A note on the $GI/GI/\infty$ system with identical service and interarrival-time distributions

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Dedicated to Frits Göbel on the occasion of his 70th birthday

Abstract. We study the stationary distribution of the number of busy servers in a $GI/GI/\infty$ system in which the service-time distribution is identical to the interarrival-time distribution, and obtain several representations for the variance. As a result we can verify an expression for the variance, conjectured by Rajaratnam and Takawira (*IEEE Trans. Vehicular Technol.* **50** (2001) 954-970), when the common distribution of interarrival and service times is a gamma distribution.

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1 Introduction

We are interested in the steady-state distribution – and in particular the first two moments – of the number of busy servers in a $GI/GI/\infty$ system in which the service time distribution is identical to the interarrival-time distribution. This particular model comes about when an infinite pool of customers has to be served by a tandem service system consisting of a finite group of servers (of size N, say) followed by an infinite-server group. If service times at all servers are independent and identically distributed random variables, then the arrival process at the second group is simply a superposition of N independent renewal processes, each with an interarrival-time distribution equal to the service-time distribution. The number of busy servers in the infinite-server system is therefore the sum of N independent and identically distributed random variables, each representing the number of busy servers in the infinite-server group when $N = 1$. It is the stationary distribution of the latter random variable which is our concern in this note.

The model of a tandem system consisting of a finite-server group followed by an infinite-server group, but with a Poisson arrival process to the first group, has been proposed by Rajaratnam and Takawira in [6] and earlier papers (see the references in [6]) in the performance analysis of cellular mobile networks. In this setting it is of interest to know the stationary distribution – and in particular the mean and variance – of the number of busy servers in the infinite-server group, but this distribution seems difficult to obtain. Rajaratnam and Takawira [6] therefore resort to an approximate analysis which involves the model studied in this note.

We reveal in the next section a remarkably simple expression for $M(t)$, the *time-dependent* mean number of busy servers in the $GI/GI/\infty$ system, when interarrival and service times are identically distributed. This result enables us to find V, the variance of the *stationary* number of busy servers, by exploiting a classic result of Takács [7]. Some special cases are considered in Section 3. In particular, we will verify an expression for V which was conjectured by Rajaratnam and Takawira [6] in the case that interarrival and service times have identical gamma distributions.

We finally note that an interesting representation for V has been given by Yamazaki et al. [8], but it does not seem to lead to the explicit expression of Section 2.

2 The number of busy servers

We start off with some general notation and terminology. Let us consider a $GI/GI/\infty$ system in which the interarrival times have a common distribution function F, and the service times have a common distribution function H. Both

F and H are supposed to have a finite first moment, so that the arrival and service rates

$$\lambda \equiv \left(\int_0^\infty t dF(t)\right)^{-1} \quad \text{and} \quad \mu \equiv \left(\int_0^\infty t dH(t)\right)^{-1},$$

respectively, are positive. We assume that the system starts empty at time 0 and that the time until the first arrival has distribution function G. By m we denote the renewal function associated with F, that is,

$$m(t) \equiv \sum_{n=1}^{\infty} F^{n*}(t), \quad t \geq 0,$$

where F^{n*} stands for the n-fold convolution of F. By $X(t)$ and X we denote the number of busy servers at time t and in steady state, respectively, and we let $B_n(t)$ and B_n be their respective nth binomial moments, that is,

$$B_n(t) \equiv \sum_{k=n}^{\infty} \binom{k}{n} \Pr\{X(t) = k\} \text{ and } B_n \equiv \sum_{k=n}^{\infty} \binom{k}{n} \Pr\{X = k\}, \quad n \geq 1.$$

The next theorem summarizes some classic results of Takács' [7] on these binomial moments.

Theorem 1 (*i*) *The binomial moments $B_n(t)$ exist for all n and $t \geq 0$, and, if $G = F$, satisfy the recurrence relation*

$$B_n(t) = \int_0^t B_{n-1}(t-u)(1 - H(t-u))dm(u), \quad t \geq 0, \; n = 1, 2, \ldots, \tag{1}$$

where $B_0(t) = 1$.
(*ii*) *There exists a unique steady-state distribution with binomial moments B_n satisfying*

$$B_n = \lambda \int_0^\infty B_{n-1}(t)(1 - H(t))dt, \quad n = 1, 2, \ldots, \tag{2}$$

where $B_n(t),\; n = 0, 1, \ldots,$ are the time-dependent binomial moments in the case $G = F$.

