ, the NLS-solution is sketched below:



3.2.2 Nonlinear modulated harmonic

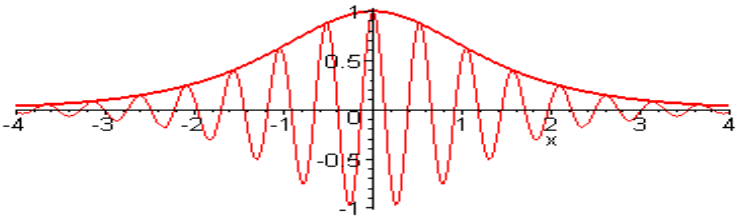
For $\mu > 0$, small amplitude periodic motions around minimal potential energy exist:



3.2.3 Soliton

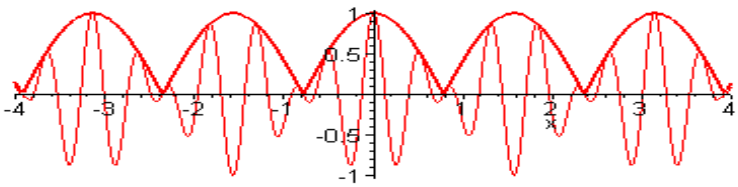
For $\mu > 0$ soliton solution as homoclinic orbit for $\mu = \gamma q^2/2$:

$$a = qsech(q\sqrt{\frac{\gamma}{2\beta}}\tau)$$



3.2.4 Nonlinear bi-harmonic

For $\mu < 0$ only periodic solutions exist; the real amplitude crosses the origin, leading to plots for the real part of the complex amplitude as below:



3.3 Intermezzo: Uniform expansion techniques

In this subsection we consider asymptotic methods for *nonlinear* equations. We restrict the analysis to ode's but all ideas and methods are equally useful for the perturbation analysis of pde's, as we shall see in the following subsections. In particular, phenomena as 'mode generation' and 'resonance' are essential in the infinite dimensional problems as well.
As classical examples to which the ideas can be best illustrated we consider the following two nonlinear extensions of the harmonic oscillator (in suitably defined normalized coordinates): the second order nonlinearity

$$\ddot{x} + x + x^2 = 0, \tag{34}$$

and the third order nonlinearity (Duffing's eqn)

$$\ddot{x} + x + x^3 = 0, \tag{35}$$

both with the initial condition

$$x(0) = \epsilon, \quad \dot{x}(0) = 0.$$

Neglecting the nonlinearity, the solution is simply the motion of the harmonic oscillator:

$$x_1(0) = \epsilon \cos t.$$

For small ϵ this may well be a good approximation of the nonlinear equations also. Note that for the linear oscillator, all solutions have the same period, 2π, independent of amplitude.

Remark 28 Writing $y = \epsilon x$ the initial values are normalised: $y(0) = 1, \dot{y} = 0$, but now the 'small' parameter ϵ appears in the equations:

$$\ddot{y} + y + \epsilon y^2 = 0, \textit{ and } \ddot{y} + y + \epsilon^2 y^3 = 0$$

This shows more clearly that the contribution of nonlinearity is 'small' (as long as y remains bounded).

Exercise 29 Before discussing perturbation methods, investigate the global properties of solutions by phase plane analysis. In most cases this is not possible but here we can exploit the standard analysis for Newtonian systems:

$$\ddot{x} = -\frac{\partial V}{\partial x}$$

where V is the potential energy. For (34), the potential V is given by:

$$V_2(x) = \frac{1}{2}x^2 + \frac{1}{3}x^3$$

while for (35) the potential is given by

$$V_3(x) = \frac{1}{2}x^2 + \frac{1}{4}x^4;$$

Sketch the graph of these potentials and investigate the phase portraits of solutions. Conclude from this which solutions will be bounded for ALL time.

We conclude that for (34), all solutions with amplitude ε smaller than a critical value are periodic, while for (35) even all solutions are periodic.
The smooth distortions of the integral curves near $\varepsilon = 0$ compared to the circles of the harmonic oscillator, gives additional confidence in the regularity of the problems.
One phenomenon that is not directly visible from the phase plane analysis is the fact that the periods of the periodic motion depend on the amplitude (on ϵ), in contrast to the linear equation for which every solution has period 2π. This can, however, be expected by observing that both equations can be written like

$$\ddot{x} + \omega(x)^2 \, x = 0$$

with

$$\omega(x)^2 = 1 + x, \text{ resp. } \omega(x)^2 = 1 + x^2$$

for (34) and (35) respectively. Here the period depends on the solution and hence is not known in advance. It can therefore be expected that perturbation-methods will also have to be able to vary the frequency with ϵ.

Exercise 30 Use MAPLE to actually calculate several solutions numerically; plot the solution as function of time; take different initial conditions (for instance: take $x(t = 0) = \varepsilon, \dot{x}(t = 0) = 0)$, and observe that (how) the period of the solution depends on the amplitude (i.e. on ε) of the solution.

Exercise 31 For small-amplitude solutions, the period will be close to that of the linear equation, which is 2π. What can you say about the period for larger amplitude solutions, will the period be larger or smaller? (Can you write down a formula for the period and analyse this?)

Naive perturbation method
With $\epsilon \cos t$ as a good approximation, we try to find a more accurate solution by substituting a series expansion for the solution:

$$x(\epsilon, t) = \epsilon \, \cos t + \epsilon^2 \, x_2(t) + \epsilon^3 \, x_3(t) + \ldots$$

Substituting this series in the equation leads for each $x_k, k \geq 1$ to an equation of the form

$$\ddot{x}_k + x_k = F_k$$

where F_k depends on the functions $x_j, j \leq k - 1$ and is therefore known in a successive treatment.
The solution of this linear eqn can be found explicitly; however, even if the forcing F_k is bounded, the solution does not have to be bounded; when the forcing is periodic with the same period as the solution of the homogeneous eqn, the phenomenon of resonance appears.
Resonance

