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Markov Processes and Related Fields

Asymptotic Period of an Aperiodic Markov Chain

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Abstract. We introduce the concept of *asymptotic period* for an irreducible and aperiodic, discrete-time Markov chain $\mathcal{X}$ on a countable state space, and develop the theory leading to its formal definition. The asymptotic period of $\mathcal{X}$ equals one – its period – if $\mathcal{X}$ is recurrent, but may be larger than one if $\mathcal{X}$ is transient; $\mathcal{X}$ is *asymptotically aperiodic* if its asymptotic period equals one. Some sufficient conditions for asymptotic aperiodicity are presented. The asymptotic period of a birth-death process on the nonnegative integers is studied in detail and shown to be equal to 1, 2 or ∞. Criteria for the occurrence of each value in terms of the 1-step transition probabilities are established.

Keywords: aperiodicity, birth-death process, harmonic function, period, transient Markov chain, transition probability

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1. Introduction

Let $P := (P(i,j),\ i,j \in S)$ be the matrix of 1-step transition probabilities of a homogeneous, discrete-time Markov chain $\mathcal{X} := \{X(n),\ n = 0,1,\ldots\}$ on a countably infinite state space S, so that the matrix $P^{(n)} := (P^{(n)}(i,j),\ i,j \in S)$ of n-step transition probabilities

$$P^{(n)}(i,j) := \Pr\{X(m+n) = j \mid X(m) = i\}, \quad i,j \in S,\ m,n = 0,1,\ldots,$$

is given by

$$P^{(n)} = P^n, \quad n = 0,1,\ldots.$$

We will assume throughout that $\mathcal{X}$ is stochastic, irreducible, and aperiodic.

Although the Markov chain $\mathcal{X}$ is aperiodic it may happen, if $\mathcal{X}$ is *transient*, that in the long run the process evolves cyclically through a finite number of sets constituting a partition of S. This phenomenon occurs for instance when $\mathcal{X}$ is a transient birth-death process on the nonnegative integers with only a finite number of positive self-transition probabilities, for in this case the process will eventually move cyclically between the even-numbered and the odd-numbered states. It seems natural then to say that the *asymptotic period* of $\mathcal{X}$ equals two or, perhaps, a multiple of two. In our general setting the *asymptotic period* of $\mathcal{X}$ may be defined as the maximum number of sets involved in the type of cyclic behaviour described above. In this paper these ideas will be formalized, and some of their consequences will be investigated.

After discussing preliminary concepts and results in Section 2 we formally define, in Section 3, the *asymptotic period* of a Markov chain that is, in a sense to be defined, *simple*. Some sufficient conditions for *asymptotic aperiodicity* will subsequently be derived. The framework developed in Section 2 draws heavily on the work of Blackwell [2] on transient Markov chains, while our definition of asymptotic period resembles in some aspects the definition of *period of an irreducible positive operator* by Moy [11], and is directly related to the definition of *asymptotic period of a tail sequence of subsets of* S, proposed by Abrahamse [1] in a setting that is more general than ours. Actually, Abrahamse introduces the concept of asymptotic period while generalizing Blackwell's results. Our further elaboration of the concept in a more restricted setting makes it more convenient for us to build directly on the foundations laid down by Blackwell.

In Section 4 we investigate asymptotic periodicity in the specific setting of a birth-death process on the nonnegative integers. We show that the asymptotic period equals 1, 2 or ∞, and identify the circumstances under which each value occurs in terms of the 1-step transition probabilities of the process. In particular, we establish a necessary and sufficient condition for asymptotic aperiodicity.

Our motivation for introducing the concept of *asymptotic aperiodicity* has been our aim to gain more insight into the *strong ratio limit property*, which is said to prevail if there exist positive constants R, $\mu(i),\ i \in S$, and $f(i),\ i \in S$, such that

$$\lim_{n\to\infty} \frac{P^{(n+m)}(i,j)}{P^{(n)}(k,l)} = R^{-m}\frac{f(i)\mu(j)}{f(k)\mu(l)}, \quad i,j,k,l \in S,\ m \in \mathbb{Z}. \tag{1.1}$$

The strong ratio limit property was enunciated in the setting of recurrent Markov chains by Orey [12], and introduced in the more general setting at hand by Pruitt [13]. More recently, Kesten [10] and Handelmann [7] have made substantial contributions, but a satisfactory solution to the problem of finding conditions for the strong ratio limit property is still lacking. Since *aperiodicity* is necessary and sufficient for a *positive recurrent* Markov chain to possess the strong ratio limit property, it is to be expected that *asymptotic aperiodicity* is a relevant property in the more general setting at hand. Actually, the relation between asymptotic aperiodicity and the strong ratio limit property is – at least in the general setting – not clear-cut. However, in a more restricted setting asymptotic aperiodicity has been shown in [5] to be sufficient for the strong ratio limit property.

We end this introduction with some notation and terminology. Namely, when $\mathcal{X}$ is a discrete-time birth-death process on the nonnegative integers – a process often encountered in what follows – we write

$$p_i := P(i, i+1), \quad q_{i+1} := P(i+1, i) \quad \text{and} \quad r_i := P(i, i), \quad i = 0, 1, \ldots, \tag{1.2}$$

for the birth, death and self-transition probabilities, respectively. It will be convenient to define $q_0 := 0$. Since $\mathcal{X}$ is stochastic, irreducible and aperiodic, we have $p_i > 0$, $q_{i+1} > 0$, and $r_i \geq 0$ for $i \geq 0$, with $r_i > 0$ for at least one state i, while $p_i + q_i + r_i = 1$ for $i \geq 0$. In what follows a *birth-death process* will always refer to a discrete-time birth-death process on the nonnegative integers.

2. Preliminaries

We start off by introducing some further notation and terminology related to the Markov chain $\mathcal{X} = \{X(n),\ n = 0, 1, \ldots\}$. By $\mathbb{P}$ we denote the probability measure on the set of sample paths induced by P and the (unspecified) initial distribution. Recall that a nonzero function f on S is called a *harmonic function* (or *invariant vector*) for P (or, for $\mathcal{X}$) if

$$Pf(i) := \sum_{j \in S} P(i, j) f(j) = f(i), \quad i \in S. \tag{2.1}$$

Evidently, in our setting the constant function is a harmonic function for P.

For $C \subset S$ we define the events

$$U(C) := \cap_{n=0}^{\infty} \cup_{k=n}^{\infty} \{X(k) \in C\} \quad \text{and} \quad L(C) := \cup_{n=0}^{\infty} \cap_{k=n}^{\infty} \{X(k) \in C\},$$

and we let

$$\mathcal{T} := \{C \subset S \,|\, U(C) \overset{\text{a.s.}}{=} \varnothing\},$$

that is, $C \in \mathcal{T}$ if $\mathbb{P}(X(n) \in C \text{ infinitely often}) = 0$, and

$$\mathcal{R} := \{C \subset S \,|\, U(C) \overset{\text{a.s.}}{=} L(C)\},$$

that is, $C \in \mathcal{R}$ if the events $\{X(n) \in C$ infinitely often$\}$ and $\{X(n) \in C$ for n sufficiently large$\}$ are almost surely equal. In the terminology of Revuz [14, Sect. 2.3] $\mathcal{T}$ is the collection of *transient* sets and $\mathcal{R}$ is the collection of *regular* sets. Evidently, $\mathcal{T} \subset \mathcal{R}$, while it is not difficult to see that $\mathcal{R}$ is closed under finite union and complementation, and hence a field. Note that $\mathcal{T}$ and $\mathcal{R}$ are independent of the initial distribution, since, by the irreducibility of $\mathcal{X}$, $\mathbb{P}(U(C))$

and $\mathbb{P}(U(C)\backslash L(C))$ are zero or positive for all initial states (and hence all initial distributions) simultaneously.

We will say that two regular sets C_1 and C_2 are *equivalent* if their symmetric difference $C_1 \Delta C_2 := (C_1 \cup C_2)\backslash(C_1 \cap C_2)$ is transient, and *almost disjoint* if their intersection $C_1 \cap C_2$ is transient. Following Blackwell [2] (see also Chung [3, Section I.17]), we call a subset $C \subset S$ *almost closed* if $C \notin \mathcal{T}$ and $C \in \mathcal{R}$. An almost closed set C is said to be *atomic* if C does not contain two disjoint almost closed subsets. The relevance of these concepts comes to light in the next theorem.

