Title **Project Waveprofiles of maximal crest height**

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MRI -Variational Structures and Methods

Project aim: find profiles of waves that have maximal crest height, for waves on the whole real line with finite energy, and for periodic waves.
We first use a linear model (quadratic energy and momentum), and then a few aspects of a nonlinear KdV (Korteweg-de Vries) model.

1 Methodology

We will denote by $\eta(x,t)$ (real valued) wave fields depending on the spatial and temporal variables x,t; when being concerned with the profiles we will simply write $\eta(x)$. The interest will be in waves described by a Hamiltonian system with Hamiltonian H and a momentum integral M. For definiteness we take as governing evolution equation the Hamiltonian system

$$\partial_t \eta = \partial_x \delta H(\eta).$$

Here the Hamiltonian H is a translation invariant integral which is linearly independent from the momentum functional

$$M = \int \frac{1}{2}\eta^2 dx,$$

which itself is an integral of the motion, with translation as its Hamiltonian flow.
For the extremal crest formulation we suppress the dependence on time, and denote the maximal crest height functional for functions η by

$$C(\eta) := \max_x \eta(x).$$

Then the extremal problem can be written as

$$\max_\eta \{ \ C(\eta) \ \mid \eta \in \mathcal{C} \ \},$$

where the constaint set is given by:

- for finite energy waves on the whole real line:

$$\mathcal{C} = \mathcal{C}(h,m) := \{ \ \eta \mid H(\eta) = h, M(\eta) = m \ \}$$

- for waves that are periodic with period L

$$\mathcal{C} = \mathcal{C}(h,m) := \{ \ \eta \mid \eta \text{ is } L\text{-periodic}, \operatorname{av}(\eta) = 0, H(\eta) = h, M(\eta) = m \ \}$$

  where the average is defined as $\operatorname{av}(\eta) = \int_0^L \eta(x)\,dx$.

We denote the value function as the extremal crest height for given constraint values (m,h) :

$$V(h,m) := \max_\eta \{ \ C(\eta) \ \mid \eta \in \mathcal{C}(h,m) \ \}.$$

Show that from the time invariance of M, H the dynamic result is that at each position and time the evolution from an intial value $\eta_0(x)$ will be such that

$$\eta(x,t) \le V(m_0, h_0) \text{ where } m_0 = M(\eta_0), h_0 = H(\eta_0).$$

In all these cases, the constraint set is nonempty only if the values (m,h) are 'feasible' by satisfying certain conditions.

2 Linear Equations

Take for the Hamiltonian the quadratic functional

$$H(\eta) = \int (\partial_x \eta)^2 \, dx$$

1. Write down the dynamic equation, and investigate the periodic solutions on the whole real line. Dispersion is the fact that waves with different wavelength travel at different speed (the so-caleld wave speed). Investigate if the equation has dispersion or not.
   Show directly that M is indeed an invariant functional.

2. Now investigate the maximal crest height problem on the whole real line for finite energy profiles. You will be able to find these profiles explicitly, and find that none of them is smooth (differentiable).

3. Investigate the maximal crest height problem for periodic profiles with given period L. You will be able to find two branches of profiles (a harmonic branch and a catenary branch... why these names?), depending on the value of the quotient h/m. Observe that on one branch, there is for a specific value of h/m a smooth solution; for all other values the profiles will have corners. Show that the smooth solutions are actually relative equilibrium solutions of the dynamic system.

3 KdV extremal profiles of finite energy

Now take as Hamiltonian

$$H(\eta) = \int \left[\frac{1}{2} (\partial_x \eta)^2 - \frac{1}{3} \eta^3 \right] dx$$

which means that we are dealing with the KdV equation. We will only consider finite energy solutions on the whole real line.

1. Show directly that M is indeed an invariant functional.

2. Find the relative equilibrium solutions explicitly, which are known as the KdV 'solitons'.

3. Investigate the maximal crest height problem and show that extremal profiles are 'cornered ' solitons that can be found explicitly.

