

Variational water wave model -with accurate dispersion and vertical vorticity

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Cooker: "Commemoration of Howell Peregrine." JEM & Qua Art Qua Science 2010

Cotter & B: "Variational water wave model." JEM 2010

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Free University A'dam 09-12-2009

- Introduction
- Variational principle for water waves
 - Nonlinear oscillator
 - Euler equations incompressible fluid with free surface
 - Potential flow water waves
 - New model: potential flow water waves with horizontal circulation
- Discontinuous Galerkin finite element models
 - Potential flow water waves
- Conclusions and future work

Introduction

- **Depth-averaged shallow water equations** workhorse in coastal and marine engineering; **shallow** means horizontal scales \gg vertical ones:



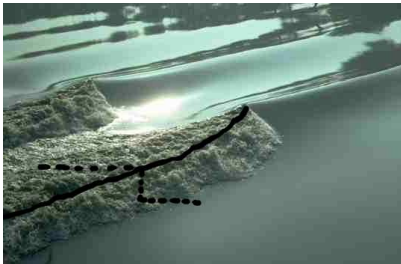
(a) Prediction of wave run-up/flooding in surf zone. Courtesy: J.-W. Wolf.



(b) Hydrodynamics and bed erosion. Pet-temerzeewering. Courtesy: wiki.

Shallow water equations are **simplified** with horizontal coordinates x, y and time t :

- variables are **depth** $h(x, y, t)$ and depth-averaged horizontal **velocity** $\mathbf{v}(x, y, t)$
- **bores** and hydraulic jumps arise as simplified model of breaking waves ...



(c) Undular bore in River Severn. Courtesy: D. Howell Peregrine



(d) Mascaret/bore River Risle; breaking/nonbreaking. Courtesy: Malandain '88

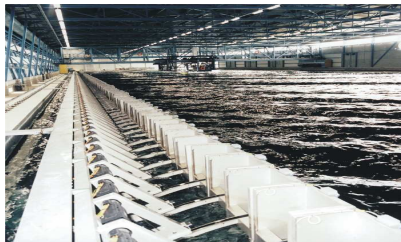
- **Free surface potential flow water wave model** under gravity is widely used for offshore and marine engineering problems:

→ accurate dispersion: (deep-water) waves of different wave length travel with different speeds

→ velocity irrotational $\mathbf{u} = \nabla\phi$



(e) Offshore oil platform “Thunder Horse” tilted by hurricane Dennis.



(f) Wave maker at MARIN, The Netherlands.

Both models approximations of 3D Euler equations with free-surface dynamics.

Advantages:

- inviscid (avoid numerical dissipation wave amplitude)
- SWE: less degrees of freedom due to depth-averaging
- Potential flow: simpler flow field

Disadvantages:

- SWE: no dispersion of waves
- Potential flow: no circulation/vortices, and bores.

Objective 1: Derive a hybrid model with accurate dispersion, bores, and horizontal depth-averaged circulation, via variational/Hamiltonian techniques.

Why variational/Hamiltonian techniques?

Preserve mathematical properties of PDE's and associated conservation laws, energy, mass, wave amplitude

Objective 2: Derive a conservative DGFEM for this new hybrid model to be compatible, no loss of wave amplitude except locally at bores.

Why DGFEM?

Flexibility at moving boundaries, differential geometry, flexibility regarding compatibility: crux lies in flux & function spaces.

Make waves!

with accurate dispersion, no amplitude loss, horizontal circulation and, locally, allow bores as breaking waves.

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Waves are a source of delight. They also cause enormous destruction.

We need to understand how they form and how they propagate, and find ways to harness their energy safely.

Maths holds the key to this understanding.

$$\frac{\partial A}{\partial t} + (c+A)\frac{\partial A}{\partial x} + \frac{\partial^3 A}{\partial x^3} = 0$$

maths makes waves

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Waves Photo: Howell Peregrine (Mathematics, Univ. of Bristol)

The Queen's
Alexander Palace
1999

(g) Photograph: Howell Peregrine; design: Andrew Burbanks

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Ockham: "Frusta fit per plura quod potest fieri per pauciora"

—it is futile to employ many principles when it is possible to employ fewer.