Takács requires F to be non-lattice for (2), but, as observed in [8], this condition can be dropped. For generalizations of Takács' findings and related results we refer to Pakes and Kaplan [5], Kaplan [3], Liu et al. [4] and Ayhan et al. [2], and the references there.

As announced, we wish to obtain the mean M and variance V of the stationary number of busy servers in the case that $F = H$. To this end we first observe a surprisingly simple corollary to Theorem 1 concerning $M(t)$, the mean number of busy servers at time t.

Corollary 2 *If $G = F = H$ then $M(t) = H(t)$ for all $t \geq 0$.*

Proof From Theorem 1 (i) we see that

$$M(t) = B_1(t) = \int_0^t (1 - H(t-u))dm(u) = m(t) - H * m(t).$$

But since $m(t) \equiv \sum_{n=1}^{\infty} F^{n*}(t) = \sum_{n=1}^{\infty} H^{n*}(t)$, the result follows immediately. □

Remark A more direct (and perhaps more appealing) argument leads to a generalization of Corollary 2 in which we do not require $G = F$. Indeed, let T_i be the arrival time of the ith customer, and S_i his service time. Then we can write

$$X(t) = \sum_{i=1}^{\infty} I_{[T_i, T_i+S_i)}(t), \quad t \geq 0,$$

where I_A denotes the indicator function of a set A. Taking expectations on both sides we get

$$M(t) = \sum_{i=1}^{\infty} \Pr\{T_i \leq t < T_i + S_i\} = \sum_{i=1}^{\infty} \Pr\{T_i \leq t < T_{i+1}\} = \Pr\{T_1 \leq t\},$$

since $T_{i+1} - T_i$ and S_i are independent and identically distributed, and also independent of T_i. So we actually have $M(t) = G(t)$ for all $t \geq 0$.

If $F = H$ then Theorem 1 (ii) (or Little's formula) tells us $M = B_1 = 1$, since $\lambda^{-1} = \mu^{-1} = \int_0^\infty (1 - H(t)dt$. Moreover, Corollary 2 tells us that $B_1(t) = M(t) = H(t)$, which upon substitution in (2) gives us B_2 and hence $V = 2B_2 + M - M^2 = 2B_2$. Summarizing we have the following.

Theorem 3 *If $F = H$ then the mean M and variance V of the stationary number of busy servers in the system $GI/GI/\infty$ are given by*

$$M = 1 \quad \text{and} \quad V = 2\mu \int_0^\infty H(t)(1 - H(t))dt. \tag{3}$$

We easily see that changing the unit of time does not affect the value of V (which is obvious also on physical grounds). We also note that

$$0 < V < 2\mu \int_0^\infty (1 - H(t))dt = 2.$$

The special case of a gamma distribution (discussed in Section 3.2) may be used to show that both lower and upper bound can be approached arbitrarily close by choosing the parameter c in (8) sufficiently large and small, respectively.

It is interesting to observe that V can be represented as

$$V = \mu \int_0^\infty \int_0^\infty |t_1 - t_2| dH(t_1) dH(t_2), \tag{4}$$

so that V/μ may be interpreted as the expected absolute value of the difference of two service times.

If H has a continuous density h on $(0, \infty)$, with $h(t) = \mathcal{O}(t^{a-1})$ for some $a > 0$ and $t \downarrow 0$, and $h(t) = \mathcal{O}(t^{-b-2})$ for some $b > 0$ and $t \to \infty$, we may also express V in terms of the Mellin transform of h, given by

$$M(h, z) \equiv \int_0^\infty t^{z-1} h(t) dt, \quad 1 - a < \Re(z) < 2 + b.$$

To do so we use the general Parseval relation for Mellin transforms (and integration by parts), and end up with the contour-integral representation

$$V = \frac{\mu}{\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \frac{M(h, z+1) M(h, 2-z)}{z(z-1)} dz, \quad -\min(a, b) < \kappa < 0. \tag{5}$$

3 Special cases

In this section we will look more closely at two special cases, namely H is a mixture of a degenerate and an exponential distribution, and H is a gamma distribution. Since V is independent of the unit of time it is no restriction of generality to assume $\mu = 1$ in the remainder of this section.