The explicit solution, of

$$\ddot{u} + \omega_0^2 u = a \cos \omega t$$

is given for $\omega \neq \omega_0$ by

$$u(t) = \frac{a}{\omega_0^2 - \omega^2} \cos \omega t;$$

this shows that the solution is bounded, and is forced to be periodic with the same period of the driving force. Quite different is the situation for $\omega = \omega_0$:

$$u(t) = \frac{a}{2\omega_0} t \sin \omega_0 t \ ;$$

then the solution is unbounded, growing linearly in time. The unboundedness is a consequence of the forcing with the same period as the natural frequency ω_0 of the unforced system; this phenomenon is called *resonance*. Resonance can be useful to obtain solutions of large amplitude (when pushing a swing) but is often undesirable.
Now, given a force f, how can we investigate if resonance occurs in the equation

$$\ddot{u} + \omega_0^2 \, u = f(t),$$

or, stated differently, can we give conditions on f that guarantee that the solution is uniformly bounded for all time?
The answer will be clear if f is periodic with period T_0, where T_0 is the fundamental period $T_0 = 2\pi \, / \, \omega_0$. Then, f can be expanded as a Fourier series, and the requirement must be that the coefficients in front of the term with $e^{\imath \omega_0 t}$ vanish, i.e.

$$\int_0^{T_0} f(t) \cos \omega_0 t \, dt = 0, \quad \int_0^{T_0} f(t) \sin \omega_0 t \, dt = 0.$$

These conditions on f are known as *solvability conditions*.

Remark 32 A somewhat different way to derive these conditions is as follows. Since $\sin \omega_0 t, \cos \omega_0 t$ are solutions of the homogeneous eqn, multiply the equation with $\sin \omega_0 t$ and $\cos \omega_0 t$ and integrate over a fundamental period T_0. Then the left hand sides vanish, demanding the right hand sides to vanish. This leads to the conditions above as being necessary.

Remark 33 *Fredholm alternative* (analogy with Linear Algebra)
A deeper way to interpret the conditions is as follows. Write the equation in operator form by introducing the operator L such that $L \, u \equiv \ddot{u} + \omega_0^2 \, u$. Then the equation can be written like

$$L \, u = f$$

Furthermore, the operator is symmetric with respect to the integral inner-product on the set of T_0-periodic functions

$$\int_0^{T_0} v(t) \, L \, u(t) \, dt = \int_0^{T_0} L \, v(t) \, u(t) \, dt$$

The kernel of L is given above:

$$\ker L = \{ \sin \omega_0 t, \cos \omega_0 t \}$$

Then the solvability conditions are nothing but the condition that the force f is perpendicular to the kernel of L. Compare this with the Fredholm alternative in Linear Algebra: for (symmetric) matrices A the solvability condition for $Ax = b$ is that $b \perp \ker(A^*)$.

Remark 34 In case f is not periodic, but square integrable, Fourier-integral methods can be applied. Let's use complex notation for simplicity:

$$\dot{z} + i\omega_0 z = f(t)$$

Writing

$$f(t) = \int F(\omega) e^{-i\omega t} d\omega; \quad F(\omega) = \frac{1}{2\pi} \int f(t) e^{i\omega t} d\omega$$

the solution is given by

$$z(t) = \int \frac{F(\omega)}{-i\omega + i\omega_0} e^{-i\omega t} d\omega$$

The last integral seems to have a singularity at $\omega = \omega_0$. To investigate the behaviour near ω_0, write $\omega = \omega_0 + \sigma$. Then observe

$$z(t) = e^{-i\omega_0 t} \int \frac{F(\omega_0 + \sigma)}{-i\sigma} e^{-i\sigma t} d\sigma$$

and, with

$$\frac{d}{dt} \int \frac{F(\omega_0 + \sigma)}{-i\sigma} e^{-i\sigma t} d\sigma = \int F(\omega_0 + \sigma) e^{-i\sigma t} d\sigma = e^{i\omega_0 t} \int F(\omega) e^{-i\omega t} d\omega = e^{i\omega_0 t} f(t),$$

we find the explicit solution

$$z(t) = e^{-i\omega_0 t} \int^t e^{i\omega_0 s} f(s) ds,$$

known as the *'variation of constant formula'*. This shows that now NO resonance will appear: the solution remains bounded even when $t \to \infty$, and $F(\omega_0) \neq 0.$, i.e. even when the fundamental frequency ω_0 is present in the driving force. Of course, when the driving is periodic, resonance appears as we have seen above,

$$\dot{z} + i\omega_0 z = A e^{-i\omega_0 t} \Longrightarrow z(t) = A t e^{-i\omega_0 t}$$

which also follows from the variation of constant formula:

$$z(t) = e^{-i\omega_0 t} \int^t e^{i\omega_0 s} A e^{-i\omega_0 s} ds = A t e^{-i\omega_0 t},$$

The explanation of this seemingly discrepancy is the difference in energy-content in the resonant forcing: the energy is finite for square integrable functions, and hence infinitesimally small per unit time, while the energy is infinite for the periodic forcing (finite per unit time). This is also seen from looking at the increase in 'energy' over a time interval T :

$$|z(T)|^2 - |z(0)|^2 = \int_0^T \partial_t |z(t)|^2 dt = 2\,\mathrm{Re} \int_0^T z(t)\overline{f(t)} dt.$$

Remark 35 In practical situations the above may lead to some confusion when energy content is not taken into account properly. We will have to deal with this in the anlysis for the NDR for nonlinear wave equations furtheron, and therefore make the follwoing remarks.
In a practical situation, we are dealing with (results of) signals that were measured, 'given' only for a finite time, say

$$s(t),\ t \in [0,T].$$

Then, in principle there are two ways to continue the mathematical reasoning:

1. Extend the signal by zero outside the given time-interval. This then leads to an L_2-function

$$s_0(t) = \begin{cases} s(t),\ t \in [0,T]. \\ 0,\ t \notin [0,T]. \end{cases}$$

for which

$$\begin{aligned} s_0(t) &= \int \hat{s}_0(\omega) e^{-i\omega t} d\omega, \\ \hat{s}_0(\omega) &= \frac{1}{2\pi} \int s_0(t) e^{i\omega t} dt = \frac{1}{2\pi} \int_0^T s_0(t) e^{i\omega t} dt \end{aligned}$$

Approximating the integrals by Riemann-sums, there results

$$s_0(t) = \sum_n \hat{s}_0(n.\Delta\omega) e^{-i\Delta\omega.t.n} \Delta\omega$$

2. Make a $T-$periodic continuation of the signal, and use Fourier series:

$$s(t) = \sum c_n e^{-i\frac{2\pi}{T}t.n}, \quad c_n = \frac{1}{T} \int_0^T s(t) e^{i\frac{2\pi}{T}t.n} dt$$

The relation between the methods will be clear now: the choice $\Delta\omega = \frac{2\pi}{T}$ leads to complete agreement, provided

$$c_n = \hat{s}_0(n.\Delta\omega) \cdot \Delta\omega \quad \textit{for} \quad \Delta\omega = \frac{2\pi}{T}$$

which shows the appearance of the length T of the time interval in the relation between the discrete and continuous spectra.

