1 About existence

1.1 Main theorems

1. We consider a functional $F$ defined on a Hilbert space (or reflexive Banach space) $H$. A subset (loosely called ‘manifold’, assumed to be non-empty) $M$ is given and we consider the minimization problem

$$\mu := \inf \{ F(u) \mid u \in M \}$$

If $\mu$ is finite, and there exists a minimizer $\hat{u}$ we will write

$$\hat{u} \in \inf \{ F(u) \mid u \in M \},$$

and then say that the optimization problem is well-defined (or well-posed).

Prove that the problem is well-defined in the following cases:

(a) $H$ is finite dimensional, $F$ is continuous, and $M$ is compact.
(b) $H$ is infinite dimensional, $F$ is continuous, and $M$ is compact.
(c) $H$ is infinite dimensional, $F$ is lower-semi-continuous, and $M$ is weakly compact.
(d) $H$ is infinite dimensional, $F$ is lower-semi-continuous, and $M$ is weakly closed and $F$ is coercive, meaning:

$$F(u) \to \infty \text{ for } \|u\| \to \infty.$$  

2. For applications (that I will deal with) the cases 1c and 1d. above are most interesting. Prove that: A Banach space is reflexive iff (if and only if) the unit ball is weakly compact.

1.2 Applications (examples)

In the following exercises, investigate the existence of solutions. That is to say,

- either show that the problem is well-defined, (you do not have to find the minimizer, however in some cases it can be found explicitly; see next section);
- or show that $\mu = -\infty$ by constructing a sequence $u_n$ for which $F(u_n) \to -\infty$ for $n \to \infty$,
- or show $\mu$ = finite, but no solution exists, by showing that for each sequence $u_n$ for which $F(u_n) \to \mu$ for $n \to \infty$ the sequence has no convergent subsequence.
In all exercises you have to find a suitable formulation (choose the functional, the set $M$, and in particular the Hilbert space yourself when it is not given).

**Exercises:**

1. Investigate the well-posedness of the catenary optimization problem.

2. Investigate the well-posedness (and find solution) of

   \[
   \inf \left\{ \int_0^1 (\partial_x u)^2 \, dx \ \bigg| \ \int_0^1 u^2 \, dx = 3 \right\}
   \]

3. Investigate the well-posedness of

   \[
   \inf \left\{ \int_0^1 (\partial_x u)^2 \, dx \ \bigg| \ \int_0^1 u^2 \, dx = 3, \int_0^1 x \, u \, dx = 7 \right\}
   \]

4. Investigate the well-posedness of

   \[
   \inf \left\{ \int_0^1 (\partial_x u)^2 \, dx \ \bigg| \ \int_0^1 u^2 \, dx = 3, u(0) = 0 \right\}
   \]

5. Investigate the well-posedness of

   \[
   \inf \left\{ \int_0^1 (\partial_x u)^2 \, dx \ \bigg| \ \int_0^1 u^2 \, dx \geq 3, u(0) = 0 \right\}
   \]

6. Investigate the well-posedness of

   \[
   \inf \left\{ \int_0^1 u^2 \, dx \ \bigg| \ \int_0^1 (\partial_x u)^2 \, dx = 3, u(0) = 0 \right\}
   \]

7. Investigate the well-posedness of

   \[
   \inf \left\{ \int_0^1 u^2 \, dx \ \bigg| \ \int_0^1 (\partial_x u)^2 \, dx \leq 3, u(0) = 0 \right\}
   \]

8. Investigate the well-posedness of (KdV-soliton case)

   \[
   \inf \left\{ \int_0^1 ((\partial_x u)^2 - u^3) \, dx \ \bigg| \ \int_0^1 u^2 \, dx = 3, u(0) = 0 \right\}
   \]

and (KdV-soliton case)

\[
\inf \left\{ \int_{-\infty}^{\infty} ((\partial_x u)^2 - u^3) \, dx \ \bigg| \ \int_{-\infty}^{\infty} u^2 \, dx = 3, u \to 0 \text{ for } |x| \to \infty \right\}
\]

9. In a finite dimensional approximation with Fourier-truncations, the quadratic functionals above are approximated by finite sums of the (complex) Fourier coefficients. Verify that (up to some constant):

   \[
   \int_0^1 u^2 \, dx \approx \sum_{-N \leq k \leq N} |c_k|^2
   \]

   \[
   \int_0^1 (\partial_x u)^2 u^2 \, dx \approx \sum_{-N \leq k \leq N} k^2 |c_k|^2
   \]
Now consider the finite dimensional problems:
\[
\inf \left\{ \sum_{N \leq k \leq N} k^2 |c_k|^2 \left| \sum_{N \leq k \leq N} |c_k|^2 = 1 \right. \right\},
\]
\[
\sup \left\{ \sum_{N \leq k \leq N} k^2 |c_k|^2 \left| \sum_{N \leq k \leq N} |c_k|^2 = 1 \right. \right\}
\]
and show that both problems are well-posed. Related to the infinite dimensional problem, which of these finite dim approximations (or problem) would you call ‘spurious’?