3.1 Mixtures of degenerate and exponential distributions

Let us assume that the interarrival and service times in the $GI/GI/\infty$ system are all drawn from a mixture of a degenerate and an exponential distribution with means 1, that is, for some p, $0 \le p < 1$,

$$H(t) = F(t) = pI_{[1,\infty)}(t) + (1-p)(1 - \mathrm{e}^{-t}), \quad t \ge 0. \tag{6}$$

Substitution of this distribution function in (3) readily yields

$$V \equiv V(p) = (1-p)(1 + (4/\mathrm{e} - 1)p). \tag{7}$$

Rajaratnam and Takawira [6] observed that $V(0) = 1$ and $V(1) = 0$, and found via simulation that $V(1/2) \approx 5/8 = 0.6250$ (in reality, $V(1/2) = 1/4 + 1/\mathrm{e} \approx 0.6179$). Hence, they proposed the quadratic interpolation formula

$$V(p) \approx (1-p)(1 + p/2),$$

which is pretty close to (7) since $4/\mathrm{e} - 1 \approx 0.4715$.

3.2 Gamma distributions

Now suppose that the interarrival and service times in the $GI/GI/\infty$ system are drawn from a common gamma distribution with mean 1, that is,

$$F(t) = H(t) = \frac{1}{\Gamma(c)} \int_0^t c(cu)^{c-1} \mathrm{e}^{-cu} du, \quad t \geq 0, \tag{8}$$

where c is some positive constant and Γ is the gamma function

$$\Gamma(a) \equiv \int_0^\infty u^{a-1} \mathrm{e}^{-u} du, \quad a > 0.$$

Analysing the $E_c/E_c/\infty$ queue by standard Markovian techniques, Rajaratnam and Takawira [6] found that for small integral values of c the variance V of the stationary number of busy servers is given by

$$V \equiv V(c) = 2^{1-2c} \frac{\Gamma(2c+1)}{\Gamma(c+1)^2}, \tag{9}$$

and they conjectured the validity of this expression for arbitrary $c > 0$. Before proving this in two different ways, we note that by the duplication formula for the gamma function (see [1, (6.1.18)]) $V(c)$ may also be written as

$$V(c) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(c + \frac{1}{2})}{\Gamma(c+1)}. \tag{10}$$

As a consequence, by [1, (6.1.47)],

$$V(c) \sim \frac{2}{\sqrt{\pi c}} \left\{ 1 - \frac{1}{8} c^{-1} + \dots \right\} \quad \text{as} \quad c \to \infty.$$

Moreover $V(c)$ is a decreasing function of c by the log-convexity of the gamma function on $(0, \infty)$. In fact, letting $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ (the *digamma function*), we have

$$\frac{d}{dc} \log V(c) = \psi(c + \tfrac{1}{2}) - \psi(c+1) < 0,$$

since, by [1, (6.4.10)],

$$\psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2} > 0, \quad z \notin \{0, -1, -2, \dots\}.$$

Theorem 4 *The variance of the stationary number of busy servers in the system $GI/GI/\infty$ when interarrival and service times have a common distribution function (8) is given by (9), or (10), for all $c > 0$.*

Proof Elementary substitution of (8) in (3), followed by appropriate changes of variables and a change in the order of integration, leads to

$$V(c) = \frac{2}{(\Gamma(c))^2}\int_0^\infty \left(\int_0^t c(cu)^{c-1}\mathrm{e}^{-cu}du\right)\left(\int_t^\infty c(cv)^{c-1}\mathrm{e}^{-cv}dv\right)dt$$

$$= \frac{2}{(\Gamma(c))^2}\int_0^\infty \left(\int_0^1 (ct)^c x^{c-1}\mathrm{e}^{-cxt}dx\right)\left(\int_1^\infty (ct)^c y^{c-1}\mathrm{e}^{-cyt}dy\right)dt$$

$$= \frac{2}{(\Gamma(c))^2}\int_0^1\int_1^\infty (xy)^{c-1}\int_0^\infty (ct)^{2c}\mathrm{e}^{-c(x+y)t}dt\ dy\ dx$$

$$= \frac{2}{c(\Gamma(c))^2}\int_0^1\int_1^\infty \frac{(xy)^{c-1}}{(x+y)^{2c+1}}\int_0^\infty u^{2c}\mathrm{e}^{-u}du\ dy\ dx$$

$$= \frac{2\Gamma(2c+1)}{(\Gamma(c+1))^2}\int_0^1\int_1^\infty \frac{c(xy)^{c-1}}{(x+y)^{2c+1}}dy\ dx,$$