3.3.1 Third-order nonlinearity: Duffing's eqn

We first consider the oscillator with third order nonlinearity

$$Eqn \equiv \ddot{x} + x + x^3 = 0, \quad x(0) = \epsilon, \quad \dot{x}(0) = 0$$

Naive perturbation technique
Substituting the series expansion

$$x(\epsilon, t) = \epsilon\, x_1(t) + \epsilon^2\, x_2(t) + \epsilon^3\, x_3(t) + \dots$$

in the equation, there results

$$\begin{aligned} \ddot{x}_1 + x_1 = 0, \quad & x_1(0) = 1, \qquad \dot{x}_1(0) = 0 \\ \ddot{x}_2 + x_2 = 0, \quad & x_2(0) = 0, \qquad \dot{x}_2(0) = 0 \\ \ddot{x}_3 + x_3 = -x_1^3, \quad & x_3(0) = 0, \qquad \dot{x}_3(0) = 0 \end{aligned}$$

Clearly the solution for x_1 is given by $x_1 = \cos t$. The solution for x_2 vanishes identically: $x_2 \equiv 0$. The forcing of the eqn for x_3 can then be expanded (after some straightforward calculations) to

$$x_1^3 = (\cos\, t)^3 = \frac{1}{4}\, \cos\, 3t + \frac{3}{4}\, \cos\, t$$

The solution can be written down, but the appearance of the source term $\frac{3}{4}\, \cos\, t$ indicates that resonance will appear:

$$x_3(t) = -\frac{3}{8}\, t\, \sin\, t + \frac{1}{32}\, \cos\, 3t$$

Hence, up to third order, the solution reads

$$x(\epsilon, t) = \epsilon\, \cos\, t + \epsilon^3\, [\, -\frac{3}{8}\, t\, \sin\, t + \frac{1}{32}\, \cos\, 3t\,] + \dots\, .$$

Interpretation.
For small values of $t > 0$, in fact for t in any interval of prescribed length, $t \in [0, T]$, this is a good approximation for ϵ sufficiently small, since the third order term, although increasing linearly in time, is bounded by *constant* $\epsilon^3\, T$, and a same result holds for the other terms.
However, in practice we want to go further and have a solution that is also correct for larger values of time. Since the solution is known to be bounded for all time (from phase plane analysis), we would even like to have an approximation that holds for *all* time, also $t \mapsto \infty$; an approximation with this property is called a *uniformly valid* approximation. Clearly the approximation constructed above does not satisfy this criterion, since $\epsilon^3\, t$ will explode for $t \mapsto \infty$ at fixed ϵ.

Remark 36 From symmetry of the equation (if $x(t)$ is a solution with $x(0) = \epsilon$, then $y(t) \equiv -x(t)$ is the solution with $y(0) = -\epsilon$) it follows that the solution $x(\epsilon, t)$ should be odd in ϵ, and hence all terms $\epsilon^k\, x_k(t)$ with k even should vanish.

3.3.2 Improved perturbation techniques

The naive method above (series expansion in ϵ) does not give a possibility to prevent the linear growth in time, since this is caused by the resonance phenomenon in the equation for x_3. All improved perturbation techniques are based on creating more flexibility in the Ansatz for the solution so that this resonance can be prevented.
One improved technique is as follows; it is often referred to as Lindstedt-Poincaré technique, and is closely related to other methods like 'uniformization', 'renormalization', 'averaging technique'(van der Pol, or Krylov-Bogoliubov method), and even 'multiple time scale' method (although in this problem no two time scales can be distinguished).
We will demonstrate one technique, which will be called the *amplitude-phase method*, for reasons to become clear soon.
Neglecting the nonlinear term, the solution is a simple harmonic function:

$$x(t) = \epsilon\, A\, \cos(\theta)$$

where the amplitude A is constant, and the phase θ is simply $\theta = t + \alpha$, where α is a phase constant. In order to choose the constants such that the initial value is satisfied, A and α have to satisfy

$$A\, \cos(\alpha) = 1.$$

Of course, $A = 1, \alpha = 0$ is one solution, but not unique.
A straightforward generalization of this form of solution to the nonlinear system would be to let A and θ depend on time (and ϵ) in a way to be decided, such that the approximation satisfies the equation as good as possible (variation of 'parameters'). Substituting the 'Ansatz'

$$x(\epsilon, t) = \epsilon\, A(\epsilon, t)\, \cos(\theta(\epsilon, t))$$

into the equation, will lead to equations for A and θ which may then be solved approximately. Observe that in this Ansatz, the 'amplitude' and 'phase' are not uniquely defined, just as above.
Let us now investigate the details.
Substitution in the eqn leads to

$$Eqn = \epsilon\, R_c\, \cos(\theta) + \epsilon\, R_s \sin(\theta) - \epsilon^3\, \frac{1}{4}\, A^3\, \cos(3\,\theta)$$

with

$$R_c = \ddot{A} - A\,\dot{\theta}^2 + A + \epsilon^2\, \frac{3}{4}\, A^3$$

and

$$R_s = -2\dot{A}\dot{\theta} - A\ddot{\theta}$$

In order to reduce the residue as much as possible, one now chooses A, θ in such a way that R_c and R_s vanish; this leads to the socalled *phase-amplitude equations*

$$\ddot{A} - A\,\dot{\theta}^2 + A + \epsilon^2\, \frac{3}{4}\, A^3 = 0$$

and

$$-2\dot{A}\dot{\theta} - A\ddot{\theta} = 0.$$

A solution of these eqns can now be found easily: A independent of time and θ such that

$$\dot{\theta}^2 = 1 + \epsilon^2 \frac{3}{4} A^2$$

which leads to

$$\dot{\theta} = 1 + \epsilon^2 \frac{3}{8} A^2 + \mathcal{O}(\epsilon^4),$$

i.e. a linear dependence on time for θ. Taking A=1, the approximate solution becomes

$$x(\epsilon, t) \approx \epsilon \cos(\omega(\epsilon)\, t), \text{ with } \omega(\epsilon) = 1 + \frac{3}{8}\epsilon^2.$$