4. Discuss dynamic consequences of the results found.

Title **Project Periodic Hamiltonian Motions**

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MRI -Variational Structures and Methods

Project aim: find (characterise) periodic solutions of Hamiltonian systems in Classical Mechanics. Study the attached 5 pages, and make all exercises (including proof of proposition 158).
Observe that by restricting the function space and/or by adding cleverly chosen natural constraints saddle-point type of critical points are transformed to (constrained) extremisers.

8.4 Periodic Hamiltonian motions

We now give a few, more difficult, examples of optimization problems in dynamics for which the existence of a critical point can be proved. In some way or another, we prove the existence of a periodic motion of a classical Hamiltonian system by using (variants of) the action functional.
Note that the restriction to *periodic* solutions (whether the period is prescribed in advance, or when it is not prescribed but has to found as part of the solution) makes the problem to one with *boundary* conditions, different from the *initial* value problem that is usually considered; for an initial value problem *local existence* on a sufficiently *small* time interval around the initial time can be proved with the ususal contraction argument. But for a boundary value problem, the difficulty in the proof of existence is the fact that existence of a solution on the periodic interval is a problem of existence of a *global solution*.
A second observation is that usually the periodic solution is a saddle point of the action functional, while Weierstrasz theorem can only give results for minimizers (or maximizers). By transforming the problem in various ways, we will be able to reformulate the saddle point as a minimizer on a suitably restricted subset.

In historic perspective, this specific application has lead to many new developments in variational methods over a period of more than two centuries.

We recall briefly the general setting.
The *position* of mass points (all masses normalized to 1 for simplicity) is described by a vector from the *configuration space*: $q \in IR^n$. The *state* of the dynamic system is described by a point in the state-, or *phase space*: $(q,p) \in IR^n \times IR^n$, where p has the meaning of momentum (velocity $\dot{q}$).
For given (smooth) potential energy function $V = V(q,t)$, defined on configuration space and possibly depending on time (forcing), consider solutions of the following *Hamiltonian system*, equivalently formulated on configuration space (as Newton-like equation, 2-nd order in time) and on phase space (Hamilton formulation, 1-st order in time):

$$-\ddot{q} = V'(q,t), \quad \text{resp} \quad \left\{ \begin{array}{rcl} \dot{q} &=& p \\ \dot{p} &=& -V'(q,t) \end{array} \right. \tag{8.14}$$

These systems arise from a dynamic variational principle: the equations are the Euler-Lagrange equations of the *action functional* on configuration space:

$$\mathcal{A}(q) = \int [\frac{1}{2}|\dot{q}|^2 - V(q,t)]dt \tag{8.15}$$

resp the *canonical action functional* on phase space:

$$\mathcal{CA}(q,p) = \int [p \cdot \dot{q} - V(q,t)]dt. \tag{8.16}$$

Periodic solutions

When V depends periodically on time, period T say, it is possible to look for solutions that are T-periodic. (When V is autonomous, T is not given a priori.) Finding T-periodic solutions reduces the problem to a boundary-value problem on the compact interval $t \in [0,T]$, with boundary conditions:

$$\left\{ \begin{array}{lcl} q(0) & = & q(T) \\ \dot{q}(0) & = & \dot{q}(T) \end{array} \right. \quad \textit{resp.} \left\{ \begin{array}{lcl} q(0) & = & q(T) \\ p(0) & = & p(T) \end{array} \right. \tag{8.17}$$

and are obtained as critical points of

$$\text{Crit } \left\{\mathcal{A}(q) \middle| \, q(0) = q(T)\right\}, \quad \text{resp} \;\; \text{Crit } \left\{\mathcal{CA}(q,p) \middle| \, q(0) = q(T)\right\}.$$

Note that the remaining boundary conditions arise as *natural boundary conditions*.

In particular when V is autonomous, special periodic solutions can be considered that are simple continuations of the motion on part of the period:
Brake orbits: motion between restpoints, satisfying

$$\dot{q}(0) = \dot{q}(T/2) = 0, \;\text{resp} \;\; .\; p(0) = p(T/2) = 0; \tag{8.18}$$

these boundary conditions are obtained as natural boundary conditions by not prescribing any conditions at all.