Nonlinear oscillator

- Dynamics of nonlinear oscillator contained in action integral

$$\begin{aligned} 0 = \delta L[\phi, \chi] &:= \delta \int_0^T \phi \dot{\chi} - \left(\frac{1}{2} \omega^2 \phi^2 + V(\chi) \right) dt \\ &= \delta \int_0^T \phi \dot{\chi} - \underbrace{\left(\frac{1}{2} \omega^2 \phi^2 + V(\chi) \right)}_{\text{kinetic+potential energy}} dt \end{aligned} \quad (1)$$

position $\chi(t)$, momentum $\phi(t)$, frequency ω , $\dot{\chi} = d\chi/dt$, and endpoint variations $\delta\chi(0) = \delta\chi(T) = 0$.

- Given arbitrariness of $\delta\chi, \delta\phi$:

$$\begin{aligned}
 0 = \delta L[\phi, \chi] &:= \frac{d}{d\epsilon} L[\phi + \epsilon\delta\phi, \chi + \epsilon\delta\chi] \\
 &= \int_0^T \delta\phi\dot{\chi} + \phi\delta\dot{\chi} - (\omega^2\phi\delta\phi + V'(\chi)\delta\chi)dt \\
 &= \int_0^T (\dot{\chi} - \omega^2\phi)\delta\phi - (\dot{\phi} + V'(\chi))\delta\chi dt
 \end{aligned} \tag{2}$$

- Hamilton's first order equations emerge:

$$\dot{\chi} - \omega^2\phi = 0 \quad \text{and} \quad \dot{\phi} + V'(\chi) = 0. \tag{3}$$

- Combined into one second-order equation derivable from

$$\begin{aligned}
 0 = \delta L[\chi, \dot{\chi}] &:= \delta \int_0^T K - V dt = \omega^{-2} \delta \int_0^T \frac{1}{2} \dot{\chi}^2 - \omega^2 V(\chi) dt \\
 &= -\omega^{-2} \int_0^T (\ddot{\chi} + \omega^2 V'(\chi)) \delta\chi dt.
 \end{aligned} \tag{4}$$

Variational principles: Euler fluid with free surface

Lagrangian action principle incompressible Euler equations with free surface:

- reads, in extension of finite-dimensional case $\chi \rightarrow \boldsymbol{\chi}$ & $V(\chi) \rightarrow g\chi_3$:

$$0 = \delta L[\boldsymbol{\chi}, \partial_\tau \boldsymbol{\chi}, p] = \delta \int_0^T \int_{\Omega_0} \frac{1}{2} |\partial_\tau \boldsymbol{\chi}|^2 - g\chi_3 + p(1/D - 1) \, d\mathbf{a} d\tau \quad (5)$$

- fluid parcel position: $\mathbf{x} = (x, y, z)^T = \boldsymbol{\chi}(\mathbf{a}, \tau)$ and time $t = \tau$
- continuum of fluid labelled: $\mathbf{a} = (a, b, c)^T = \boldsymbol{\chi}(\mathbf{a}, 0)$
- Jacobian density D : $D^{-1} = \partial(\boldsymbol{\chi})/\partial(\mathbf{a})$; $D \, d\mathbf{x} = d\mathbf{a}$
- endpoint conditions: $\delta \boldsymbol{\chi}(\mathbf{a}, 0) = \delta \boldsymbol{\chi}(\mathbf{a}, T) = 0$
- acceleration of gravity g
- Lagrange multiplier and pressure p incompressibility constraint $1/D - 1 = 0$.
- Variations yield:

$$\delta \boldsymbol{\chi} : \frac{\partial^2 \boldsymbol{\chi}}{\partial \tau^2} = -(\nabla p + g\mathbf{z}) \circ \boldsymbol{\chi} \quad \text{and} \quad \delta p : D = 1 \quad (6)$$