The term that remains in the residue, $\frac{1}{4}\epsilon^3 \cos(3\theta)$, is of higher order and does not lead to resonance; hence the solution constructed above reads

$$x(\epsilon, t) = \epsilon \cos\left(\left(1 + \frac{3}{8}\epsilon^2 + \mathcal{O}(\epsilon^4)\right) t\right) + \mathcal{O}(\epsilon^3).$$

This approximate solution is valid on a much larger interval of time then the approximation with the linearly growing resonant term: $x(\epsilon, t) = \epsilon \cos(t) + \mathcal{O}(\epsilon^3, t\varepsilon^3)$.The last approximation has order term smaller than $O(\varepsilon)$ provided $t\varepsilon^3 << \varepsilon$, i.e. $t << \varepsilon^{-2}$. The improved approximation has error $O(\varepsilon^3, \varepsilon^5 t)$ which is smaller than $O(\varepsilon)$ for $t < \varepsilon^{-4} = (\varepsilon^{-2})^2$, a considerably longer time interval. In a case like this one often says that the improved approximation is *uniformly valid*, although, strictly speaking, 'uniformity' should not be interpreted as 'valid for all time'; it is just valid on a much larger time-interval. In higher order terms, other resonances will turn up. These can then be avoided by adjusting once again the frequency in the lowest order solution, i.e. by carefully choosing ω_4 in $x(\epsilon, t) = \epsilon \cos\left(\left(1 + \frac{3}{8}\epsilon^2 + \omega_4 \varepsilon^4 + \mathcal{O}(\epsilon^6)\right) t\right)$; doing that leads to a solution that is accurate in order ε on a time-interval $t < (\varepsilon^{-2})^3$. (To determine ω_4 one first needs to find the third order contribution explicitly.)

Remark 37 Compare the solution obtained above with the solution from series expansion, and observe that the series expansion solution is precisely the expansion in ϵ of the phase-amplitude solution! This explains the nonuniformity of the series expansion solution.

3.3.3 Uniform Approximation Procedure

The above example may have shwn the line of reasoning, which can be summarised as follows. Expand both amplitude and argument of an oscillating function (frequency) in a power-series of ε :

- take as 'Ansatz':

$$x(\varepsilon, t) = [\varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 + ...] \cos\left([\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + ...]t\right) + \varepsilon^2\xi_2 + \varepsilon^3\xi_3 + ..$$

- substitute in the equation,
- expand in powers of ε
- equate to zero the coefficient for each power of ε
- determine successively the constants A_k and ω_k such that the functions ξ_2 (ξ_3...) are bounded (from non-resonance conditions for ξ and ξ_2...).

For the Duffing equation, substitute (simplifying somewhat):

$$x(\varepsilon, t) = [\varepsilon A_1 + ...] \cos\left([\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + ...]t\right) + \varepsilon^3\xi$$

and find

$$\begin{aligned} O(\varepsilon) &: \quad A_1[-\omega_0 + 1] = 0 \Rightarrow \omega_0 = 1 \\ O(\varepsilon^2) &: \quad 2\omega_0\omega_1 = 0 \Rightarrow \omega_1 = 0 \\ O(\varepsilon^3) &: \quad -2A_1\omega_0\omega_2 \cos(\omega_0 t) + A_1^3 \cos^3(\omega_0 t) + \ddot{\xi} + \xi = 0 \end{aligned}$$

From this last equation follows the condition for boundedness of ξ, i.e. to prevent resonance in ξ :

$$-2A_1\omega_0\omega_2 \int \cos^2(\omega_0 t)dt + A_1^3 \int \cos^4(\omega_0 t)dt = 0,$$

hence the same result as above:

$$\omega_2 = \frac{1}{2}A_1^2 \frac{\int \cos^4(\omega_0 t)dt}{\int \cos^2(\omega_0 t)dt} = \frac{3}{8}A_1^2$$

Remark 38 There is another, somewhat deeper, but actually more natural way, to derive the above results. For Duffing's equation this starts with the observation that the equation $\ddot{x} + x + x^3 = 0$ transforms after a time transformation $\tau = \lambda t \in [0, 2\pi]$ to

$$\lambda^2 u'' + u + u^3 = 0$$

Interpreting λ^2 as the Lagrange multiplier of a constrained variational problem:

$$\lambda^2 = \frac{d}{d\gamma}\left[Min\left\{\int u^2 + \frac{1}{2}u^4 \middle| \int u'^2 = \gamma\right\}\right],$$

the correct result is obtained by approximating the minimising value by the value of the trial function that is the solution of the linear equation, i.e. $U = a\cos(\tau)$, with $a = \sqrt{\gamma/\pi}$:

$$\begin{aligned} \lambda^2 &= \frac{d}{d\gamma}\left[\gamma + \frac{1}{2}\gamma^2 \int \cos(\tau)^4 d\tau/\pi^2\right] = 1 + \gamma\frac{3}{4\pi}, \\ \textit{hence} \ \lambda &\approx 1 + \gamma\frac{3}{8\pi} = 1 + \frac{3}{8}a^2 \end{aligned}$$

leading to the correct asymptotically valid first order solution

$$x(t) = a\cos\left(\left[1 + \frac{3}{8}a^2\right]t\right) + O(a^3).$$

3.3.4 Second-order nonlinearity

Now consider the oscillator with second order nonlinearity

$$\ddot{x} + x + x^2 = 0, \quad x(0) = \epsilon, \quad \dot{x}(0) = 0$$

A *Naive perturbation technique* will show as follows that resonance is absent in second order terms, but will now appear in the third order term. Substituting the series expansion

$$x(\epsilon, t) = \epsilon\, x_1(t) + \epsilon^2\, x_2(t) + \epsilon^3\, x_3(t) + \ldots$$

in the equation, there results

$$\begin{aligned} \ddot{x}_1 + x_1 = 0, &\quad x_1(0) = 1, &\quad \dot{x}_1(0) = 0 \\ \ddot{x}_2 + x_2 = -x_1^2, &\quad x_2(0) = 0, &\quad \dot{x}_2(0) = 0 \\ \ddot{x}_3 + x_3 = -2x_1\, x_2, &\quad x_3(0) = 0, &\quad \dot{x}_3(0) = 0 \end{aligned}$$

