Title **Homework**

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MRI -Variational Structures and Methods

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It is NOT required to make all exercises; as discussed, during the lectures I often say 'this could be a nice exercise', so there are many. But on student's request, I also formulate some exercises in detail below.

(Note: there may (will) be errors in the following; this is unintentional, and you are requested to be critical as always.)

1 About existence

1.1 Main theorems

1. We consider a functional F defined on a Hilbertspace (or reflexive Banach space) H. A subset (loosely called 'manifold', assumed to be non-empty) M is given and we consider the minimization problem

$$\mu := \inf \ \{F(u) \ \mid u \in M\}$$

If μ is finite, and there exists a minimizer $\hat{u}$ we will write

$$\hat{u} \in \inf \ \{ F(u) \ \mid u \in M \} ,$$

and then say that the optimization problem is well-defined (or well-posed).
Prove that the problem is well-defined in the following cases:

   (a) H is finite dimensional, F is continuous, and M is compact.

   (b) H is infinite dimensional, F is continuous, and M is compact.

   (c) H is infinite dimensional, F is lower-semi-continuous, and M is weakly compact.

   (d) H is infinite dimensional, F is lower-semi-continuous, and M is weakly closed and F is *coercive*, meaning:

$$F(u) \to \infty \text{ for } \|u\| \to \infty.$$

2. For applications (that I will deal with) the cases 1c and 1d. above are most interesting. Prove that: *A Banach space is reflexive iff (if and only if) the unit ball is weakly compact.*

1.2 Applications (examples)

In the following exercises, investigate the existence of solutions. That is to say,

- either show that the problem is well-defined, (you do not have to find the minimizer, however in some cases it can be found explicitly; see next section);
- or show that $\mu = -\infty$ by constructing a sequence u_n for which $F(u_n) \to -\infty$ for $n \to \infty$,
- or show $\mu =$ finite, but no solution exists, by showing that for each sequence u_n for which $F(u_n) \to \mu$ for $n \to \infty$ the sequence has no convergent subseqence.

In all exercises you have to find a suitable formulation (choose the functional, the set M, and in particular the Hilbert space yourself when it is not given).

Exercises:

1. Investigate the well-posedness of the catenary optimization problem.

2. Investigate the well-posedness (and find solution) of

$$\inf\left\{ \int_0^1 (\partial_x u)^2 \, dx \;\middle|\; \int_0^1 u^2 dx = 3 \right\}$$

3. Investigate the well-posedness of

$$\inf\left\{ \int_0^1 (\partial_x u)^2 \, dx \;\middle|\; \int_0^1 u^2 dx = 3, \int_0^1 x.u dx = 7 \right\}$$

4. Investigate the well-posedness of

$$\inf\left\{ \int_0^1 (\partial_x u)^2 \, dx \;\middle|\; \int_0^1 u^2 dx = 3, u(0) = 0 \right\}$$

5. Investigate the well-posedness of

$$\inf\left\{ \int_0^1 (\partial_x u)^2 \, dx \;\middle|\; \int_0^1 u^2 dx \geq 3, u(0) = 0 \right\}$$

6. Investigate the well-posedness of

$$\inf\left\{ \int_0^1 u^2 dx \;\middle|\; \int_0^1 (\partial_x u)^2 \, dx = 3, u(0) = 0 \right\}$$

7. Investigate the well-posedness of

$$\inf\left\{ \int_0^1 u^2 dx \;\middle|\; \int_0^1 (\partial_x u)^2 \, dx \leq 3, u(0) = 0 \right\}$$

8. Investigate the well-posedness of (KdV-soliton case)

$$\inf\left\{ \int_0^1 \left((\partial_x u)^2 - u^3 \right) dx \;\middle|\; \int_0^1 u^2 dx = 3, u(0) = 0 \right\}$$

   and (KdV-soliton case)

$$\inf\left\{ \int_{-\infty}^{\infty} \left((\partial_x u)^2 - u^3 \right) dx \;\middle|\; \int_{-\infty}^{\infty} u^2 dx = 3, u \to 0 \text{ for } |x| \to \infty \right\}$$

9. In a finite dimensional approximation with Fourier-truncations, the quadratic functionals above are approximated by finite sums of the (complex) Fourier coefficients. Verify that (up to some constant):

$$\begin{aligned} \int_0^1 u^2 dx &\approx \Sigma_{-N \leq k \leq N} \, |c_k|^2 \\ \int_0^1 (\partial_x u)^2 u^2 dx &\approx \Sigma_{-N \leq k \leq N} \, k^2 |c_k|^2 \end{aligned}$$

Now consider the finite dimensional problems:

$$\inf \left\{ \sum_{-N\leq k\leq N} k^2 |c_k|^2 \;\middle|\; \sum_{-N\leq k\leq N} |c_k|^2 = 1 \right\},$$

$$\sup \left\{ \sum_{-N\leq k\leq N} k^2 |c_k|^2 \;\middle|\; \sum_{-N\leq k\leq N} |c_k|^2 = 1 \right\}$$

and show that both problems are well-posed. Related to the infinite dimensional problem, which of these finite dim approximations (or problem) would you call 'spurious'?