If, moreover, V is even in q: $V(q) = V(-q)$, *normal mode solutions* are determined by the motion during a quarter period:

$$q(0) = \dot{q}(T/4) = 0, \quad \text{resp} \quad q(0) = p(T/4) = 0, \tag{8.19}$$

obtained by prescribing only $q(0) = 0$ in the variational principle.

Except for normal modes solutions (when $q(0) = 0$ is prescribed), the functional $\int |\dot{q}|^2$ is not equivalent to the H^1-norm (it vanishes for constant vector functions). In those cases it is natural to split H^1 in an orthogonal way:

$$H^1_T([0,T]) \equiv \{q \in H^1_T([0,T]) \mid q(0) = q(T)\} = IR^n + Y_T,$$

with

$$Y_T = \left\{y \in H^1_T([0,1]) \middle| \int y = 0\right\},$$

so $q = c + y$, with a constant vector $c \in IR^n$ and $y \in Y_T$. Then $\int |\dot{q}|^2 = \int |\dot{y}|^2$ is (equivalent to) the norm in Y_T.

8.4.1 Periodic motions with prescibed period

Proposition 153 *If $V(q)$ is even and subquadratic in q there exists a normal mode solution related to the minimizer (when nontrivial) of*

$$Min \left\{ \int_0^{T/4} [\frac{1}{2}|\dot{q}|^2 - V(q)]dt \middle| \; q(0) = 0 \right\}$$

The proof of the existence is contained in the previous examples and will not be repeated.

Proposition 154 Periodic potential and forcing.
Suppose $q \mapsto V(q)$ is periodic in each component of q. Let f be a T-periodic function with $\int_0^T f(t)dt = 0$. Then the minimization problem

$$Min \left\{ \int_0^{T} [\frac{1}{2}|\dot{q}|^2 - V(q) - f(t)q(t)]dt \middle| \; q(0) = q(T) \right\} \tag{8.20}$$

has a solution which is a T-periodic solution of the forced equation:

$$-\ddot{q} = V'(q) + f(t).$$

Exercise 155 *1.* Prove the proposition. Observe that by writing $q(t) = c + y(t)$

$$\int_0^T [\frac{1}{2}|\dot{q}|^2 - V(q) - f(t)q(t)]dt = \int_0^T [\frac{1}{2}|\dot{y}|^2 - V(c+y) - f(t)y(t)]dt.$$

Then show that a minimizing sequence $q_k = c_k + y_k$, $\{y_k\}$ is bounded (since V is bounded); finally, from the periodicity of V, the elements $\{c_k\}$ can be taken to be bounded.

2. Note that the special case $f \equiv 0$ leads to a solution for any T. Only for certain values of T, however, the minimizer is non-trivial.

3. A specific example for which the minimizers are not trivial, are periodic solutions of the forced pendulum equation:

$$-\ddot{q} = \sin(q) + f(t).$$

As stated in the introduction, typically, periodic solutions correspond to *saddle points* of the (canonical) action functional. As a first example, the following result; in this case the saddle point can be characterized explicitly (analytically), and transformed to a *naturally constrained minimizer*, i.e. the constraint does not contribute to the governing equation for the critical point.

Proposition 156 *Suppose V is strictly convex and subquadratic at infinity. Then there exists a periodic solution corresponding to a saddle point:*

$$Sad \left\{ \int_0^T [\frac{1}{2}|\dot{q}|^2 - V(q)]dt \right| q(0) = q(T)\}$$

$$= Min_{\ y \in Y_T}\ Max_{\ c \in \mathcal{R}^n} \int_0^T [\frac{1}{2}|\dot{y}|^2 - V(c+y)]dt$$

$$= Min \left\{ \int_0^T [\frac{1}{2}|\dot{q}|^2 - V(q)]dt \right| q \in H^1_T, \ \int_0^T V'(q) = 0 \right\}.$$

Exercise 157 Prove the proposition above; first prove the existence, then investigate the governing equation to show that the constraint $\int_0^T V'(q) = 0$ is a natural constraint: the multiplier in the equation vanishes.