Theorem 2.1 (Blackwell [2]). *Associated with the Markov chain $\mathcal{X}$ is a finite or countable collection $\{C_1, C_2, \ldots\}$ of disjoint almost closed sets, which is unique up to equivalence and such that*

(i) *every C_i, except at most one, is atomic;*

(ii) *the nonatomic C_i, if present, contains no atomic subsets and consists of transient states;*

(iii) $\sum_i \mathbb{P}(L(C_i)) = 1$.

A collection of sets $\{C_1, C_2, \ldots\}$ satisfying the conditions in the theorem will be called a *Blackwell decomposition* (of S) for $\mathcal{X}$. A set $C \subset S$ is a *Blackwell component* (of S) for $\mathcal{X}$ if there exists a Blackwell decomposition for $\mathcal{X}$ such that C is one of the almost closed sets in the decomposition. The uniqueness up to equivalence of the Blackwell decomposition for $\mathcal{X}$ implies that if C_1 and C_2 are Blackwell components, then they are either equivalent or almost disjoint. The number of almost closed sets in the Blackwell decomposition for $\mathcal{X}$ will be denoted by $\beta(\mathcal{X})$. If $\beta(\mathcal{X}) = 1$ then $\mathcal{X}$ is called *simple*, and a simple Markov chain is called *atomic* or *nonatomic* according to the type of its state space. Evidently, if $\mathcal{X}$ is simple and nonatomic then S does not contain atomic subsets, but infinitely many disjoint almost closed subsets. It will be useful to observe the following.

Lemma 2.1. *Let $S = \{0, 1, \ldots\}$ and $\mathcal{X}$ have jumps that are uniformly bounded by M. Then $\beta(\mathcal{X}) \leq M$, and every Blackwell component for $\mathcal{X}$ is atomic.*

Proof. Let C be an almost closed set for $\mathcal{X}$ and let $s_1 < s_2 < \ldots$ denote the states of C. We claim that there exists a constant N such that for every $n \geq N$ we have $s_{n+1} \leq s_n + M$. Indeed, if $s_{n+1} > s_n + M$, then the process will leave C when it leaves the set $\{s_1, s_2, \ldots, s_n\}$. The irreducibility of S insures that a visit to this finite set of states will almost surely be followed by a departure from the set. So if, for each N, there is an integer $n \geq N$ such that $s_{n+1} > s_n + M$, then each entrance in C is almost surely followed by a departure from C, and hence $\mathbb{P}(L(C)) = 0$, contradicting the fact that C is almost closed.

Next, let $\{C_1, C_2, \ldots, C_\beta\}$, with $\beta \equiv \beta(\mathcal{X})$, be a Blackwell decomposition for $\mathcal{X}$, $s_1^{(i)} < s_2^{(i)} < \ldots$ the states of C_i, and N_i such that for every $n \geq N_i$ we have $s_{n+1}^{(i)} \leq s_n^{(i)} + M$. If $\beta > M$, then, choosing

$$s = \max_{1 \leq i \leq M+1} \{s_{N_i}^{(i)}\},$$

the set $\{s+1, s+2, \ldots, s+M\}$ must have a nonempty intersection with each of the disjoint sets $C_1, C_2, \ldots, C_{M+1}$, which is clearly impossible. Hence, $\beta \leq M$.

Finally, let C be a Blackwell component for $\mathcal{X}$ and suppose C is nonatomic. Then C contains infinitely many disjoint almost closed subsets, so we can choose $M+1$ disjoint almost closed subsets $C_1, C_2, \ldots, C_{M+1}$ of C. By the same argument as before there must be a state s in C such that each of the disjoint sets $C_1, C_2, \ldots, C_{M+1}$ shares a state with the set $\{s+1, s+2, \ldots, s+M\}$. This is impossible, so C must be atomic. $\Box$

A criterion for deciding whether a Markov chain is simple and atomic is given in the next theorem.

Theorem 2.2 (Blackwell [2]). *The Markov chain $\mathcal{X}$ is simple and atomic if and only if the only bounded harmonic function for $\mathcal{X}$ is the constant function.*

As an aside we note that when $\mathcal{X}$ is transient – the setting of primary interest to us – and the constant function is the only bounded harmonic function, then there is precisely one escape route to infinity, or, in the terminology of Hou and Guo [8] (see, in particular, Sections 7.13 and 7.16), the *exit space* of $\mathcal{X}$ contains exactly one *atomic exit point.*

Of course, the existence, up to a multiplicative constant, of a unique *bounded* harmonic function does not, in general, preclude the existence of an *unbounded* harmonic function. But when $\mathcal{X}$ is recurrent the constant function happens to be the only (bounded or unbounded) harmonic function (see, for example, Chung [3, Theorem I.7.6]). It follows in particular that $\mathcal{X}$ is simple and atomic if $\mathcal{X}$ is recurrent.

A function f on the space $\Omega := \{(\omega_0, \omega_1, \ldots) \,|\, \omega_i \in S,\ i = 0, 1, \ldots\}$ will be called *m-invariant* if, for every $\omega := (\omega_0, \omega_1, \ldots) \in \Omega,\ f(\omega) = f(\theta^m \omega)$, where θ is the shift operator $\theta(\omega_0, \omega_1, \ldots) = (\omega_1, \omega_2, \ldots)$, and $\theta^m \omega = \theta(\theta^{m-1}\omega)$. We also use the notation $\theta^m E := \{\theta^m \omega \,|\, \omega \in E\}$, for $E \subset \Omega$. An event is called *m-invariant* if its indicator function is m-invariant. A 1-invariant event is simply referred to as *invariant.* Evidently, the collection of invariant events constitutes a σ-field. We shall need another result of Blackwell's, involving invariant events (see [1, Theorem 5] for a generalization).

Theorem 2.3 (Blackwell [2]). *For any invariant event E there is a $C \in \mathcal{R}$ such that $E \stackrel{\text{a.s.}}{=} U(C)$.*

Note that the event $U(C)$ is actually invariant for *any* subset C of S, so for every $C \subset S$ there must be a *regular* set $\tilde{C}$ such that $U(C) \overset{\text{a.s.}}{=} U(\tilde{C})$. It follows in particular that every invariant event has probability zero or one if $\mathcal{X}$ is simple and atomic.

The regular set corresponding to an invariant event is unique up to equivalence. For if C_1 and C_2 are regular sets satisfying $U(C_1) \overset{\text{a.s.}}{=} U(C_2)$, then

$$U(C_1 \backslash C_2) \subset U(C_1) \backslash L(C_2) \overset{\text{a.s.}}{=} U(C_2) \backslash L(C_2) \overset{\text{a.s.}}{=} \varnothing,$$

and similarly with C_1 and C_2 interchanged. Since $U(C_1 \Delta C_2) \subset U(C_1 \backslash C_2) \cup U(C_2 \backslash C_1)$, it follows that $C_1 \Delta C_2$ must be transient. So, up to events of probability zero, the σ-field of invariant events is identical with the σ-field of events of the form $U(C)$ with $C \in \mathcal{R}$.

Theorem 2.3 plays a crucial role in the proof of Theorem 2.4, which involves $\mathcal{X}^{(m)} := \{X^{(m)}(n) \equiv X(nm),\ n = 0, 1, \ldots\}$, the m-step Markov chain associated with $\mathcal{X}$, and is instrumental in our definition of asymptotic period. For $C \subset S$ we let

$$U^{(m)}(C) := \cap_{n=0}^{\infty} \cup_{k=n}^{\infty} \{X(km) \in C\} \text{ and } L^{(m)}(C) := \cup_{n=0}^{\infty} \cap_{k=n}^{\infty} \{X(km) \in C\},$$

so that $U^{(1)}(C) = U(C)$ and $L^{(1)}(C) = L(C)$. Since $E = \theta^m E$ if (and only if) E is m-invariant, we have $\theta^m U^{(m)}(C) = U^{(m)}(C)$. But actually we have, more generally,

$$\theta^{m-j} U^{(m)}(C) = \cap_{n=0}^{\infty} \cup_{k=n}^{\infty} \{X(km + j) \in C\}, \quad j = 0, 1, \ldots, m-1, \tag{2.2}$$

and

$$\theta^{m-j} L^{(m)}(C) = \cup_{n=0}^{\infty} \cap_{k=n}^{\infty} \{X(km + j) \in C\}, \quad j = 0, 1, \ldots, m-1, \tag{2.3}$$

as can easily be verified. The following simple observation will prove useful.