- Note that, with $\partial_\tau \boldsymbol{\chi} = \mathbf{u} \circ \boldsymbol{\chi}$:

$$\partial_\tau D^{-1} = D^{-1} \nabla \cdot \mathbf{u} \circ \boldsymbol{\chi} \stackrel{D=1}{\Rightarrow} \nabla \cdot \mathbf{u} = 0$$

Eulerian action principle, $\chi(\mathbf{a}, \tau) \rightarrow \mathbf{a}(\mathbf{x}, t)$:

- reads

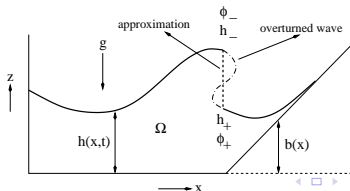
$$0 = \delta L[\mathbf{a}, \partial_t \mathbf{a}, p] = \delta \int_0^T \int_{\Omega} \frac{1}{2} D |\mathbf{u}|^2 - D g z + p(1 - D) \, dx dt \quad (7)$$

- velocity $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u, v, w)^T$.
- Variations yield incompressible Euler equations, incompressibility condition, and kinetic free-surface condition at S : $z = h + b$, depth $h = h(x, y, t)$ and topography $b = b(x, y)$ with \mathbf{v}_s at S :

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla(p + gz) = 0 \quad (8a)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (8b)$$

$$\partial_t h + \mathbf{v}_s \cdot \nabla(b + h) - w_s = 0 \quad (8c)$$



Eulerian action principle

- with explicit constraints reads

$$\begin{aligned}
 0 = \delta \int_0^T \int_{\Omega} & \frac{1}{2} D |\mathbf{u}|^2 - g D z + p(1 - D) - \underbrace{D \boldsymbol{\pi} \cdot (\partial_t \mathbf{a} + (\mathbf{u} \cdot \nabla) \mathbf{a})}_{\text{labels term}} \\
 & + \phi (\partial_t D + \nabla \cdot (\mathbf{u} D)) \, d\mathbf{x} + \int_{\partial\Omega_s} \lambda (\partial_t h + \mathbf{v}_s \cdot \nabla (h + b) - w_s) \, dx dy dt \quad (9)
 \end{aligned}$$

- 3D parcel labels $\mathbf{a}(\mathbf{x}, t)$
- Lagrange multipliers $D \boldsymbol{\pi}(\mathbf{x}, t)$ constraining label advection by velocity \mathbf{u}
- Lagrange multiplier ϕ constraining density D to satisfy mass continuity
- Lagrange multiplier λ for kinematic free surface condition.
- Lagrange multiplier p for incompressibility constraint $D - 1 = 0$.

Strategy

- with explicit constraints:

$$\begin{aligned}
 0 = \delta \int_0^T \int_{\Omega} \frac{1}{2} D |\mathbf{u}|^2 - g D z + p(1 - D) - \underbrace{D \boldsymbol{\pi} \cdot (\partial_t \mathbf{a} + (\mathbf{u} \cdot \nabla) \mathbf{a})}_{\text{labels term}} \\
 + \underbrace{\phi (\partial_t D + \nabla \cdot (\mathbf{u} D))}_{\text{density}} dx dt \quad (10)
 \end{aligned}$$

- Potential flow:** remove labels term; constraint: $\mathbf{u} = \nabla \phi$
- New model:** approximate labels term to horizontal/depth-averaged ones

$$\frac{1}{h} \int_b^{h+b} D dz \partial_t \mathbf{a} + \frac{1}{h} \int_b^{h+b} D \mathbf{u} dz \cdot \nabla \mathbf{a} = 0. \quad (11)$$

Luke's variational principle: potential flow water waves

- Variation of

$$0 = \delta \int_0^T \int_{\Omega} \frac{1}{2} D |\mathbf{u}|^2 - g D z + p(1 - D) + \phi (\partial_t D + \nabla \cdot (\mathbf{u} D)) \, dx dt, \quad (12)$$

