Variational Water Wave Model

with accurate Dispersion and Vertical Vorticity

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1 Motivation

On the one hand:

- **depth-averaged shallow water equations** (SWE) are a workhorse in coastal engineering
- **shallow** means horizontal scales $\gg$ vertical ones
- SWE are **simplified** with horizontal 2D coordinates x, y and time t
- variables are **depth** $h(x,y,t)$ and depth-averaged horizontal **velocity** $\mathbf{v}(x,y,t)$
- **bores** and hydraulic jumps arise as simplified model of breaking waves . . . .

Fig. 1: Steepening & breaking waves in the surf zone. Photo: D.H. Peregrine.

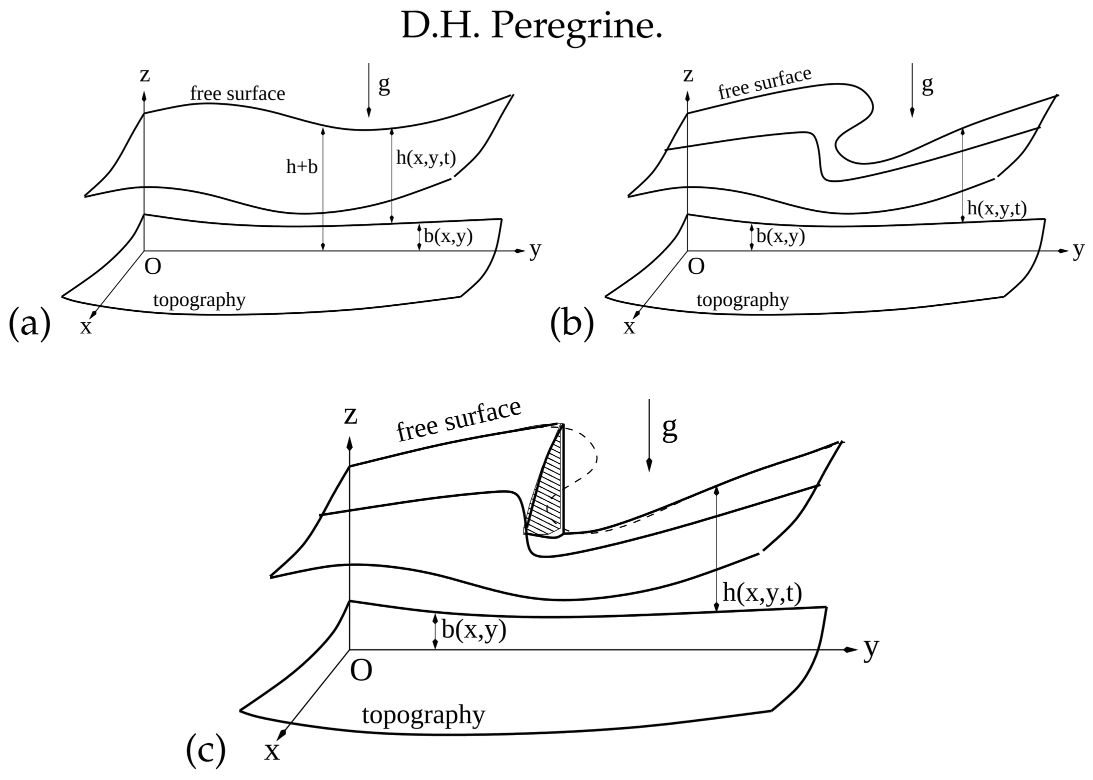


Fig. 2: Sketch of a) set-up, b) triple-valued $h(x,y,t)$, & c) its bore approximation.

On the other hand:

- **3D free surface potential flow water wave model** under gravity is widely used for marine engineering problems
- has accurate dispersion: (deep-water) waves of different wave length travel with different speeds
- has 3D irrotational velocity $\mathbf{u} = \nabla\phi$.