2 Constraints and Lagrange’s Multiplier Rule

1. Let $F$ and $G$ be functionals, and suppose that the minimization problem
\[
U (\gamma) \in \inf \{ F (u) \ | \ G (u) = \gamma \} =: V (\gamma)
\]
is well posed and that the constraint set is regular for all parameters $\gamma$ in an interval. Denoting the solution by $U (\gamma)$, the LMR reads
\[
\delta F (U (\gamma)) = \lambda (\gamma) \delta G (U (\gamma)),
\]
where the multiplier depends on $\gamma$ also. Suppose that $\gamma \to V (\gamma) = F (U (\gamma))$ (this is called the value-function) is smooth.

(a) Prove that
\[
\lambda (\gamma) = \frac{d}{dp} F (U (\gamma)).
\]

(b) Illustrate this result for $F (u) = \int_0^1 (\partial_x u)^2 \ dx, G (u) = \int_0^1 u^2 \ dx$.

(c) Show: If $\gamma \to V (\gamma)$ is convex (in the interval considered), then the constrained solution is a minimizer of the unconstrained problem:
\[
U (\gamma) \in \inf \{ F (u) - \lambda (\gamma) G (u) \}
\]

(d) Show: If $\gamma \to V (\gamma)$ is NOT convex (in the interval considered), then the constrained solution is NOT a (local) minimizer of the unconstrained problem above.

2. Determine the explicit solution(s?) of the following problems:

(a) The catenary problem.

(b) The problem:
\[
\inf \left\{ \int_0^1 (\partial_x u)^2 \ dx \left| \int_0^1 u^2 \ dx = 3, \int x.\ dx = 0; u (0) = u (1) = 0 \right. \right\}
\]
... as expected.

(c) The problem:
\[
\inf \left\{ \int_0^1 (\partial_x u)^2 \ dx \left| \int_0^1 u^2 \ dx = 3, \int \sin (\pi x) . u \ dx = 0; u (0) = u (1) = 0 \right. \right\}
\]
What happens? One or two (non-vanishing) multipliers?
3 Boundary & interface conditions, effective boundary cd’s

1. Consider on the bounded domain \( \Omega \) (with boundary \( \Gamma \)) the problem

\[
\inf \left\{ \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Gamma} u \, d\sigma \ \bigg| \int_{\Omega} u^2 \, dx = 3 \right\}
\]

Investigate the well-posedness. Write down the boundary value problem.

2. Consider on the bounded domain \( \Omega \) (with boundary \( \Gamma = \Gamma_1 \cup \Gamma_2 \)) the problem

\[
\inf \left\{ \int_{\Omega} |\nabla u|^2 \, dx \ \bigg| \int_{\Omega} u^2 \, dx = 1, u = 0 \text{ on } \Gamma_1 \right\}
\]

(a) Investigate the well-posedness.
(b) Write down the boundary value problem.
(c) Give the solution in case \( \Omega \) is the unit square in \( \mathbb{R}^2 \), and \( \Gamma_1 \) the upper and lower side.
(d) Give the solution in case \( \Omega \) is the unit square in \( \mathbb{R}^2 \), and \( \Gamma_1 = \partial \Omega \).

3. Consider the optical-wave guide eigenvalue problem in \( \mathbb{R}^2 \):

\[
\inf \left\{ \int_{\mathbb{R}^2} \left[ |\nabla E|^2 - k_0^2 n^2 E^2 \right] \, dx \, dz \ \bigg| \ u \to 0 \text{ for } |x| \to \infty \right\}
\]

with index

\[
n = n(x) = \begin{cases} 
n_0 & \text{for } |x| > W \\
 \text{different} & \text{for } |x| < W
\end{cases}
\]

(a) Use separation of variables \( E(x,z) = \phi(x)f(z) \) and formulate the eigenvalue problem for \( \phi \).
(b) Investigate interface conditions.
(c) Formulate (with complete argumentation) effective boundary conditions at \( |x| = W \) for the problem on \( |x| \leq W \).
(d) Give the complete variational formulation of this problem for \( \phi \).

4. Consider the functional

\[
F(u) = \int \left[ r(x) \left( \partial_{xx} u \right)^2 - p(x) \left( \partial_x u \right)^2 + q(x) u^2 - f(x) u \right] \, dx,
\]

where \( r, p, q, f \) are given functions, on the set of functions on \([0,L]\) with boundary conditions

\[u(0) = \partial_x u(L) = 0.\]

(a) Write down the Euler-Lagrange equation and the (natural) boundary conditions.
(b) Give sensible meaning to this equation when there may be (at most) jump-discontinuities in (one of) the functions \( r, p, q \).
(c) Furthermore, derive the interface conditions at the points of discontinuity of these functions.