- wrt \mathbf{u} yields potential flow: $\mathbf{u} = \nabla \phi$; substitute into variational principle
- to get **Luke's variational principle** $0 = \delta L[\phi, \phi_s, h]$ (when we set $D = 1$):

$$0 = \delta \int_0^T \iint \int_b^{b+h} \frac{1}{2} D |\nabla \phi|^2 + D g z + D \partial_t \phi + p(1 - D) \, dx dy dz dt \quad (13)$$

- Variations give **water wave equations** and Bernoulli:

$$\begin{array}{ll} \text{Bernoulli:} & p = \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g z \\ \text{3D Laplace:} & \nabla^2 \phi = 0 \\ \text{free surface} & (\partial_t \phi)_s + \frac{1}{2} |\nabla \phi|_s^2 + g(h + b) = 0 \\ \text{dynamics} & \partial_t h + (\nabla \phi)_s \cdot \nabla (h + b) - (\partial_z \phi)_s = 0. \end{array} \quad (14)$$

New model with horizontal circulation

- Horizontal labels $\mathbf{a} = \mathbf{a}(x, y, t)$ and multipliers $\boldsymbol{\pi} = \boldsymbol{\pi}(x, y, t)$ are kept

$$0 = \delta \int_0^T \int_{\Omega} \frac{1}{2} D |\mathbf{u}|^2 - g D z + p(1 - D) + \phi (D_t + \nabla \cdot (\mathbf{u} D)) \\ - D \boldsymbol{\pi} \cdot (\partial_t \mathbf{a} + \mathbf{u} \cdot \nabla \mathbf{a}) d\mathbf{x} + \int_{\partial\Omega_s} \lambda (\partial_t h + \mathbf{v}_s \cdot \nabla (h + b) - w_s) dx dy dt$$

- \mathbf{u} -variation yields: $\mathbf{u} = \nabla \phi + \mathbf{v} \equiv \nabla \phi + (\nabla \mathbf{a})^T \boldsymbol{\pi}$ with $\mathbf{v} = \mathbf{v}(x, y, t)$
- Substitution

- Substitute $\mathbf{u} = \nabla\phi + \mathbf{v} \equiv \nabla\phi + (\nabla\mathbf{a})^T\boldsymbol{\pi}$ and $\lambda = D_s\phi_s$ into principle and clean up:

$$\begin{aligned}
 0 &= \delta \int_0^T \int_{\Omega} D\partial_t\phi + D\boldsymbol{\pi} \cdot \partial_t\mathbf{a} + \frac{1}{2}D|\nabla\phi + \mathbf{v}|^2 + Dgz + p(D-1) \, dxdt \\
 &\equiv \delta \int_0^T \left(\int_{\Omega_H} \int_b^{b+h} D\partial_t\phi + D\boldsymbol{\pi} \cdot \partial_t\mathbf{a} \, dx dy dz + \mathcal{H} \right) dt \quad (15)
 \end{aligned}$$

- with Hamiltonian/energy \mathcal{H} .
- Structure similar to canonical one: $0 = \delta \int_0^T -p\dot{q} + H(p, q) dt$.
- Variation wrt pairs $\{D, \phi\}, \{(\phi)_s, h\}, \{D\boldsymbol{\pi}, \mathbf{a}\} \dots$ yields modification of water wave equations.