Lemma 2.2. *Let E be an m-invariant event for some $m \geq 1$. Then, for all $i \geq 1$,*

$$E \overset{\text{a.s.}}{=} \varnothing \iff \theta^i E \overset{\text{a.s.}}{=} \varnothing.$$

Proof. If $\mathbb{P}(E) > 0$ there must be a state s, say, such that $\mathbb{P}(E \,|\, X(0) = s) > 0$. Moreover, aperiodicity and irreducibility of the chain imply that there is an integer k such that $\mathbb{P}(X(km-i) = s) > 0$. Since, by Theorem 2.3, $E \overset{\text{a.s.}}{=} U^{(m)}(C)$ for some set C, we obviously have $\mathbb{P}(\theta^i E \,|\, X(km-i) = s) = \mathbb{P}(E \,|\, X(0) = s)$. Hence, if $\mathbb{P}(E) > 0$, then

$$\begin{aligned} \mathbb{P}(\theta^i E) &\geq \mathbb{P}(\theta^i E \,|\, X(km-i) = s) \mathbb{P}(X(km-i) = s) \\ &= \mathbb{P}(E \,|\, X(0) = s) \mathbb{P}(X(km-i) = s) > 0. \end{aligned}$$

The same argument with E and $\theta^i E$ interchanged and $km-i$ replaced by $km+i$ yields the converse. $\square$

Before stating and proving Theorem 2.4 we establish some additional auxiliary lemmas. In what follows we write $E \overset{\text{a.s.}}{\subset} F$ for $E\backslash F \overset{\text{a.s.}}{=} \varnothing$.

Lemma 2.3. *Let E_1 and E_2 be m-invariant events for some $m \geq 1$. Then, for all $i \geq 0$ and $j \geq 0$,*

(i) $E_1 \overset{\text{a.s.}}{\subset} \theta^j E_2 \iff \theta^i E_1 \overset{\text{a.s.}}{\subset} \theta^{i+j} E_2,$

(ii) $E_1 \overset{\text{a.s.}}{=} \theta^j E_2 \iff \theta^i E_1 \overset{\text{a.s.}}{=} \theta^{i+j} E_2.$

Proof. The event $E_1\backslash\theta^j E_2$ is m-invariant, so, by Lemma 2.2, we have

$$E_1\backslash\theta^j E_2 \overset{\text{a.s.}}{=} \varnothing \iff \theta^i(E_1\backslash\theta^j E_2) \overset{\text{a.s.}}{=} \varnothing,$$

which implies the first statement. Moreover, the first statement remains valid, by a similar argument, if we interchange the sets E_1 and $\theta^j E_2$. Combining both results yields the second statement. □

Note that the second statement of this lemma generalizes Lemma 2.2. The next auxiliary result is a straightforward corollary of the previous lemma.

Lemma 2.4. *Let E be an m-invariant event for some $m \geq 1$. Then, for all $j \geq 0$ and $k_2 \geq k_1 \geq 0$,*

(i) $E \overset{\text{a.s.}}{\subset} \theta^j E \Rightarrow \theta^{k_1 j} E \overset{\text{a.s.}}{\subset} \theta^{k_2 j} E,$

(ii) $E \overset{\text{a.s.}}{=} \theta^j E \Rightarrow \theta^{k_1 j} E \overset{\text{a.s.}}{=} \theta^{k_2 j} E.$

Our final preparatory lemma is the following.

Lemma 2.5. *Let C_1 and C_2 be subsets of S that are regular with respect to $\mathcal{X}^{(m)}$ for some $m \geq 1$. Then*

$$U^{(m)}(C_1 \cap C_2) \overset{\text{a.s.}}{=} U^{(m)}(C_1) \cap U^{(m)}(C_2).$$

Proof. We clearly have

$$U^{(m)}(C_1 \cap C_2) \subset U^{(m)}(C_1) \cap U^{(m)}(C_2) \overset{\text{a.s.}}{=} L^{(m)}(C_1) \cap L^{(m)}(C_2).$$

Since

$$L^{(m)}(C_1) \cap L^{(m)}(C_2) = L^{(m)}(C_1 \cap C_2) \subset U^{(m)}(C_1 \cap C_2),$$

the result follows. □

Theorem 2.4. *If $\mathcal{X}$ is simple and atomic and $m > 1$, then $\beta \equiv \beta(\mathcal{X}^{(m)})$ is a divisor of m and the Blackwell decomposition for $\mathcal{X}^{(m)}$ consists of a collection $\{C_0, C_1, \ldots, C_{\beta-1}\}$ of disjoint atomic almost closed sets, which can be chosen such that, for each $i = 0, 1, \ldots, \beta - 1$,*

$$\mathbb{P}(\theta^j L^{(m)}(C_{i+1\,(\mathrm{mod}\,\beta)}) \,|\, \theta^{j+1} L^{(m)}(C_i)) = 1, \quad j = 0, 1, \ldots, m-1. \tag{2.4}$$

If $\mathcal{X}$ is simple and nonatomic, then $\mathcal{X}^{(m)}$ is simple and nonatomic for all $m \geq 1$.

Proof. First suppose $\mathcal{X}$ is simple and atomic. Let C_0 be a Blackwell component for $\mathcal{X}^{(m)}$ and assume, for the time being, that C_0 is atomic. Since $\theta^i U^{(m)}(C_0)$ is m-invariant for all i, we can apply Theorem 2.3 to $\mathcal{X}^{(m)}$ and conclude that there is a sequence $C_1, C_2, \ldots$ of regular sets (with respect to $\mathcal{X}^{(m)}$) such that

$$\theta^i U^{(m)}(C_0) \overset{\text{a.s.}}{=} U^{(m)}(C_i), \quad i = 1, 2, \ldots. \tag{2.5}$$

By Lemma 2.2 the sets C_i are almost closed, since C_0 is almost closed. Also, by Lemma 2.3, we have

$$U^{(m)}(C_{i+1}) \overset{\text{a.s.}}{=} \theta^{i+1} U^{(m)}(C_0) \overset{\text{a.s.}}{=} \theta U^{(m)}(C_i),$$

and hence

$$L^{(m)}(C_{i+1}) \overset{\text{a.s.}}{=} \theta L^{(m)}(C_i), \quad i = 0, 1, 2, \ldots. \tag{2.6}$$

Next defining

$$b := \min\{i \geq 1 \,|\, \theta^i U^{(m)}(C_0) \overset{\text{a.s.}}{=} U^{(m)}(C_0)\}, \tag{2.7}$$

we have $b \leq m$ since $U^{(m)}(C_0)$ is m-invariant. Also, b must be a divisor of m, for otherwise, by Lemma 2.4, we would have

$$U^{(m)}(C_0) \overset{\text{a.s.}}{=} \theta^{\ell b} U^{(m)}(C_0) = \theta^{m+i} U^{(m)}(C_0) = \theta^i U^{(m)}(C_0),$$

with $\ell = \min\{k \in \mathbb{N} \,|\, kb > m\}$ and $i = \ell b - m < b$, contradicting (2.7). For $i \geq b$ we have, by Lemma 2.3,

$$U^{(m)}(C_i) \overset{\text{a.s.}}{=} \theta^i U^{(m)}(C_0) \overset{\text{a.s.}}{=} \theta^{i-b} U^{(m)}(C_0) \overset{\text{a.s.}}{=} U^{(m)}(C_{i-b}),$$

so that C_i and C_{i-b} are equivalent (with respect to $\mathcal{X}^{(m)}$). We can therefore replace (2.6) by

$$L^{(m)}(C_{i+1\,(\mathrm{mod}\,b)}) \overset{\text{a.s.}}{=} \theta L^{(m)}(C_i), \quad i = 0, 1, 2, \ldots, b-1.$$

But in view of Lemma 2.3 we can actually write, for any value of j,

$$\theta^j L^{(m)}(C_{i+1\,(\mathrm{mod}\,b)}) \overset{\text{a.s.}}{=} \theta^{j+1} L^{(m)}(C_i), \quad i = 0, 1, 2, \ldots, b-1, \tag{2.8}$$

so that (2.4) prevails for $i = 0, 1, \ldots, b-1$.