With $x_1 = \cos t$, the equation for x_2 becomes

$$\ddot{x}_2 + x_2 = -(\cos t)^2 \equiv -\frac{1}{2}\,(1 + \cos\, 2t)$$

with solution (satisfying the homogeneous initial data)

$$x_2(t) = -\frac{1}{2} + \frac{1}{6}\,\cos\, 2t + \frac{1}{3}\,\cos\, t$$

So, in second order the solution is uniformly bounded. Continuing with the third order calculation, the forcing of the eqn for x_3 can be expanded (after some straightforward calculations) to

$$-2x_1\, x_2 = -\frac{1}{3} + \frac{5}{6}\,\cos\, t - \frac{1}{3}\,\cos\, 2t - \frac{1}{6}\,\cos\, 3t$$

The solution can be written down, but the appearance of the source term $\frac{5}{6}\,\cos\, t$ indicates that resonance will appear:

$$x_3(t) = \frac{5}{12}\, t\, \sin\, t + (\ \textit{periodic terms})$$

Hence, up to third order, the solution reads

$$\begin{aligned} x(\epsilon, t) \quad = \quad & \epsilon\,\cos\, t + \epsilon^2\,[\,-\frac{1}{2} + \frac{1}{6}\,\cos\, 2t + \frac{1}{3}\,\cos\, t\,] \\ & + \epsilon^3\,[\,\frac{5}{12}\, t\, \sin\, t + (\ \textit{periodic terms})\,]. \end{aligned}$$

Just as above for the Duffing eqn, series expansion leads to resonance and then to a nonuniformly valid approximation.

Exercise 39 Construct a uniformly valid approximation (up to and including third order) by replacing the 'first order' solution $\epsilon\,\cos\, t$ by

$$\varepsilon\,\cos([1 + \varepsilon^2\omega_2]t)$$

for an appropriate value of ω_2.

3.3.5 Resonance in dispersive wave equations

The same phenomena (resonance leading to non-uniform approximations) and methods (direct perturbation method, and improved method) will appear in partial differential equations. For nonlinear equations (KdV; the same is possible for NLS) this will be dealt with in the next subsection. Here we prepare by investigating resonance effects in linear dispersive wave equations.
Consider the forced equation

$$\partial_t u + i\Omega(-i\partial_x)u = f(x,t).$$

Formally, a double Fourier transform (for spatial, x, and temporal, t, variable) leads to

$$f(x,t) = \int\int F(k,\omega)e^{i(kx-\omega t)}dkd\omega$$

and for the equation:

$$\left[-i\omega + i\Omega(k)\right]U(k,\omega) = F(k,\omega)$$

and hence

$$\begin{aligned} u(x,t) &= \int\int U(k,\omega)e^{i(kx-\omega t)}dkd\omega \\ &= \int\int \frac{F(k,\omega)}{-i(\omega-\Omega(k))}e^{i(kx-\omega t)}dkd\omega \end{aligned}$$

From this formal expression the 'resonance' is seen to take place at the dispersion curve $\omega = \Omega(k)$ in the ω, k-plane when the spectrum of the forcing doesn't vanish there.
As a specific example, suppose that the forcing is a solution of the homogeneous equation, say

$$f(x,t) = Ae^{i(k_0x-\Omega_0t)} \text{ with } \Omega_0 = \Omega(k_0);$$

then the resonant solution is given by

$$u_{res} = Ate^{i(k_0x-\Omega_0t)}$$

and increases without bound in time. For a continuous spectrum, or a 'finite time signal', the same remarks apply as made in the ODE-case.

3.4 Distortion of bi-chromatic waves

This subsection is for the most part contained in a paper with Edi Cahyono and Agus Suryanto [6]. We use the notation of that paper, but restrict at the end to the dispersion for surface waves.
The standard equation to be used is the spatial KdV-equation,

$$\begin{aligned} \partial_z u - iK(i\partial_t)u - \partial_t\left[u^2\right] &= 0 \qquad (36)\\ \text{or}\quad \partial_z u + \partial_t\left[Ru - u^2\right] &= 0. \qquad (37)\end{aligned}$$

The spatial parameter is denoted by z instead of x.

3.4.1 Nonlinear harmonics

The effect of the nonlinearity, and the way how the 'nonlinear dispersion relation' is introduced to prevent resonance, can already be seen from an investigation of the simplest example, an initially harmonic mode, of frequency $\bar{\omega}$ say. In linear theory the solution is

$$u_{lin} = ae^{i\Theta(\bar{\omega})} + cc = 2a\cos(\Theta(\bar{\omega}))$$

where here and in the following we use the notation $\Theta(\omega)$ to denote the phase of a mode with frequency ω satisfying the dispersion relation:

$$\Theta(\bar{\omega}) = K(\bar{\omega})z - \bar{\omega}t.$$

We start with a straightforward series expansion in the amplitude:

$$u = u^{(1)} + u^{(2)} + u^{(3)}$$

where the function $u^{(k)}$ is of the order k in the amplitude.
The first order solution is the linear harmonic

$$u^{(1)} = ae^{i\Theta(\bar{\omega})} + cc = 2a\cos(\Theta(\bar{\omega}))$$

Then for the second order term there results the equation and corresponding solution

$$\begin{aligned}(\partial_z - iK)u^{(2)} &= \partial_t\left(u^{(1)2}\right)\\ u^{(2)} &= -2a^2\sigma_2 e^{2i\Theta(\bar{\omega})} + cc.\end{aligned}$$

where $\sigma_2 = \frac{\bar{\omega}}{2K(\bar{\omega}) - K(2\bar{\omega})}$ is the (transfer) coefficient from the generation of the second harmonic term in the solution. The third order equation