8.4.2 Periodic motions with prescribed energy

In the following V is autonomous. Then the "energy" is conserved (constant in time) for all solutions:

$$\frac{1}{2}|\dot{q}|^2 + V(q) = E, \quad \text{resp . } H(q,p) \equiv \frac{1}{2}|p|^2 + V(q) = E. \tag{8.21}$$

We try to find periodic solutions, with a priori unknown period T, that have a prescribed value of the energy. It is simplest to normalize the time: $\tau = t/T \in [0,1]$.

Proposition 158 Variational principles on phase space
Up to time-scaling, periodic solutions with prescribed energy E are critical points of

$$Crit \left\{ \int_0^1 p \cdot \dot{q} d\tau \right| H(q,p) = E, q(0) = q(1)\} \tag{8.22}$$

and also of

$$Crit \left\{ \int_0^1 p \cdot \dot{q} \right| \int_0^1 H(q,p)d\tau = E, q(0) = q(1) \right\} \tag{8.23}$$

In the last case, the (constant) multiplier λ arising from the energy constraint is precisely the period T of the motion.

For fixed $q \in IR^n$ such that $V(q) \leq E$, the supremum over p in the above principles is obtained for the vector collinear with $\dot{q}$:

$$\dot{q} = \lambda p, \quad \text{and} \quad |p| = \sqrt{2(E - V(q))}.$$

Hence, solutions of these optimization problems are obtained if solutions can be found of the following variational problems.

Proposition 159 Variational principles on configuration space
Up to time-scaling, periodic solutions with prescribed energy E are critical points of the following Jacobi functional

$$Crit \left\{ \int_0^1 \sqrt{2(E - V(q))}|\dot{q}|d\tau \middle| \quad q(0) = q(1) \right\} \tag{8.24}$$

and also of

$$Crit \ \{\mathcal{J}_E(q)| \ q(0) = q(1)\} \tag{8.25}$$

where J_E is the following modified Jacobi functional

$$\mathcal{J}_E(q) = [\int_0^1 \frac{1}{2}|\dot{q}|^2 d\tau] * [E - \int_0^1 V(q)d\tau] \tag{8.26}$$

The modified Jacobi functional is essentially easier to investigate (with standard Hilbert space techniques) than the original Jacobi functional (in which the factor $\sqrt{2(E - V(q))}$ can vanish at certain times, and hence does not easily define a norm).
As an example of the use of $\mathcal{J}_E$, the following proposition in which we use "polar coordinates": for $q \in \{q \in H^1([0,1]) \mid q(0) = 0\}$, $q = \rho\omega$ with $\omega \in S$, S the unit ball: $S = \{q \mid \int |\dot{q}|^2 = 1\}$.

Proposition 160 *Assume that $V(x) \geq 0, V(0) = 0$ and that V is* strictly convex *and* even. *For any $E > 0$ there is a periodic solution with energy E; this solution is a* normal mode *and corresponds to a* saddle point *of J_E:*

$$Sad \ \{\mathcal{J}_E(q)| \quad q(0) = 0\} \tag{8.27}$$

The character of the saddle point can be found explicitly, and can be transformed to a (naturally) constrained minimizer as follows:

$$Min_{\ \omega \in S} \ Max_{\ \rho \geq 0} \ \mathcal{J}_E(\rho\omega) \equiv Min \quad \{\mathcal{J}_E(q)| \quad q \in N_E\} \tag{8.28}$$

The set N_E is a natural constraint, explicitly given by

$$N_E = \left\{ \ q \ \middle| \ \int [V(q) + \frac{1}{2}V'(q) \cdot q] = E \right\}. \tag{8.29}$$

Exercise 161 Prove the proposition above; first existence, then verify that N_E is a natural constraint.