- System:

$$\nabla^2 \phi + \nabla \cdot \mathbf{v} = 0 \quad (16a)$$

$$(\partial_t \phi)_s + \frac{1}{2} |(\nabla \phi)_s + \mathbf{v}|^2 + g(h + b) - \mathbf{v} \cdot \bar{\mathbf{u}} = 0 \quad (16b)$$

$$\partial_t h + \nabla \cdot (h \bar{\mathbf{u}}) = 0 \quad (16c)$$

$$\partial_t (h \mathbf{v}) + \nabla \cdot (h \bar{\mathbf{u}} \mathbf{v}) + h \mathbf{v} \nabla \bar{\mathbf{u}} = 0 \quad (16d)$$

$$h \bar{\mathbf{u}} = \int_b^{b+h} \nabla_H \phi dz + h \mathbf{v}. \quad (16e)$$

- Depth-averaged $\mathbf{v} = \mathbf{v}(x, y, t)$ and **potential vorticity** $q(x, y, t) = \nabla \times \mathbf{v}/h$:

$$(\partial_t + \bar{\mathbf{u}} \cdot \nabla) q = 0.$$

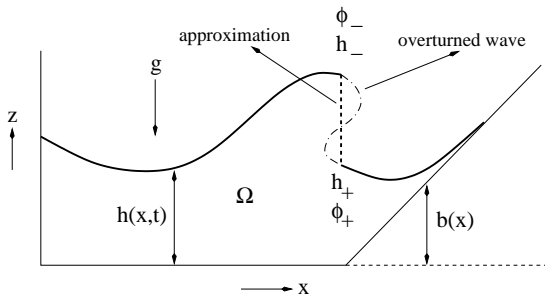
- **Potential flow** limit: $\mathbf{v} = 0$ or $\mathbf{v} = \nabla_H \varphi$.

- **Shallow water limit:** $\phi \rightarrow \phi_s$, follows from rewritten form

$$\partial_t h + \nabla \cdot (h \bar{\mathbf{u}}) = 0 \quad (17a)$$

$$\partial_t (h \bar{\mathbf{u}}) + \nabla_H \cdot (h \bar{\mathbf{u}} \bar{\mathbf{u}} + \frac{1}{2} g h^2) + g h \nabla b + h \nabla_H (\partial_t \phi)_s - h \partial_t (\overline{\nabla_H \phi}) + \frac{1}{2} h \nabla_H (|\mathbf{u}_s|^2 - |\bar{\mathbf{u}}|^2) + h (\bar{\mathbf{u}} \nabla \overline{\nabla_H \phi} - \bar{\mathbf{u}} \cdot \nabla \overline{\nabla_H \phi}) = 0. \quad (17b)$$

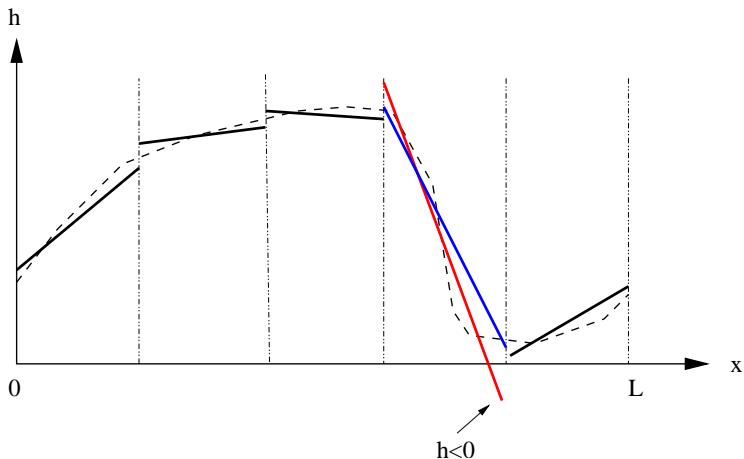
- Also leads to weak formulation for **bore and hydraulic jumps**:



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Discontinuous Galerkin finite element models

- Variables are represented by continuous functions/polynomials within an element K but are discontinuous across
- Example, depth field $h = h(x, t)$:



- Space and space-time discretizations.

- Domain Ω , $\partial\Omega_S$ mean free surface, $\partial\Omega_L$ lateral boundaries, $\partial\Omega_B$ rigid bed.

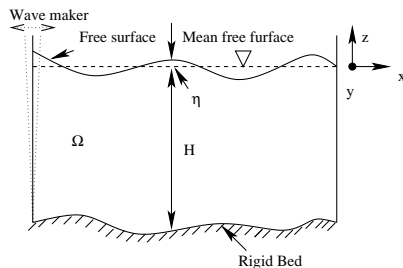


Figure: Sketch of flow domain and its boundaries.