$$(\partial_z - iK)u^{(3)} = 2\partial_t\left(u^{(1)}u^{(2)}\right)$$

contains a resonant term which can be dealt with by correcting the dispersion relation. The correction to a non-linear dispersion relation reads in this case

$$k = K(\omega) + 4\bar{\omega}\sigma_2 a^2.$$

This leads to a uniformly valid solution given by

$$u = 2a\cos[K_{NDR}z - \bar{\omega}t] + hot,$$
$$\text{with } K_{NDR} = K(\bar{\omega}) + 4\bar{\omega}\sigma_2 a^2.$$

where *hot* denote quadratic and higher order terms. The effect of the phase correction is that this nonlinear harmonic has slightly different wave number (and as a consequence, slightly different phase speed) that depends on the amplitude. This correction for the dispersion relation to a nonlinear dispersion relation (NDR) is important to obtain the correct speed, and therfore the correct position of the wave (in particular relevant when the solution is determined over longer distances).

3.4.2 Uniformly valid approximate bi-harmonic evolution

We now consider the evolution of an initially bi-chromatic wave group; the frequency difference can be arbitrary, but most interesting will be the case of small frequency difference. The primary motivation is to study properties of the distortions during the evolution, which show several peculiarities and, implicitly, shows that special care is required in finding accurate analytical approximations.
The signal at $z = 0$ is the interference of two mono-chromatic waves of the same amplitude q and slightly different frequencies $\omega_{\pm 1} = \bar{\omega} \pm \nu$, with ν small,

$$\begin{aligned} u(0,t) &= qe^{-i\omega_1 t} + qe^{-i\omega_{-1} t} + cc \\ &= 4q\cos(\nu t)\cos(\bar{\omega}t)\,. \end{aligned}$$

Linear dispersive theory would lead to the solution

$$u(z,t) = qe^{i\Theta_+} + qe^{i\Theta_-} + cc = 4q\cos(\kappa z - \nu t)\cos\left(\bar{k}z - \bar{\omega}t\right) \tag{38}$$

where

$$\begin{aligned} \Theta_\pm &= \Theta(\omega \pm \nu); \kappa = (K(\bar{\omega}+\nu) - K(\bar{\omega}-\nu))/2; \\ \bar{k} &= (K(\bar{\omega}+\nu) + K(\bar{\omega}-\nu))/2 \end{aligned}$$

This solution describes the characteristic modulation of the carrier wave $\cos\left(\bar{k}z - \bar{\omega}t\right)$ by the envelope $4q\cos(\kappa z - \nu t)$ which travels with the group velocity ν/κ. It can be observed that the modulation in the moving frame of reference is given by

$$\begin{aligned} 4q\cos(\kappa z - \nu t) &= 4q\cos(\delta\zeta - \nu\tau), \\ \text{where } \delta &= \kappa - K'(\bar{\omega})\nu \approx \frac{1}{6}K'''(\bar{\omega})\nu^3 \end{aligned}$$

The nonlinear effects will be determined in first instance by the value of the amplitude q, but, for this specific example, just as well by the frequency difference. In fact, for the standard NLS-equation it was shown in [10] that the distortion of the bi-chromatic wave is determined by the characteristic parameter which is the quotient $\frac{q}{\nu}$: the larger this quotient the larger the

distortion[4]. Different from a direct series expansion in powers of q, this will lead effectively to terms of order q, q^2 (the 'second order' contribution) and the 'third order' contribution of the form $q(\frac{q}{\nu})^2$; clearly this third order term can dominate the second order term when q/ν is not small.
We will now derive the approximate solution in detail.
The quadratic nonlinearity generates second order terms with amplitudes q^2 and phases $\pm[\Theta_+ \pm \Theta_-]$ and third order terms with amplitudes q^3 and phases $\pm[[\Theta_+ \pm \Theta_-] \pm [\Theta_+ \pm \Theta_-]]$. Each of these terms is multiplied by a specific coefficient, but the value of these coefficients differs substantially depending on ν. As before, the resonant term in third order is taken care of by modifying the dispersion relation with suitable nonlinear terms. The other terms in third order that are most dominant are the phases corresponding to the nearest side band frequencies, which are $\omega_{\pm 3} = \bar{\omega} \pm 3\nu$. The coefficient in front of these terms are inversely proportional to ν^2, leading to the two-parameter expansion described above. In a direct series expansion method, the first order solution is the linear solution (38). The second order solution is found to be

$$u^{(2)} = -q^2 \left[2s_+ e^{2i\Theta_+} + 2s_- e^{2i\Theta_-} + 4s e^{i(\Theta_+ + \Theta_-)} + 2s_0 e^{i(\Theta_+ - \Theta_-)} \right] + cc$$

where

$$s = \frac{\bar{\omega}}{K(\omega_{+1}) + K(\omega_{-1}) - K(2\bar{\omega})}, s_{(\pm)} = \frac{\omega_{(\pm 1)}}{2K(\omega_{(\pm 1)}) - K(2\omega_{(\pm 1)})}, s_0 = \frac{2\nu}{K(\omega_1) - K(\omega_{-1}) - K(2\nu)}.$$

Remark 40 Observe that in the limit $\nu \to 0$ these (transfer) coefficients become $s, s_{(\pm)} \to \sigma_2$ and $s_0 \to \sigma_0$ where

$$\sigma_0 = \frac{1}{K'(\bar{\omega}) - K'(0)}, \sigma_2 = \frac{\bar{\omega}}{2K(\bar{\omega}) - K(2\bar{\omega})} \tag{39}$$

and the second order solution becomes

$$u^{(2)} \approx -q^2 \left[8\sigma_2 e^{2i\Theta} + 2\sigma_0 e^{i(\Theta_+ - \Theta_-)} \right] + cc.$$

Compared to the result for a nonlinear harmonic, with $a = 2q$, the same second order contribution is found, but now there is a nonvanishing long wave component $2\sigma_0 e^{i(\Theta_+ - \Theta_-)}$ which has to be taken into account in calculating the third order; see Fig. 1.