Our next step will be to prove that the sets $C_0, C_1, \ldots, C_{b-1}$ are almost disjoint. Since the collection of sets that are regular with respect to $\mathcal{X}^{(m)}$ constitutes a field, the sets $C_0 \backslash C_i$ and $C_0 \cap C_i$, with $0 < i < b$, are regular. But C_0, being an atomic Blackwell component for $\mathcal{X}^{(m)}$, cannot contain two almost closed subsets, so that either $C_0 \backslash C_i$ or $C_0 \cap C_i$ must be transient. If $C_0 \backslash C_i$ is transient, then

$$U^{(m)}(C_0) \backslash U^{(m)}(C_i) \subset U^{(m)}(C_0 \backslash C_i) \overset{\text{a.s.}}{=} \varnothing,$$

which implies

$$U^{(m)}(C_0) \overset{\text{a.s.}}{=} U^{(m)}(C_0) \cap U^{(m)}(C_i) \subset U^{(m)}(C_i) \overset{\text{a.s.}}{=} \theta^i U^{(m)}(C_0),$$

that is, $U^{(m)}(C_0) \overset{\text{a.s.}}{\subset} \theta^i U^{(m)}(C_0)$. But then, by Lemma 2.4,

$$\theta^i U^{(m)}(C_0) \overset{\text{a.s.}}{\subset} \theta^{bi} U^{(m)}(C_0) \overset{\text{a.s.}}{=} U^{(m)}(C_0),$$

so that $U^{(m)}(C_0) \overset{\text{a.s.}}{=} \theta^i U^{(m)}(C_0)$, contradicting (2.7). So we conclude, for $0 < i < b$, that $C_0 \cap C_i$ is transient, and hence that C_0 and C_i, are almost disjoint. It subsequently follows that C_i and C_j, with $0 \le i < j < b$, are also almost disjoint. Indeed, C_0 and C_{j-i} being almost disjoint, we have, by Lemma 2.5,

$$\begin{aligned} U^{(m)}(C_0) \cap \theta^{j-i} U^{(m)}(C_0) &\overset{\text{a.s.}}{=} U^{(m)}(C_0) \cap U^{(m)}(C_{j-i}) \\ &\overset{\text{a.s.}}{=} U^{(m)}(C_0 \cap C_{j-i}) \overset{\text{a.s.}}{=} \varnothing. \end{aligned}$$

Hence, by Lemma 2.2 and Lemma 2.5,

$$\begin{aligned} U^{(m)}(C_i \cap C_j) &\overset{\text{a.s.}}{=} U^{(m)}(C_i) \cap U^{(m)}(C_j) \\ &\overset{\text{a.s.}}{=} \theta^i \left(U^{(m)}(C_0) \cap \theta^{j-i} U^{(m)}(C_0) \right) \overset{\text{a.s.}}{=} \varnothing, \end{aligned}$$

establishing our claim. It is no restriction of generality to assume that the sets $C_0, C_1, \ldots, C_{b-1}$ are actually *disjoint* (rather than *almost disjoint*), since replacing C_i by the equivalent set C'_i, where $C'_0 = C_0$ and $C'_i = C_i \backslash \cup_{j<i} C_j$, $i = 1, \ldots, b-1$, does not disturb the validity of (2.5).

Our next step will be to show that $\{C_0, C_1, \ldots, C_{b-1}\}$ constitutes a Blackwell decomposition for $\mathcal{X}^{(m)}$, still assuming the Blackwell component C_0 to be atomic. First note that $\cup_{i=0}^{b-1} C_i$ is regular with respect to $\mathcal{X}$. Indeed, by definition of C_i and in view of (2.2) and (2.3), we have

$$\begin{aligned} U(\cup_{i=0}^{b-1} C_i) &= \cup_{i=0}^{b-1} \cup_{j=0}^{m-1} \theta^{m-j} U^{(m)}(C_i) \\ &\overset{\text{a.s.}}{=} \cup_{i=0}^{b-1} \cup_{j=0}^{m-1} \theta^{m-j} L^{(m)}(C_i) \overset{\text{a.s.}}{\subset} L(\cup_{i=0}^{b-1} C_i), \end{aligned}$$

so that $U(\cup_{i=0}^{b-1}C_i) \stackrel{\text{a.s.}}{=} L(\cup_{i=0}^{b-1}C_i)$. Moreover,

$$\mathbb{P}(U(\cup_{i=0}^{b-1}C_i)) \geq \mathbb{P}(U^{(m)}(\cup_{i=0}^{b-1}C_i)) \geq \mathbb{P}(U^{(m)}(C_0)) > 0,$$

so $\cup_{i=0}^{b-1}C_i$ is in fact almost closed. It follows, $\mathcal{X}$ being simple and atomic, that $\cup_{i=0}^{b-1}C_i$ and S are equivalent with respect to $\mathcal{X}$. As a consequence

$$\mathbb{P}(U^{(m)}(S\setminus \cup_{i=0}^{b-1} C_i)) \leq \mathbb{P}(U(S\setminus \cup_{i=0}^{b-1} C_i)) = 0,$$

that is, $\cup_{i=0}^{b-1}C_i$ and S are also equivalent with respect to $\mathcal{X}^{(m)}$. Hence

$$\sum_{i=0}^{b-1}\mathbb{P}(U^{(m)}(C_i)) \geq \mathbb{P}(U^{(m)}(\cup_{i=0}^{b-1}C_i)) \geq 1 - \mathbb{P}(U^{(m)}(S\setminus \cup_{i=0}^{b-1} C_i)) = 1,$$

so that

$$\sum_{i=0}^{b-1}\mathbb{P}(L^{(m)}(C_i)) = \sum_{i=0}^{b-1}\mathbb{P}(U^{(m)}(C_i)) = 1.$$

If $b = 1$ then C_0 and S are equivalent with respect to $\mathcal{X}^{(m)}$, so that $\beta(\mathcal{X}^{(m)}) = 1$, and we are done. So suppose $b > 1$ and let Γ be an arbitrary almost closed subset of $C_i,\ 0 < i < b$. Since $\theta^{b-i}U^{(m)}(\Gamma)$ is invariant with respect to $\mathcal{X}^{(m)}$, there exists, by Theorem 2.3, a regular set Γ_0 such that $\theta^{b-i}U^{(m)}(\Gamma) \stackrel{\text{a.s.}}{=} U^{(m)}(\Gamma_0)$. Lemma 2.2 implies that Γ_0 is almost closed, while, by (2.8),

$$U^{(m)}(\Gamma_0) \stackrel{\text{a.s.}}{\subset} U^{(m)}(C_0).$$

But since C_0 is atomic, we must actually have $U^{(m)}(\Gamma_0) \stackrel{\text{a.s.}}{=} U^{(m)}(C_0)$. Hence, by Lemma 2.3,

$$U^{(m)}(\Gamma) = \theta^i(\theta^{b-i}U^{(m)}(\Gamma)) \stackrel{\text{a.s.}}{=} \theta^i U^{(m)}(\Gamma_0) \stackrel{\text{a.s.}}{=} \theta^i U^{(m)}(C_0) \stackrel{\text{a.s.}}{=} U^{(m)}(C_i),$$

so that Γ and C_i are equivalent. Hence C_i is atomic. So we conclude that if C_0 is atomic then $\{C_0, C_1, \ldots, C_{b-1}\}$ constitutes a Blackwell decomposition for $\mathcal{X}^{(m)}$ (with atomic components) and hence $\beta \equiv \beta(\mathcal{X}^{(m)}) = b$, a divisor of m.