- Velocity field \mathbf{u} **irrotational**.
- Introduce **velocity potential** ϕ as $\mathbf{u} = \bar{\nabla}\phi$ such that $\bar{\nabla} \times \mathbf{u} = 0$.
- Water depth: $H(x, y) + \eta(t, x, y) = h(t, x, y)$.

Dynamics of free surface waves (Luke, 1967 and Miles, 1977) contained in following functional:

$$\mathcal{L}_f[\phi, \phi_s, h] := \int_0^T \int_{\Omega_H} \int_b^{b+h} \partial_t \phi + \frac{1}{2} |\bar{\nabla} \phi|^2 + gz \, dx \, dy \, dz \, dt + \int_0^T \int_{\partial\Omega_L} g_N \phi \, dx \, dy \, dt \quad (18)$$

- Water depth and topography: $h = h(t, x, y)$ and $b = b(x, y)$.
- Prescribed normal velocity at lateral boundaries g_N .
- Velocity potential at the mean free surface: $\phi_s = \phi(t, x, y, z = b + h)$.

J.C. Luke: "A variational principle for a fluid with a free surface." JFM 1967.

J.W. Miles: "On Hamilton's principle for surface waves." JFM 1977.

Klopman, Van Groesen, Dingemans: "Variational Boussinesq model." Subm. JFM 2009

- Governing nonlinear equations:

$$\bar{\nabla}^2 \phi = 0 \quad \text{on } \Omega \quad (19)$$

with $\bar{\nabla} = (\partial_x, \partial_y, \partial_z)$ and ϕ the velocity potential.

- Free surface boundary conditions:

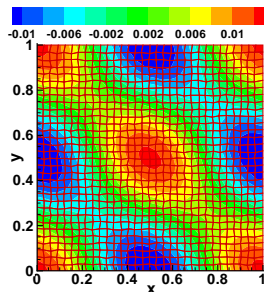
$$\partial_t \phi + \frac{1}{2} |\bar{\nabla} \phi|^2 + g(h + b) = 0 \quad \text{and} \quad (20)$$

$$\partial_t h + \bar{\nabla}_H \phi \cdot \bar{\nabla}_H (h + b) - \partial_z \phi = 0 \quad (21)$$

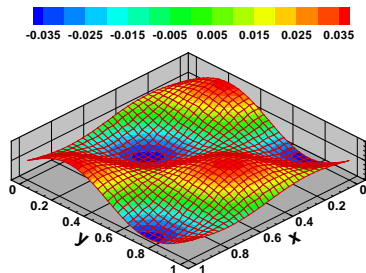
at the free surface $\partial\Omega_S : z = h(t, x, y) + b(x, y)$.

Harmonic Waves

- Two harmonic wave modes: for $p = 1, 2$ verified 2nd, 3rd-order.



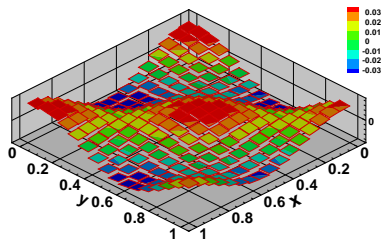
(a) Velocity potential.



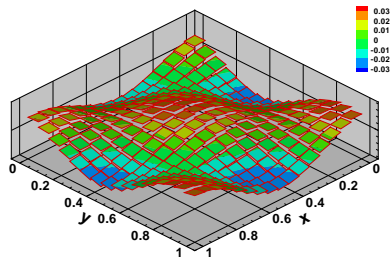
(b) Free surface wave height

Figure: Contour plots of the velocity potential and the free surface wave height at time $t = 0$ on an irregular grid of size $32 \times 32 \times 8$.

Comparison of variational and standard space-time DG schemes.