The equation in third order contains resonant terms which leads to definition of the nonlinear dispersion relation: replace $K(\omega)$ by K^{KdV}

$$K^{KdV} = K(\omega) + K_{corr}(\omega)$$
$$\text{where } K_{corr}(\bar{\omega} \pm \nu) = 4q^2(\bar{\omega} \pm \nu)(s_0 + s_\pm + 2s)$$

Note that in the limit for $\nu \to 0$ this becomes

$$K_{corr} \sim 4q^2\bar{\omega}(\sigma_0 + 3\sigma_2). \tag{40}$$

[4] In [5] it was shown by using exact formula for the NLS-bi-soliton solution that the opposite is the case for the interaction of solitons, the non-periodic variant of the bi-chromatic interference.

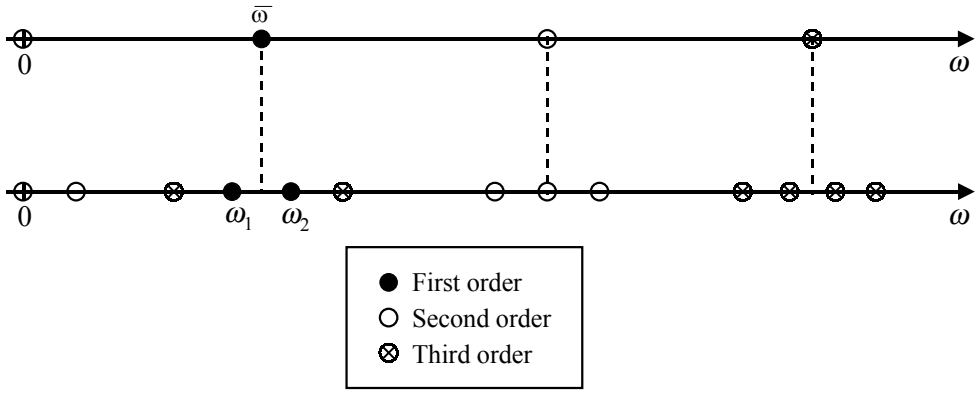


Figure 1: Schematic diagram of mode generation from nonlinearity for the mono-chromatic (above) and the bi-chromatic wave (below).

The corrected 'first' order linear solution then becomes

$$u^{(1)} = qe^{i\Theta_+}e^{iK_{corr}(\omega+\nu)z} + qe^{i\Theta_-}e^{iK_{corr}(\omega-\nu)z} + cc$$

which in a good approximation reads

$$u^{(1)} = 4q\cos(\bar{k}z - \bar{\omega}t + K_{corr}z)\cos(\kappa z - \nu t).$$

As stated above, the bounded part of the third order solution, although being small of order q^3, and therefore easily neglected in most cases, contains some surprise, as we show now. The third order contribution to the solution contains terms from side band frequencies $\omega \pm 3\nu$ and low and high frequency components, see Figure 1. We will restrict to the most important terms which are the side bands. These are given by

$$u^{(3)}_{sb} = q^3\left[a_3 e^{i(2\Theta_+ - \Theta_-)} + a_{-3}e^{i(2\Theta_- - \Theta_+)}\right] + cc$$

where

$$a_{\pm 3} = 4(s_0 + s_\pm)\frac{\omega \pm 3\nu}{2K(\omega \pm \nu) - K(\omega \mp \nu) - K(\omega \pm 3\nu)}$$

Observe, that in the limit $\nu \to 0$

$$a_{\pm 3} \sim \frac{-4\bar{\omega}\,(\sigma_0 + \sigma_2)}{4\,[K''\nu^2 \pm K'''\nu^3]},$$

which is why we introduce $\alpha_{\pm 3} = \nu^2 a_{\pm 3}$ and $\beta_{\pm 3} = \alpha_3 \pm \alpha_{-3}$, hence $\beta_3 \sim \frac{-\gamma}{2K''}, \beta_{-3} \sim \frac{1}{2}\frac{\gamma K'''}{K''^2}\nu$, and get

$$\begin{aligned} u^{(3)}_{sb} &= 2q\left(\frac{q}{\nu}\right)^2\left[\alpha_3\cos(2\Theta_+ - \Theta_-) + \alpha_{-3}\cos(2\Theta_- - \Theta_+)\right] \\ &= q\left(\frac{q}{\nu}\right)^2\left[\beta_3\cos(\bar{k}z - \bar{\omega}t)\cos(3(\delta\zeta - \nu\tau)) + \beta_{-3}\sin((\bar{k}z - \bar{\omega}t)\sin(3(\delta\zeta - \nu\tau))\right] \end{aligned}$$

Summarising so far, the first and third order side band contributions together can be written (if we introduce in third order the higher order correction from the nonlinear dispersion relation in the carrier wave) like

$$u^{(1)} + u^{(3)}_{sb} = q\cos(\bar{k}z - \bar{\omega}t + K_{corr}z)\left(4\cos(\delta\zeta - \nu\tau) + \left(\frac{q}{\nu}\right)^2 \beta_3 \cos(3(\delta\zeta - \nu\tau))\right)$$
$$+ q\left(\frac{q}{\nu}\right)^2 \beta_{-3} \sin(\bar{k}z - \bar{\omega}t)\sin(3(\delta\zeta - \nu\tau))$$

The effect of the side band contributions is clearly seen:

- the carrier wave of the linear solution with phase correction $\cos(\bar{k}z - \bar{\omega}t + K_{corr}z)$ is modulated by the linear modulation $4\cos(\delta\zeta - \nu\tau)$ and a triple periodic contribution $\left(\frac{q}{\nu}\right)^2 \beta_3 \cos(3(\delta\zeta - \nu\tau))$; the amplitude of this last contribution may become substantial for small ν;
- in addition, the signal contains a contribution from the out-of phase carrier wave that is modulated by a same triple periodic function. This last effect is proportional to $q^2 K'''(\bar{\omega})/\nu$; when $K'''(\bar{\omega}) \neq 0$ this term will cause skew-symmetric distortions of the envelope.