We will now show that, in fact, each component in the Blackwell decomposition for $\mathcal{X}^{(m)}$ has to be atomic if $\mathcal{X}$ is simple and atomic. If $\beta(\mathcal{X}^{(m)}) > 1$, we could replace C_0 in the preceding argument by an atomic Blackwell component for $\mathcal{X}^{(m)}$, and subsequently reach a contradiction, since all the components in the Blackwell decomposition for $\mathcal{X}^{(m)}$ have to be atomic if C_0 is atomic. So it remains to consider the case $\beta(\mathcal{X}^{(m)}) = 1$. Assuming S to be nonatomic with respect to $\mathcal{X}^{(m)}$, there are almost closed sets that are *not* equivalent to S. Let Γ_0 be such a set. Then, by Theorem 2.3, there are sets Γ_i, regular with respect to $\mathcal{X}^{(m)}$ and unique up to equivalence, such that

$$\theta^i U^{(m)}(\Gamma_0) \stackrel{\text{a.s.}}{=} U^{(m)}(\Gamma_i), \quad i = 1, 2, \ldots.$$

Copying the argument following (2.5) up to and including (2.8) with C_i replaced by Γ_i, we conclude from the analogue of (2.8) that $\cup_{i=0}^{b-1}\Gamma_i$ is regular with respect to $\mathcal{X}$, while

$$\mathbb{P}(U(\cup_{i=0}^{b-1}\Gamma_i)) \geq \mathbb{P}(U^{(m)}(\cup_{i=0}^{b-1}\Gamma_i)) \geq \mathbb{P}(U^{(m)}(\Gamma_0)) > 0.$$

So $\cup_{i=0}^{b-1}\Gamma_i$ is in fact almost closed, and it follows, $\mathcal{X}$ being simple and atomic, that $\cup_{i=0}^{b-1}\Gamma_i$ and S are equivalent with respect to $\mathcal{X}$.

It is no restriction of generality to assume that the sets $\Gamma_0, \Gamma_1, \ldots, \Gamma_{b-1}$ are disjoint. Indeed, $\Gamma_0 \backslash \Gamma_i$ cannot be transient, by the same argument we have used earlier for $C_0 \backslash C_i$. Hence, the collection of regular sets constituting a field, $\Gamma_0 \backslash \Gamma_i$ must be almost closed with respect to $\mathcal{X}^{(m)}$. So, if $\Gamma_0 \cap \Gamma_i$ is *not* transient, we may replace Γ_0 by $\Gamma_0 \backslash \Gamma_i$ in the preceding argument and end up with new sets $\Gamma_0, \Gamma_1, \ldots, \Gamma_{b-1}$ such that $\Gamma_0 \cap \Gamma_i$ is transient. Repeating the procedure if necessary, we reach, after less than b steps, a situation in which $\Gamma_0 \cap \Gamma_i$ is transient for each $i < b$. It follows, by the same argument we have used before for the C_i's, that *all* Γ_i's are almost disjoint and by a similar adaptation as before for the C_i's we can actually make them disjoint without essentially changing the situation. But if $\Gamma_0, \Gamma_1, \ldots, \Gamma_{b-1}$ are disjoint almost closed sets such that the analogue of (2.8) is satisfied, and $\cup_{i=0}^{b-1}\Gamma_i$ and S are equivalent with respect to $\mathcal{X}$, then $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_{b-1}\}$ constitutes a Blackwell decomposition for $\mathcal{X}^{(m)}$, which, since $\beta(\mathcal{X}^{(m)}) = 1$, implies $b = 1$, and hence that Γ_0 and S are equivalent, contradicting our assumption on Γ_0. So if $\mathcal{X}$ is simple and atomic and $\beta(\mathcal{X}^{(m)}) = 1$, then S has to be atomic. Summarizing we conclude that every component in the Blackwell decomposition of S for $\mathcal{X}^{(m)}$ must be atomic if $\mathcal{X}$ is simple and atomic.

Finally, suppose $\mathcal{X}$ is simple and nonatomic. Evidently, each subset of S that is almost closed with respect to $\mathcal{X}$ contains a subset that is almost closed with respect to $\mathcal{X}^{(m)}$, and it follows that a nonatomic almost closed set with respect to $\mathcal{X}$ must contain a nonatomic almost closed set with respect to $\mathcal{X}^{(m)}$. So S must contain a nonatomic almost closed set with respect to $\mathcal{X}^{(m)}$. We have seen that all components in the Blackwell decomposition of S for $\mathcal{X}^{(m)}$ must be atomic if $\beta(\mathcal{X}^{(m)}) > 1$, so the only remaining possibility is that $\mathcal{X}^{(m)}$ is simple and nonatomic. □

Note that (2.4) is equivalent to stating that for $j = 0, 1, \ldots, m-1$,

$$\begin{aligned} &\{X(km + j) \in C_i \text{ for } k \text{ sufficiently large}\} \\ &\qquad \stackrel{\text{a.s.}}{=} \{X(km + j + 1) \in C_{i+1\,(\mathrm{mod}\,\beta)} \text{ for } k \text{ sufficiently large}\}. \end{aligned}$$

In what follows we will refer to a Blackwell decomposition of S for $\mathcal{X}^{(m)}$ with this property as a *cyclic* decomposition.

Theorem 2.4 provides the framework for the formal definition of the asymptotic period of a simple Markov chain in the next section. We conclude this section with a series of lemmas and corollaries, which supply further information on $\beta(\mathcal{X}^{(m)})$.

Lemma 2.6. *Let $\mathcal{X}$ be simple and atomic, and $m \geq 1$. Then a Blackwell component for $\mathcal{X}^{(m)}$ is almost closed with respect to $\mathcal{X}^{(k\beta)}$ for all $k \geq 1$, where $\beta \equiv \beta(\mathcal{X}^{(m)})$. Also, $\beta(\mathcal{X}^{(\beta)}) = \beta$.*

Proof. Let C be a Blackwell component for $\mathcal{X}^{(m)}$. As a consequence of (2.4) we have $U^{(\beta)}(C) \stackrel{\text{a.s.}}{=} L^{(\beta)}(C)$, and hence $U^{(k\beta)}(C) \stackrel{\text{a.s.}}{=} L^{(k\beta)}(C)$ for any $k \geq 1$. Also,

$$\mathbb{P}(L^{(k\beta)}(C)) \geq \mathbb{P}(L^{(\beta)}(C)) = \mathbb{P}(U^{(\beta)}(C)) \geq \mathbb{P}(U^{(m)}(C)) > 0,$$

since β is a divisor of m. So we conclude that C is almost closed with respect to $\mathcal{X}^{(k\beta)}$. It follows in particular that a Blackwell component for $\mathcal{X}^{(m)}$ must contain a Blackwell component for $\mathcal{X}^{(\beta)}$. Hence $\beta(\mathcal{X}^{(\beta)}) \geq \beta$, and so $\beta(\mathcal{X}^{(\beta)}) = \beta$, since $\beta(\mathcal{X}^{(\beta)})$ is a divisor of β. $\square$

The following corollary is immediate.

Corollary 2.1. *Let $\mathcal{X}$ be simple. If $\beta(\mathcal{X}^{(m)}) < m$ for all $m > 1$, then $\beta(\mathcal{X}^{(m)}) = 1$ for all m.*

Lemma 2.7. *Let $\mathcal{X}$ be simple and $k, \ell \geq 1$. Then $\beta(\mathcal{X}^{(k\ell)}) = \kappa\beta(\mathcal{X}^{(\ell)})$, where κ is a divisor of $\beta(\mathcal{X}^{(k)})$.*

Proof. If $\mathcal{X}$ is nonatomic then, by Theorem 2.4, $\beta(\mathcal{X}^{(m)}) = 1$ for all m, so that the statement is trivially true. So let us assume that $\mathcal{X}$ is simple and atomic. We write $\beta_\ell \equiv \beta(\mathcal{X}^{(\ell)})$, and denote the (atomic) Blackwell components for $\mathcal{X}^{(\ell)}$ by $B_0, B_1, \ldots, B_{\beta_\ell}$. By the previous lemma these sets are almost closed with respect to $\mathcal{X}^{(k\ell)}$, so each B_i must contain at least one Blackwell component for $\mathcal{X}^{(k\ell)}$. Let $C_0 \subset B_0$ be such a Blackwell component and consider the sets C_i defined in the proof of Theorem 2.4 in terms of C_0 and $m = k\ell$. We let

$$\kappa := \min\{k \geq 1 \,|\, \theta^{k\beta_\ell} U^{(k\ell)}(C_0) \stackrel{\text{a.s.}}{=} U^{(k\ell)}(C_0)\},$$

and claim that $\kappa\beta_\ell = \beta(\mathcal{X}^{(k\ell)})$.