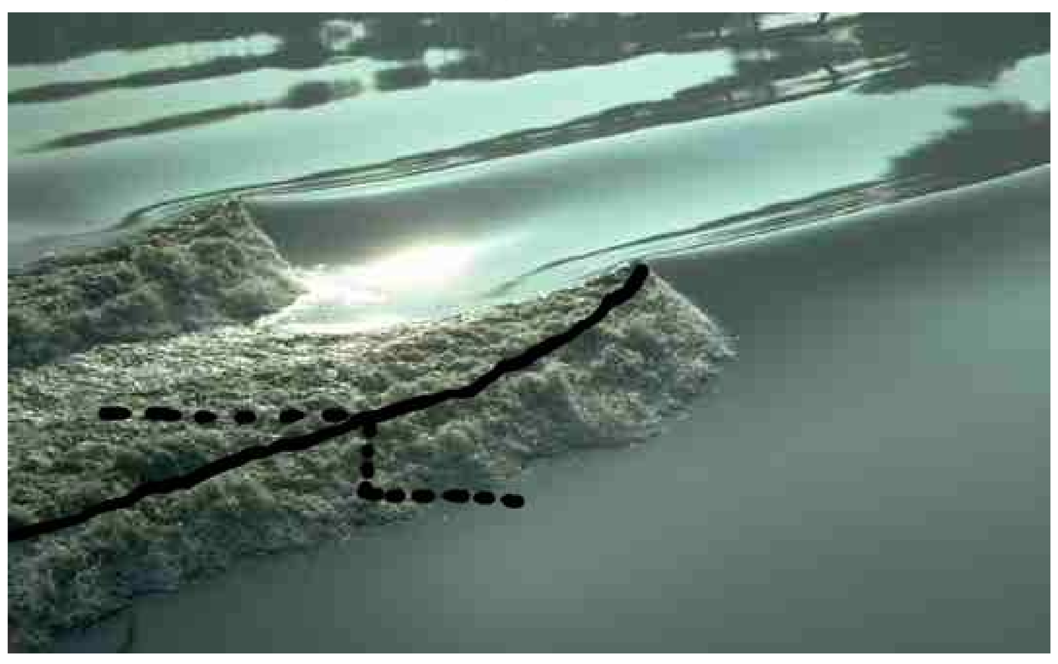


Fig. 3: Undular bore on River Severn. Photo: DHP.

2 Objectives: Make Waves!

Objective 1: Combine 2 models in hybrid wave model with accurate dispersion, bores, & vertical vorticity via variational techniques.

Preserve mathematical properties of PDE's & associated conservation laws. Fast prediction of waves & horizontal currents in coastal zone.

Objective 2: Derive fast, compatible Discontinuous Galerkin FEM (DGFEM) for new wave model: no loss of wave amplitude except locally at bores.

3 Strategy: Variational Principles

Use **Clebsch potentials** in variational principle for velocity $\mathbf{u}$. Three cases:

- (i) Full incompressible Euler fluid with free surface: $\mathbf{u} = \nabla\phi(x,y,z,t) + \boldsymbol{\pi}(x,y,z,t)\nabla \boldsymbol{l}(x,y,z,t)$
- (ii) Classical potential flow water wave model: $\mathbf{u} = \nabla\phi$
- **(iii)** New faster water wave model with vertical vorticity: $\mathbf{u} = \nabla\phi(x,y,z,t) + \boldsymbol{\pi}(x,y,t)\nabla l(x,y,t) = \nabla\phi + \mathbf{v}(x,y,t)$; note: approximation with only vertical vorticity $\nabla\times\mathbf{v}$!

Variational principle:

$$0 = \delta\int_0^T\int_\Omega \frac{1}{2}D|\boldsymbol{u}|^2 - g\,D\,z + p\,(1-D) + \underbrace{\boldsymbol{\pi}\cdot(\partial_t\boldsymbol{l} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{l})}_{\text{labels term}}$$
$$+\underbrace{\phi\,(\partial_t\mathbf{D} + \nabla\cdot(\boldsymbol{u}\mathbf{D}))}_{\text{density}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t \qquad (1)$$

- **$\boldsymbol{\pi}$: vector Lagrange multiplier to enforce label l advection**
- **ϕ: multiplier to enforce law for density D as constraint**
- **p: multiplier pressure enforces constant density $D=1$.**