(a) Variational space-time DG scheme



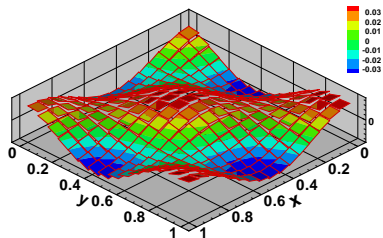
(b) Standard space-time DG scheme

Figure: Free surface evolution at $t = T$ on a grid of size $16 \times 16 \times 4$.

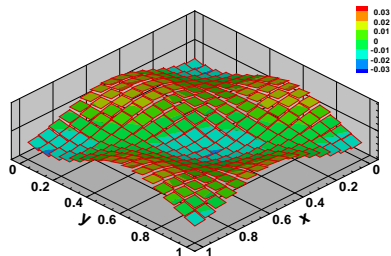
Ambati: Ph.D. Dissertation, University of Twente, eprints.eemcs.utwente.nl, 2008

VdV and Xu: "Space-time DGFEM for nonlinear water waves." JCP 224, 2007.

Comparison of variational and standard space-time DG schemes.



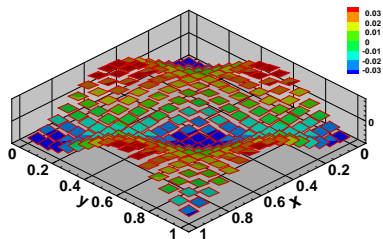
(a) Variational space-time DG scheme



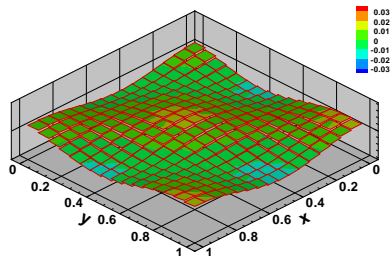
(b) Standard space-time DG scheme

Figure: Free surface evolution at $t = 2T$ on a grid of size $16 \times 16 \times 4$.

Comparison of variational and standard space-time DG schemes.



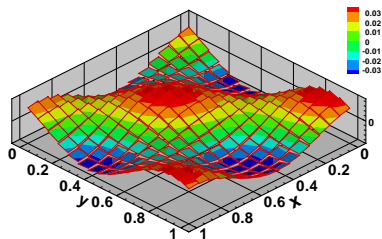
(a) Variational space-time DG scheme



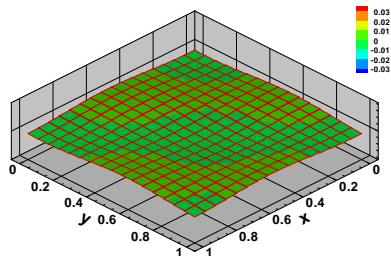
(b) Standard space-time DG scheme

Figure: Free surface evolution at $t = 4T$ on a grid of size $16 \times 16 \times 4$.

Comparison of variational and standard space-time DG schemes.



(a) Variational space-time DG scheme



(b) Standard space-time DG scheme

Figure: Free surface evolution at $t = 8T$ on a grid of size $16 \times 16 \times 4$.

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- Simplification **new model** in vertical under investigation, as well as bore relations.
- Extension variational DGFEM to **nonlinear free surface waves** underway.
- Extension to compatible DGFEM for **new water wave model with accurate dispersion and horizontal circulation**: NWO project.
- **MACE —Maths Assails Coastal Erosion**: fast/accurate flood/storm wave predictions along coast.

Conclusions

- Novel space-time DGFEM's developed for **hyperbolic systems with nonconservative products**; applied to **rotating shallow water equations**.
- **Novel variational DGFEM** potential flow free surface water waves. **No decay in amplitude**.

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We need to understand how they form and how they propagate, and find ways to harness their energy safely.

Maths holds the key to this understanding.

$$-\frac{\partial A}{\partial t} + (c+A)\frac{\partial A}{\partial x} + \frac{\partial^3 A}{\partial x^3} = 0$$

maths makes waves

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