To prove the claim we first note that part of the proof of Theorem 2.4 can be copied to show that the sets $C_0, C_{\beta_\ell}, \ldots, C_{(\kappa-1)\beta_\ell}$ are almost disjoint, while, for $i \geq \kappa$, the sets $C_{i\beta_\ell}$ and $C_{(i-\kappa)\beta_\ell}$ are equivalent with respect to $\mathcal{X}^{(m)}$. Since B_0 is a Blackwell component for $\mathcal{X}^{(\ell)}$ and $C_0 \subset B_0$, we have $\cup_{i=0}^{k-1} C_{i\beta_\ell} \subset B_0$. But, again in analogy with part of the proof of Theorem 2.4, it is easily seen that $\cup_{i=0}^{\kappa-1} C_{i\beta_\ell}$ is almost closed with respect to $\mathcal{X}^{(\ell)}$, so, B_0 being atomic, we actually have $\cup_{i=0}^{\kappa-1} C_{i\beta_\ell} \stackrel{\text{a.s.}}{=} B_0$. As in the proof of Theorem 2.4 it is no restriction to assume that the sets $C_0, C_{\beta_\ell}, \ldots, C_{(\kappa-1)\beta_\ell}$ are disjoint rather than almost disjoint.

Assuming that the Blackwell components for $\mathcal{X}^{(\ell)}$ are suitably numbered, we have $C_1 \subset B_1$ and the preceding argument can be repeated to show that the sets $C_1, C_{\beta_\ell+1}, \ldots, C_{(\kappa-1)\beta_\ell+1}$ are disjoint, while $\cup_{i=0}^{\kappa-1} C_{i\beta_\ell+1} \stackrel{\text{a.s.}}{=} B_1$. Thus proceeding it follows eventually that $\{C_0, C_1, \ldots, C_{\kappa\beta_\ell-1}\}$ constitutes a Blackwell decomposition of S for $\mathcal{X}^{(m)}$, so that $\beta(\mathcal{X}^{(m)}) = \kappa\beta_\ell$, as claimed.

We finally observe that the κ sets $\cup_{i=0}^{\beta_\ell-1} C_{i\kappa+j}$, $j = 0, 1, \ldots, \kappa-1$, are almost closed with respect to $\mathcal{X}^{(k)}$, so that κ must be a divisor of $\beta(\mathcal{X}^{(k)})$. □

This lemma has some interesting and useful corollaries, of which the first is immediate.

Corollary 2.2. *Let $\mathcal{X}$ be simple and $m > 1$. If ℓ is a divisor of m, then $\beta(\mathcal{X}^{(\ell)})$ is a divisor of $\beta(\mathcal{X}^{(m)})$.*

Corollary 2.3. *Let $\mathcal{X}$ be simple and $m > 1$. If $\beta(\mathcal{X}^{(m)}) = m$, then $\beta(\mathcal{X}^{(\ell)}) = \ell$ for all divisors ℓ of m.*

Proof. Let $m = k\ell$. Then, by Lemma 2.7,

$$\beta(\mathcal{X}^{(m)}) = k\ell = \kappa\beta(\mathcal{X}^{(\ell)}),$$

with κ a divisor of $\beta(\mathcal{X}^{(k)})$, and, hence, by Theorem 2.4, of k. Since $\beta(\mathcal{X}^{(\ell)})$ is a divisor of ℓ we must have $\kappa = k$ and $\beta(\mathcal{X}^{(\ell)}) = \ell$. □

Corollary 2.4. *Let $\mathcal{X}$ be simple and $k,\ \ell \geq 1$. Then $\beta(\mathcal{X}^{(k\ell)}) = \beta(\mathcal{X}^{(k)})\beta(\mathcal{X}^{(\ell)})$ if $\beta(\mathcal{X}^{(k)})$ and $\beta(\mathcal{X}^{(\ell)})$ are relatively prime.*

Proof. By Lemma 2.7 we have $\beta(\mathcal{X}^{(k\ell)}) = \kappa\beta(\mathcal{X}^{(\ell)}) = \lambda\beta(\mathcal{X}^{(k)})$, with κ a divisor of $\beta(\mathcal{X}^{(k)})$ and λ a divisor of $\beta(\mathcal{X}^{(\ell)})$. But if $\beta(\mathcal{X}^{(k)})$ and $\beta(\mathcal{X}^{(\ell)})$ are relatively prime this is possible only if $\kappa = \beta(\mathcal{X}^{(k)})$ and $\lambda = \beta(\mathcal{X}^{(\ell)})$. □

3. Asymptotic period

We are now ready to formally define the *asymptotic period* of a *simple* Markov chain. As in the previous section, $\mathcal{X}$ denotes the Markov chain of Section 1, and is, accordingly, stochastic, irreducible, and aperiodic.

Definition 3.1. Let the Markov chain $\mathcal{X}$ be simple. The *asymptotic period* of $\mathcal{X}$ is given by

$$d(\mathcal{X}) := \sup\{m \geq 1 \,|\, \beta(\mathcal{X}^{(m)}) = m\}; \tag{3.1}$$

$\mathcal{X}$ is *asymptotically aperiodic* if $d(\mathcal{X}) = 1$, otherwise $\mathcal{X}$ is *asymptotically periodic* with asymptotic period $d(\mathcal{X}) > 1$.

We shall see that it is possible for $\mathcal{X}$ to have $d(\mathcal{X}) = \infty$.

If, for some m, we would have $\beta \equiv \beta(\mathcal{X}^{(m)}) > d(\mathcal{X})$, then, by Lemma 2.6, $\beta(\mathcal{X}^{(\beta)}) = \beta > d(\mathcal{X})$, which is a contradiction. So we actually have the following result, which formalizes the intuitive concept of asymptotic period put forward in the introduction.

Theorem 3.1. *The asymptotic period of a simple Markov chain $\mathcal{X}$ satisfies*

$$d(\mathcal{X}) = \sup\{\beta(\mathcal{X}^{(m)}) \mid m \geq 1\}. \tag{3.2}$$

From Theorem 2.4 we immediately conclude the following.

Theorem 3.2. *If $\mathcal{X}$ is simple and nonatomic then $\mathcal{X}$ is asymptotically aperiodic.*

The next theorem confirms the intimation in the introduction that an asymptotic period larger than one requires the chain to be transient.

Theorem 3.3. *If $\mathcal{X}$ is simple and recurrent then $\mathcal{X}$ is asymptotically aperiodic.*

Proof. Suppose $d(\mathcal{X}) > 1$, so that $\beta(\mathcal{X}^{(m)}) = m$ for some $m > 1$. Let $\{C_0, C_1, \ldots, C_{m-1}\}$ be a cyclic Blackwell decomposition for $\mathcal{X}^{(m)}$, and choose $i \in C_0$. As a consequence of Theorem 2.4 and the recurrence of state i we must have $P^{(\ell m+1)}(i, C_1) = 1$ for all ℓ. On the other hand, the aperiodicity of i implies $P^{(\ell m+1)}(i, i) > 0$ for ℓ sufficiently large, contradicting the fact that C_0 and C_1 are disjoint. So $\mathcal{X}$ must be asymptotically aperiodic if it is recurrent. □

An example of a chain with an asymptotic period greater than 1 is obtained by letting $\mathcal{X}$ be a transient birth-death process on the nonnegative integers (as defined in the introduction) with self-transition probabilities $r_i = 0$ except $r_0 = 1 - p_0 > 0$. Clearly, $\mathcal{X}$ is irreducible and aperiodic, while Lemma 2.1 implies that $\mathcal{X}$ is simple (and atomic). But it is readily seen that $\beta(\mathcal{X}^{(2)}) = 2$, so that $d(\mathcal{X}) > 1$. (We will see in the next section that, actually, $d(\mathcal{X}) = 2$.)

It is possible for the asymptotic period of a Markov chain to be infinity. Indeed, let us assume that the birth probabilities p_i in a birth-death process are such that $\prod_{i=0}^{\infty} p_i > 0$. Then there is a probability $\prod_{i=j}^{\infty} p_i \geq \prod_{i=0}^{\infty} p_i$ that a visit to state j is followed solely by jumps to the right. Hence, with probability one, the process will make only a finite number of self-transitions or jumps to the left. It follows that the sets $C_i := \{i, n+i, 2n+i, \ldots\}$, $i = 0, 1, \ldots, n-1$, are (disjoint) atomic almost closed sets with respect to $\mathcal{X}^{(n)}$, so that $\beta(\mathcal{X}^{(n)}) = n$ for all n and, hence, $d(\mathcal{X}) = \infty$.