2 Constraints and Lagrange's Multiplier Rule

1. Let F and G be functionals, and suppose that the minimization problem

$$U(\gamma) \in \inf \{ F(u) \;\mid G(u) = \gamma \} =: V(\gamma)$$

is well posed and that the constraint set is regular for all parameters γ in an interval. Denoting the solution by $U(\gamma)$, the LMR reads

$$\delta F(U(\gamma)) = \lambda(\gamma)\, \delta G(U(\gamma)),$$

where the multiplier depends on γ also. Suppose that $\gamma \to V(\gamma) = F(U(\gamma))$ (this is called the *value-function*) is smooth.

   (a) Prove that
   $$\lambda(\gamma) = \frac{d}{dp} F(U(\gamma)).$$

   (b) Illustrate this result for $F(u) = \int_0^1 (\partial_x u)^2\, dx, G(u) = \int_0^1 u^2 dx$.

   (c) Show: If $\gamma \to V(\gamma)$ is *convex* (in the interval considered), then the constrained solution is a minimizer of the *unconstrained problem*:
   $$U(\gamma) \in \inf \{ F(u) - \lambda(\gamma) G(u) \quad \}$$

   (d) Show: If $\gamma \to V(\gamma)$ is NOT *convex* (in the interval considered), then the constrained solution is NOT a (local) minimizer of the *unconstrained problem* above.

2. Determine the explicit solution(s?) of the following problems:

   (a) The catenary problem.

   (b) The problem:
   $$\inf \left\{ \int_0^1 (\partial_x u)^2\, dx \;\middle|\; \int_0^1 u^2 dx = 3, \int x.u dx = 0; u(0) = u(1) = 0 \right\}$$
   ... as expected.

   (c) The problem:
   $$\inf \left\{ \int_0^1 (\partial_x u)^2\, dx \;\middle|\; \int_0^1 u^2 dx = 3, \int \sin(\pi x)\,.u dx = 0; u(0) = u(1) = 0 \right\}$$
   What happens? One or two (non-vanishing) multipliers?

3 Boundary & interface conditions, effective boundary cd's

1. Consider on the bounded domain Ω (with boundary Γ) the problem

$$\inf\left\{ \int_\Omega |\nabla u|^2\, dx - \int_\Gamma u d\sigma \;\middle|\; \int_\Omega u^2 dx = 3\right\}$$

   Investigate the well-posedness. Write down the boundary value problem.

2. Consider on the bounded domain Ω (with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$) the problem

$$\inf\left\{ \int_\Omega |\nabla u|^2\, dx \;\middle|\; \int_\Omega u^2 dx = 1, u = 0 \text{ on } \Gamma_1\right\}$$

   (a) Investigate the well-posedness.
   (b) Write down the boundary value problem.
   (c) Give the solution in case Ω is the unit square in $\mathbb{R}^2$, and Γ_1 the upper and lower side.
   (d) Give the solution in case Ω is the unit square in $\mathbb{R}^2$, and $\Gamma_1 = \partial\Omega$.

3. Consider the optical-wave guide eigenvalue problem in $\mathbb{R}^2$:

$$\inf\quad \left\{\int_{\mathbb{R}^2} \left[|\nabla E|^2 - k_0^2 n^2 E^2\right] dxdz \;\middle|\; u \to 0 \text{ for } |x| \to \infty \right\}$$

   with index

$$n = n(x) = \begin{cases} n_0 \text{ for } |x| > W \\ \text{different } > n_0 \text{ for } |x| < W \end{cases}$$

   (a) Use separation of variables $E(x,z) = \phi(x) f(z)$ and formulate the eigenvalue problem for ϕ.
   (b) Investigate interface conditions.
   (c) Formulate (with complete argumentation) effective boundary conditions at $|x| = W$ for the problem on $|x| \leq W$.
   (d) Give the complete variational formulation of this problem for ϕ.

4. Consider the functional

$$F(u) = \int \left[r(x)(\partial_{xx}u)^2 - p(x)(\partial_x u)^2 + q(x)u^2 - f(x).u\right] dx,$$

   where r, p, q, f are given functions, on the set of functions on $[0, L]$ with boundary conditions

$$u(0) = \partial_x u(L) = 0.$$

   (a) Write down the Euler-Lagrange equation and the (natural) boundary conditions.
   (b) Give sensible meaning to this equation when there may be (at most) jump-discontinuities in (one of) the functions $r.p, q$.
   (c) Furthermore, derive the interface conditions at the points of discontinuity of these functions.