where we have used $\Gamma(c+1) = c\Gamma(c)$. Finally, substitution of $y = u/v$ and $x = 1/v$ and another change in the order of integration gives us

$$\int_0^1\int_1^\infty \frac{c(xy)^{c-1}}{(x+y)^{2c+1}}dy\ dx = \int_1^\infty\int_v^\infty \frac{c\ u^{c-1}}{(1+u)^{2c+1}}du\ dv$$

$$= \int_1^\infty\int_1^u \frac{c\ u^{c-1}}{(1+u)^{2c+1}}dv\ du = \int_1^\infty \frac{c\ u^{c-1}(u-1)}{(1+u)^{2c+1}}du \tag{11}$$

$$= \int_1^\infty d\left(\frac{-u^c}{(1+u)^{2c}}\right) = \frac{1}{2^{2c}},$$

as required. □

Remark The integral in (11) is closely related to a particular value of a hypergeometric function. In fact, by [1, (15.3.1) and (15.1.21)] we have

$$\int_1^\infty \frac{c\ u^{c-1}(u-1)}{(1+u)^{2c+1}}du = \int_0^1 \frac{ct^{c-1}(t-1)}{(1+t)^{2c+1}}dt$$

$$= \frac{\Gamma(c+1)\Gamma(2)}{\Gamma(c+2)}F(2c+1, c; c+2; -1) = 2^{-2c},$$

as before.

Second proof of Theorem 4 We apply (5) with $h(t) = c^c t^{c-1}\mathrm{e}^{-ct}/\Gamma(c)$ and $b = \infty$ to obtain

$$M(h,z) = \frac{c^c}{\Gamma(c)}\int_0^\infty t^{z+c-2}\mathrm{e}^{-ct}dt = c^{1-z}\frac{\Gamma(z+c-1)}{\Gamma(c)}, \quad \Re(z) > 1-c,$$

and

$$V(c)=\frac{c^{-1}}{\Gamma(c)^2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{\Gamma(z+c)\Gamma(1-z+c)}{z(z-1)}dz,\quad -c<\kappa<0.$$

By replacing $z(z-1)$ with $\Gamma(2-z)/\Gamma(-z)$, followed by a substitution of $z=-c-s$, we get (up to a factor) a Mellin-Barnes integral for the hypergeometric function value mentioned in the above Remark. Actually, by [1, (15.3.2) and (15.1.21)] it follows that

$$\begin{aligned}V(c)&=\frac{c^{-1}}{\Gamma(c)^2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{\Gamma(s+2c+1)\Gamma(s+c)\Gamma(-s)}{\Gamma(s+c+2)}ds\\&=2\frac{\Gamma(2c+1)}{\Gamma(c+2)\Gamma(c+1)}F(2c+1,c;c+2;-1)=2^{1-2c}\frac{\Gamma(2c+1)}{\Gamma(c+1)^2},\end{aligned}$$

as before. □

We finally note that for integral values of c a third proof of (9) may be based on the interpretation (4) of $V(c)$ as the expected absolute value of the difference of two service times. Namely, imagine two service times S_1 and S_2, each consisting of c exponentially distributed phases of mean $1/c$, starting at time 0, and a counter going up (or down) one unit each time a phase of S_1 (or S_2) elapses, until one of the service times ends. If we denote the state of the counter after the nth count by X_n, and let

$$N\equiv\min\{n:|X_{c+n}|=c-n\},$$

then, for $n=1,2,\ldots,N$, X_n is distributed as $\sum_{k=1}^{n}Y_k$, where $Y_1,Y_2,\ldots$ are independent random variables taking the values $+1$ and -1 with equal probabilities. Moreover, at the time of the Nth count either S_1 or S_2 has ended, and the remaining part of the surviving service time consists of $c-N$ exponentially distributed phases with means $1/c$. Consequently,

$$V(c)=\mathbb{E}|S_1-S_2|=(c-\mathbb{E}N)/c. \tag{12}$$

A combinatorial argument shows that N has a truncated negative binomial distribution

$$\Pr\{N=n\}=2\binom{c+n-1}{n}(\tfrac{1}{2})^{c+n},\quad n=0,1,\ldots,c-1,$$

with first moment

$$\mathbb{E}N=c-c\binom{2c}{c}(\tfrac{1}{2})^{2c-1},$$

which, together with (12), gives us (9) again.

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