Some further conditions for a simple Markov chain to be asymptotically aperiodic are given next.

Theorem 3.4. *Let $\mathcal{X}$ be a simple Markov chain. Then the following are equivalent:*

(i) $\mathcal{X}$ *is asymptotically aperiodic;*

(ii) $\mathcal{X}^{(m)}$ *is simple for all* $m > 1$*;*

(iii) $\mathcal{X}^{(m)}$ *is simple for all prime numbers* m*.*

Proof. By Corollary 2.1 the first statement implies the second. Evidently, the second statement implies the third. To show that the third statement implies the first, suppose $\beta(\mathcal{X}^{(m)}) = 1$ for all primes m. If $d \equiv d(\mathcal{X}) > 1$, then $\beta(\mathcal{X}^{(d)}) = d$ and d must have a prime factor $p > 1$. But then, by Corollary 2.4, $\beta(\mathcal{X}^{(p)}) = p$, which is impossible. □

It may be desirable to have an upper bound on the asymptotic period of a Markov chain. The next theorem, involving the condition

$$\text{there exists a constant } \delta > 0 \text{ such that } P^{(n)}(i,i) \geq \delta \text{ for all but finitely many states } i \in S, \tag{3.3}$$

provides a criterion which may be used for this purpose.

Theorem 3.5. *If, for some* n*, the simple Markov chain* $\mathcal{X}$ *satisfies condition* (3.3)*, then* $d(\mathcal{X})$ *is a divisor of* n*.*

Proof. In view of Theorem 3.3 we may assume that $\mathcal{X}$ is transient. Suppose $\beta(\mathcal{X}^{(m)}) = m$ for some $m \geq 1$, and let $\{C_0, C_1, \ldots, C_{m-1}\}$ be a cyclic Blackwell decomposition for $\mathcal{X}^{(m)}$. If (3.3) holds, then $P^{(n)}(i, C_0) \geq \delta$ for all but finitely many states $i \in C_0$. As a consequence $\mathbb{P}(U^{(m)}(C_0) \,|\, \theta^n U^{(m)}(C_0)) = 1$, and hence,

$$\mathbb{P}(L^{(m)}(C_0) \,|\, \theta^n L^{(m)}(C_0)) = 1,$$

since C_0 is almost closed with respect to $\mathcal{X}^{(m)}$. But by Theorem 2.4 this is possible only if $C_0 = C_{n\,(\mathrm{mod}\,m)}$, that is, if m is a divisor of n. The result follows by definition of $d(\mathcal{X})$. □

We conclude this section with two corollaries of Theorem 3.5, the first one being evident.

Corollary 3.1. *If the simple Markov chain* $\mathcal{X}$ *is such that* $P(i,i) \geq \delta$ *for some* $\delta > 0$ *and all but finitely many states* $i \in S$*, then* $\mathcal{X}$ *is asymptotically aperiodic.*

Corollary 3.2. *If, for some* n*, the simple Markov chain* $\mathcal{X}$ *satisfies condition* (3.3) *while* $\mathcal{X}^{(n)}$ *is simple, then* $\mathcal{X}$ *is asymptotically aperiodic.*

Proof. If $\mathcal{X}$ satisfies (3.3), then, by Theorem 3.5, $d \equiv d(\mathcal{X})$ is a divisor of n, so that, by Corollary 2.2, $\beta(\mathcal{X}^{(d)})$ is a divisor of $\beta(\mathcal{X}^{(n)})$. Hence we must have $d = \beta(\mathcal{X}^{(d)}) = 1$ if $\beta(\mathcal{X}^{(n)}) = 1$. □

4. Birth-death processes

Throughout this section $S = \{0, 1, \dots\}$ and $\mathcal{X}$ is a stochastic and irreducible birth-death process on S with at least one positive self-transition probability, so that $\mathcal{X}$ is aperiodic. Note that $\mathcal{X}$ is simple and atomic by Lemma 2.1. As before, P denotes the matrix of 1-step transition probabilities of $\mathcal{X}$, and we use the notation (1.2). Letting

$$\pi_0 := 1, \quad \pi_n := \frac{p_0 p_1 \dots p_{n-1}}{q_1 q_2 \dots q_n}, \quad n \geq 1,$$

and

$$K_n := \sum_{j=0}^{n} \pi_j, \quad L_n := \sum_{j=0}^{n} \frac{1}{p_j \pi_j}. \quad 0 \leq n \leq \infty, \tag{4.1}$$

we observe that $K_\infty + L_\infty = \infty$, and recall that

$$\mathcal{X} \text{ is } \begin{cases} \text{positive recurrent} & \iff \quad K_\infty < \infty, \ \ L_\infty = \infty \\ \text{null recurrent} & \iff \quad K_\infty = \infty, \ \ L_\infty = \infty \\ \text{transient} & \iff \quad K_\infty = \infty, \ \ L_\infty < \infty. \end{cases} \tag{4.2}$$

Theorem 4.1. *The asymptotic period $d(\mathcal{X})$ of the birth-death process $\mathcal{X}$ equals* 1, 2, *or* ∞.

Proof. Suppose $2 < d \equiv d(\mathcal{X}) < \infty$, and let $\{C_0, C_1, \dots, C_{d-1}\}$ be a cyclic Blackwell decomposition of S for $\mathcal{X}^{(d)}$.

By (2.4) we have, for $\ell = 0, 1 \dots$ and k sufficiently large, $X(k+\ell) \in C_{\ell \,(\mathrm{mod}\, d)}$ if $X(k) \in C_0$, and in particular $X(k+1) \in C_1$. Since C_0 and C_1 are disjoint, $X(k+1) = X(k)$ is impossible, but also $X(k+1) = X(k) - 1$ leads to a contradiction. Indeed, if $X(k) \in C_0$ and $X(k+1) = X(k) - 1 \in C_1$ then $X(k+2) \in C_2$ and hence $X(k+2) = X(k) - 2$, since the other options would contradict the fact that C_0, C_1 and C_2 are disjoint. Thus continuing we eventually find that $X(k+X(k)-1) = 1 \in C_{X(k)-1 \,(\mathrm{mod}\, d)}$ and $X(k+X(k)) = 0 \in C_{X(k) \,(\mathrm{mod}\, d)}$. But this would imply $X(k + X(k) + 1) = 0$ or $X(k + X(k) + 1) = 1$, which is impossible since $C_{X(k)-1 \,(\mathrm{mod}\, d)}$, $C_{X(k) \,(\mathrm{mod}\, d)}$ and $C_{X(k)+1 \,(\mathrm{mod}\, d)}$ are disjoint.

So, assuming k sufficiently large and $X(k) \in C_0$, we must have $X(k+1) = X(k) + 1 \in C_1$. Repeating the argument leads to the conclusion that for k sufficiently large, $X(k) \in C_0$ implies $X(k+\ell) = s + \ell \in C_{\ell \,(\mathrm{mod}\, d)}$ for all $\ell = 0, 1, \dots$. We conclude that in the long run $\mathcal{X}$ will solely make jumps to the right, that is, the number of self-transitions or jumps to the left will be finite. But then, as we have observed in Section 3, $\beta(\mathcal{X}^{(n)}) = n$ for all n, since the sets $C_i' := \{i, n+i, 2n+i, \dots\}$, $i = 0, 1, \dots, n-1$, are (disjoint) atomic almost closed sets with respect to $\mathcal{X}^{(n)}$. Hence $d(\mathcal{X}) = \infty$, contradicting our assumption $d(\mathcal{X}) < \infty$. □

In what follows we derive necessary and sufficient conditions for $d(\mathcal{X}) = \infty$ and for $d(\mathcal{X}) = 1$ (that is, for asymptotic aperiodicity) in terms of the 1-step transition probabilities of the process $\mathcal{X}$. By the above theorem we must have $d(\mathcal{X}) = 2$ in the cases not covered by these criteria. The next theorem tells us when $d(\mathcal{X}) = \infty$.