4 New Water Wave Model

- Resulting system [1] for **case (iii)** (s=at free surface):

$$\nabla^2\phi + \nabla\cdot\mathbf{v} = 0 \qquad (2a)$$
$$(\partial_t\phi)_s + \frac{1}{2}|(\nabla\phi)_s + \mathbf{v}|^2 + g\,(h+b) - \mathbf{v}\cdot\bar{\boldsymbol{u}} = 0 \qquad (2b)$$
$$\partial_t h + \nabla\cdot(h\,\bar{\boldsymbol{u}}) = 0 \qquad (2c)$$
$$\partial_t(h\,\mathbf{v}) + \nabla\cdot(h\bar{\boldsymbol{u}}\mathbf{v}) + h\mathbf{v}\nabla\bar{\boldsymbol{u}} = 0 \qquad (2d)$$
$$h\bar{\boldsymbol{u}} = \int_b^{b+h}\nabla_H\phi\,\mathrm{d}z + h\mathbf{v}. \qquad (2e)$$

- Depth-averaged $\mathbf{v} = \mathbf{v}(x,y,t)$; **potential vorticity** q:

$$(\partial_t + \bar{\boldsymbol{u}}\cdot\nabla)\,q = 0 \quad \text{with} \quad q(x,y,t) = \frac{\nabla\times\mathbf{v}}{h}. \qquad (3)$$

- **Potential flow** limit: take $\mathbf{v} = 0$ or $\mathbf{v} = \nabla\varphi(x,y,t)$.

5 Compatible DGFEM

- Strategy potential part: directly derive **discrete variational principle** [2].
- Automaticaly guarantees discrete preservation of conservation laws.
- Strategy additional vertical vorticity: open problem for future; **discrete differential geometry?**

6 No Amplitude Decay

- Preliminary test: only potential flow part and linear.
- Compare standard and variational DGFEM: variational model has **no amplitude decay . . . !**