Remark 41 Actually, to satisfy the correct initial signal, the contribution of the third order term at $z = 0$ has to be compensated by harmonic modes ('free waves')

$$u_{free} = -2q\left(\frac{q}{\nu}\right)^2 [\alpha_3 \cos\Theta(\bar{\omega} + 3\nu) + \alpha_{-3}\cos\Theta(\bar{\omega} - 3\nu)]$$

which can be re-written as a modification of the linear solution as

$$u_{free} = -q\left(\frac{q}{\nu}\right)^2 \left[\begin{array}{c} \beta_3 \cos(\bar{k}z - \bar{\omega}t + 4K''\nu^2 z)\cos 3(\Delta_3\zeta - \nu\tau) \\ +\beta_{-3}\sin(\bar{k}z - \bar{\omega}t + 4K''\nu^2 z)\sin 3(\Delta_3\zeta - \nu\tau) \end{array}\right]$$

where $\Delta_3 = \frac{K_2(3\nu) - K_2(-3\nu)}{6} \approx 9\delta$. Observe the z-dependent shift of the carrier wave that is different from that of the nonlinear dispersion relation. The interference of the bound and the free third order carrier waves determines the spatial recurrence length of the total solution, given by $4K''\nu^2 Z_{recurr} = 2\pi$, i.e.

$$Z_{recurr} = \frac{\pi}{2K''\nu^2}.$$

3.4.3 Graphical representation

We will now show a plot of a characteristic situation in surface waves, with $\bar{\omega} = 2$ and $q = 0.05$, $\nu = 0.06$. (Actually, this case is rather extreme, since $q/\nu = 5/6$ rather large. This case is taken since it shows the deformation and break of symmetry more profound than in realsitic cases.) In the figure 2 we show the evolution of the bi-chromatic as given by the analytical approximation, plotted as time signal at various positions. The large deformation of the envelope during the propagation can be clearly seen : the initially sinusoidal profile is

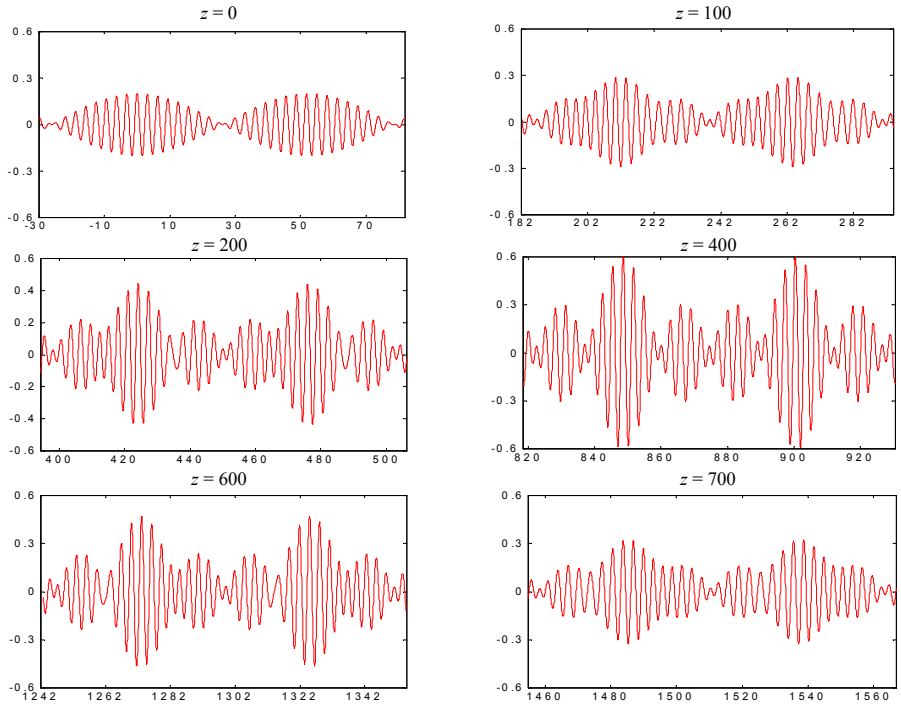


Figure 2: Analytic approximation for $q/\nu \approx 5/6$ (which is rather large). Observe the 'breathing' of the packets with periodically large intervals of small amplitude waves between large amplitude packets.

perturbed with a third harmonic of considerable amplitude, and the asymmetrization results from the third order term in the dispersion. (Comparison with numerical calculations show good agreement, including the correctness of the recurrence length theoretically predicted to be $Z_{recurr} = 824$).

Summary 42 *From a theoretical point of view, the asymptotic approximations show (starting at third order) a series expansion in the two parameters, q and q/ν. The results are correct for fixed ν and small q; when q/ν is of the order unity, the series expansion will not converge and a more detailed two-parameter analysis is required.*

References

[1] N. Akhmediev & A. Ankiewicz, Solitons, *Nonlinear pulses and beams*, Chapman & Hall, 1997.

[2] Q. Chang, E. Jia & W. Sun, Difference Scheme's for Solving the Generalized Nonlinear Schrodinger Equation, J. Comput. Phys. **148**(1999), 397-415.

[3] K.B. Dysthe, Note on a modification to the Nonlinear Schrodinger equation for application to deep water waves, *Proc. R. Soc. London*, **A369**(1979)105-114.

[4] E. van Groesen & E.M. de Jager, *Mathematical structures in continuous dynamical systems,* Elsevier North-Holland, Amsterdam (1994) 631 pages; ISBN 0-444-82151-1.

[5] E. van Groesen, T. Nusantara & E. Soewono, Large variations in NLS bi-soliton wave groups, *Optical and Quantum Electronics*, accepted 2000.

[6] E. van Groesen, E. Cahyono & A. Suryanto, Uni-directional models for narrow and broad pulse propagation in second order nonlinear media, *Optical and Quantum Electronics*, in press 2001

[7] A. Hasegawa and Y. Kodama, *Solitons in optical communications*, Clarendon Press, Oxford, 1995.

[8] A.C. Newell and J.V. Moloney, *Nonlinear optics*, Addison-Wesley, Canada,1992.

[9] C. Sulem & P-L Sulem, *The Nonlinear Schrödinger Equation*, Springer Verlag, 1999.

[10] A.Suryanto, E. van Groesen & H. Hoekstra, Deformation of modulated wave groups in third-order nonlinear media, *Optical and Quantum Electronics,* accepted 2000.

[11] T. R. Taha & M. J. Ablowitz, Analytical and Numerical Aspects of Certain Nonlinear Evolution Equations II, J. Comput. Phys. **55**(1984)203.