Theorem 4.2. *The birth-death process $\mathcal{X}$ has asymptotic period $d(\mathcal{X}) = \infty$ if and only if* $\Pi_{i=0}^{\infty} p_i > 0$.

Proof. It has been shown already in Section 3 that $d(\mathcal{X}) = \infty$ if $\Pi_{i=0}^{\infty} p_i > 0$, so it remains to prove the converse. So suppose $d(\mathcal{X}) = \infty$ and let $d > 2$ be such that $\beta(\mathcal{X}^{(d)}) = d$. The argument used in the proof of Theorem 4.1 can be copied to conclude that, with probability one, $\mathcal{X}$ will, in the long run, solely make jumps to the right, but this obviously implies $\Pi_{i=0}^{\infty} p_i > 0$. $\square$

A criterion for asymptotic aperiodicity in terms of the 1-step transition probabilities follows after having established the validity of three lemmas. The first is the following.

Lemma 4.1. *$\mathcal{X}$ is asymptotically aperiodic if and only if $\mathcal{X}^{(2)}$ is simple.*

Proof. If $\mathcal{X}$ is asymptotically aperiodic, then, by definition, $\beta(\mathcal{X}^{(2)}) < 2$, and hence $\beta(\mathcal{X}^{(2)}) = 1$, that is, $\mathcal{X}^{(2)}$ is simple. On the other hand, if $\mathcal{X}$ is *not* asymptotically aperiodic then $d(\mathcal{X}) = 2$ or $d(\mathcal{X}) = \infty$, which both imply $\beta(\mathcal{X}^{(2)}) = 2$, that is, $\mathcal{X}^{(2)}$ is *not* simple. $\square$

Note that, by Lemma 2.1, $\mathcal{X}^{(2)}$ will be atomic if it is simple.

In the next lemma a necessary and sufficient condition for $\mathcal{X}^{(2)}$ to be simple is given in terms of the polynomials $Q_n,\ n \geq 0$, that are uniquely determined by the 1-step transition probabilities of $\mathcal{X}$ via the recurrence relation

$$\begin{aligned} &xQ_n(x) = q_n Q_{n-1}(x) + r_n Q_n(x) + p_n Q_{n+1}(x), \quad n > 1, \\ &Q_0(x) = 1, \quad p_0 Q_1(x) = x - r_0. \end{aligned} \tag{4.3}$$

The result is mentioned already in [6, p. 275], but for completeness' sake we give its proof.

Lemma 4.2. *$\mathcal{X}^{(2)}$ is simple if and only if $|Q_n(-1)| \to \infty$ as $n \to \infty$.*

Proof. Writing $Q(x) := (Q_0(x), Q_1(x), \ldots)^T$ (where superscript T denotes transposition), the recurrence relation (4.3) may be succinctly represented by

$$PQ(x) = xQ(x). \tag{4.4}$$

It follows that

$$P^2 Q(x) = x^2 Q(x), \tag{4.5}$$

so that the vectors $Q(1)$ and $Q(-1)$ are two distinct solutions of the system of equations

$$P^2 y = y. \tag{4.6}$$

Moreover, P^2 being a pentadiagonal matrix, any solution to (4.6) must be a linear combination of $Q(1)$ and $Q(-1)$. It follows that the constant function is the only bounded harmonic function for P^2 if and only if $Q_n(-1)$ is unbounded. Since $|Q_n(-1)|$ is increasing (see Karlin and McGregor [9, p. 76] and Lemma 4.3 below), Theorem 2.2 leads to the required result. □

The third lemma constitutes an extension of Karlin and McGregor's result on the sequence $\{Q_n(-1)\}_n$ referred to in the proof of the previous lemma.

Lemma 4.3. *The sequence $\{(-1)^n Q_n(-1)\}_n$ is increasing, and strictly increasing for n sufficiently large. Moreover,*

$$\lim_{n\to\infty} (-1)^n Q_n(-1) = \infty \iff \sum_{j=0}^{\infty} \frac{1}{p_j \pi_j} \sum_{k=0}^{j} r_k \pi_k = \infty.$$

Proof. Writing $\bar{Q}_n(x) := (-1)^n Q_n(x)$ the recurrence relation (4.3) implies

$$\begin{aligned} p_n \pi_n (\bar{Q}_{n+1}(x) - \bar{Q}_n(x)) &= p_{n-1}\pi_{n-1}(\bar{Q}_n(x) - \bar{Q}_{n-1}(x)) \\ &\quad + (2r_n - 1 - x)\pi_n \bar{Q}_n(x), \quad n \geq 1, \\ p_0 \pi_0 (\bar{Q}_1(x) - \bar{Q}_0(x)) &= (2r_0 - 1 - x)\pi_0 \bar{Q}_0(x), \end{aligned}$$

so that

$$p_n \pi_n (\bar{Q}_{n+1}(x) - \bar{Q}_n(x)) = \sum_{k=0}^{n} (2r_k - 1 - x)\pi_k \bar{Q}_k(x), \quad n \geq 0,$$

and hence

$$\bar{Q}_{n+1}(x) = 1 + \sum_{j=0}^{n} \frac{1}{p_j \pi_j} \sum_{k=0}^{j} (2r_k - 1 - x)\pi_k \bar{Q}_k(x), \quad n \geq 0. \tag{4.7}$$

It follows in particular (as observed already by Karlin and McGregor [9, p. 76]) that

$$\bar{Q}_{n+1}(-1) = 1 + 2\sum_{j=0}^{n} \frac{1}{p_j \pi_j} \sum_{k=0}^{j} r_k \pi_k \bar{Q}_k(-1), \quad n \geq 0, \tag{4.8}$$

and hence

$$\bar{Q}_{n+1}(-1) = \bar{Q}_n(-1) + \frac{2}{p_n \pi_n} \sum_{k=0}^{n} r_k \pi_k \bar{Q}_k(-1), \quad n \geq 0. \tag{4.9}$$

Since $\bar{Q}_0(-1) = 1$ while $r_k > 0$ for at least one state k by the aperiodicity of $\mathcal{X}$, the first statement follows. So we have $\bar{Q}_n(-1) \geq 1$, which, in view of (4.8), implies the necessity in the second statement. To prove the sufficiency we let

$$\beta_j := \frac{2}{p_j \pi_j} \sum_{k=0}^{j} r_k \pi_k, \quad j \geq 0,$$

and assume that $\sum_j \beta_j$ converges. By (4.9) we then have

$$\bar{Q}_{n+1}(-1) \leq \bar{Q}_n(-1)(1 + \beta_n), \quad n \geq 0,$$

since $\bar{Q}_n(-1)$ is increasing in n. It follows that

$$\bar{Q}_{n+1}(-1) \leq \prod_{j=0}^{n} (1 + \beta_j), \quad n \geq 0.$$

But, as is well known, $\prod_j (1 + \beta_j)$ and $\sum_j \beta_j$ converge together, so we must have $\lim_{n\to\infty} \bar{Q}_n(-1) < \infty$. □

The Lemmas 4.1 – 4.3 give us a necessary and sufficient condition for $\mathcal{X}$ to be asymptotically aperiodic in terms of the 1-step transition probabilities.

Theorem 4.3. *The birth-death process $\mathcal{X}$ is asymptotically aperiodic if and only if*

$$\sum_{j=0}^{\infty} \frac{1}{p_j \pi_j} \sum_{k=0}^{j} r_k \pi_k = \infty. \tag{4.10}$$

Considering (4.2) and the fact that $r_k > 0$ for at least one state k by the aperiodicity of $\mathcal{X}$, we see that $\mathcal{X}$ is asymptotically aperiodic if $\mathcal{X}$ is recurrent, as we had observed already in the more general setting of Theorem 3.3. Another simple sufficient condition for asymptotic aperiodicity is obtained by noting that

$$\sum_{j=0}^{n} \frac{1}{p_j \pi_j} \sum_{k=0}^{j} r_k \pi_k \geq \sum_{j=0}^{n} \frac{r_j}{p_j},$$

so that $\mathcal{X}$ is asymptotically aperiodic if $\sum_{j=0}^{\infty} r_j / p_j = \infty$. Note that the latter condition is substantially weaker than the condition given, in a more general setting, in Corollary 3.1.

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