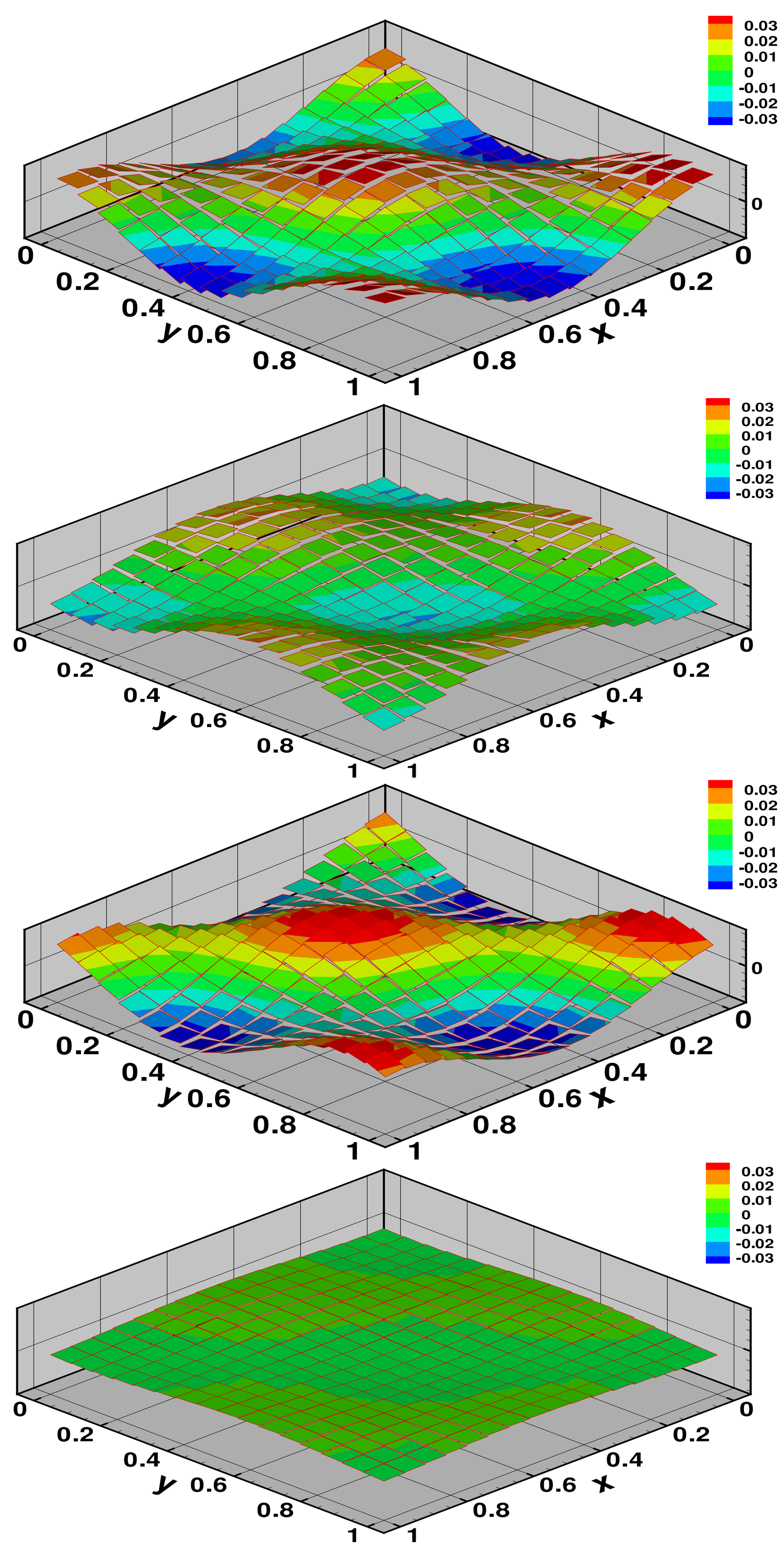


Fig. 4: Waves after 2 and 8 periods [2]. **Which results are from variational model?**

7 Future work?

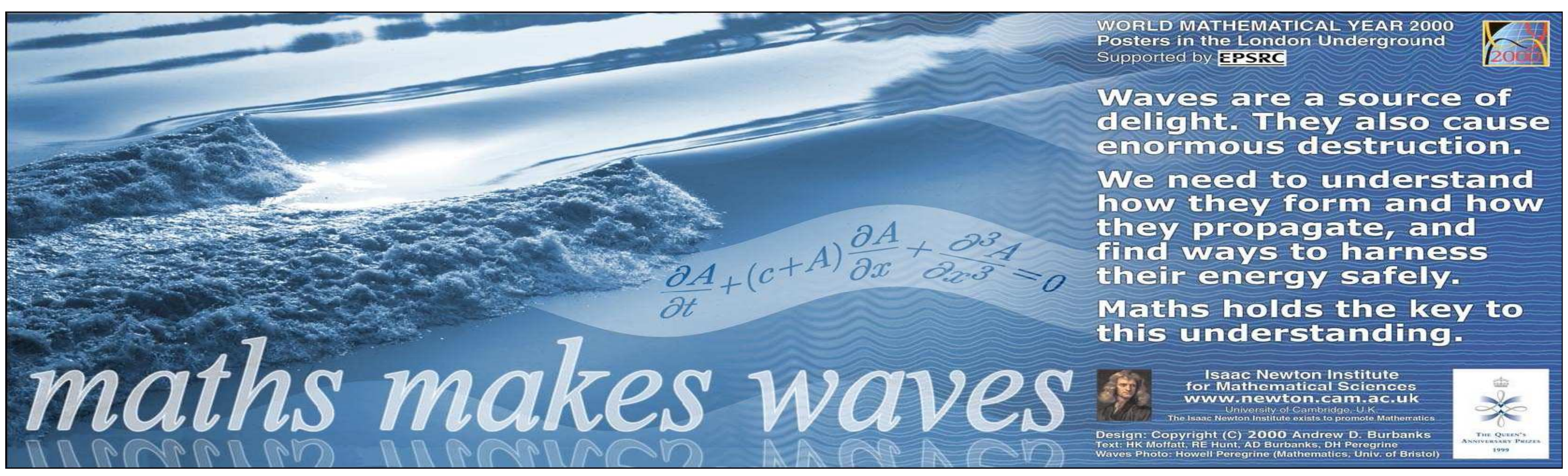
- Full implementation of **compatible** DGFEM
- with **flooding & drying** at sandbanks, dikes & beaches [3]
- **tests against wave data** from IJsselmeer and Petten dike.

References

[1] Cotter & B. 2009/10: Variational water wave model with accurate dispersion & vertical vorticity. *J. Eng. Mech.* D.H. Peregrine commemorative issue.

[2] Ambati, Van der Vegt & B. 2009: Variational space-time DGFEM for free surface waves. Revision *J. Comp. Phys.*

[3] B. 2005: Flooding and drying in DGFEM shallow-water equations. 1D. *J. Sci. Comp.* **22**.



My goal is to make waves! Photo: DHP. Design: Andrew Burbanks. Information: o.bokhove@math.utwente.nl & www.math.utwente.nl/~bokhoveo