Finite Element Methods for Partial Differential Equations

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Contents

Chapter 1

Introduction

In these class notes we will discuss finite element techniques for the solution of partial differential equations. The class notes should be used in combination with the book:

Introduction to Scientific Computing, by B. Lucquin and O. Pironneau, John Wiley & Sons, (1998), ISBN 0-471-97266-5.

The mark of the course is based on the student's performance in the following exercises and an oral examination.

- Three exercises are set: i, v, and viii and xi in Section 4. Bonus question ix.
- The oral examination occurs after making an appointment with Onno Bokhove, Room 416, Applied Mathematics, Phone 053 4893412, email: o.bokhove@math.utwente.nl; or Jaap van der Vegt.

Chapter 2

Finite Elements Methods for One-Dimensional Problems

Finite element methods provide a general solution technique for partial differential equations with a very wide range of applicability. They are especially well suited for problems which require unstructured meshes. In addition, finite element methods are based on a solid mathematical background. The basic concept of finite element methods is, however, more difficult to understand than for finite difference methods. In order to give an outline of the basic components of finite element methods we will discuss in this section their application to the solution of boundary value problems in one space dimension. We will show the basic steps to derive a finite element discretization, which will be generalized later to more dimensions and different boundary conditions.

2.1 Weak formulation for a model problem

Consider the following model problem:

$$\begin{aligned} \frac{d}{dx}\big(a(x)\frac{du}{dx}\big) + \frac{d}{dx}\big(b(x)u(x)\big) + c(x)u(x) &= s(x), \quad x \in (0,1), \\ u(0) &= g_0, \\ a(1)\frac{du(1)}{dx} + b(1)u(1) &= g_1, \end{aligned} \tag{2.1.1}$$

with $u \in C^2(0,1) \cap C^1([0,1])$. The functions $a, b \in C^1[0,1]$ and $c, s \in C^0([0,1])$ are given functions on the interval $(0,1)$ and $g_0, g_1 \in \mathbb{R}$ are given boundary data. Here, $C^n([x_0,x_1])$ denotes the space of n-times continuously differentiable functions on the interval $[x_0,x_1]$ with $x_1 > x_0$.

We can solve the boundary value problem (2.1.1) analytically or with a finite difference technique. This requires that u is at least twice continuously differentiable for the solution to exist. Also, the coefficients a and b must be sufficiently smooth. For many applications, for instance with discontinuous coefficients, and more general boundary conditions these requirements are too restrictive and we have to relax the smoothness requirements. This is accomplished by considering weak solutions and transforming the boundary value problem into a weak formulation.

In order to define the weak formulation for (2.1.1) we must first define the space:

$$V_g = \{u \in C^1([0,1]) \,|\, u(0) = g\}.$$

If we multiply (2.1.1) with arbitrary test functions $w \in V_0$, and integrate over the domain $[0,1]$ then we obtain the weighted residual formulation of the original equation (2.1.1):

$$\int_0^1 w(x)\Big(\frac{d}{dx}\big(a(x)\frac{du}{dx}\big) + \frac{d}{dx}\big(b(x)u(x)\big) + c(x)u(x) - s(x)\Big)dx = 0.$$

The weak formulation is now obtained through integration by parts:

$$\int_0^1 \Big(-a(x)\frac{dw}{dx}\frac{du}{dx} - b(x)\frac{dw}{dx}u(x) + c(x)w(x)u(x) - s(x)w(x)\Big)dx$$
$$+ \Big[w(x)\big(a(x)\frac{du}{dx} + b(x)u(x)\big)\Big]_{x=0}^{1} = 0.$$

The integration by parts results in boundary contributions, which can be replaced by either using the boundary conditions or using the restrictions imposed on the test functions w.

- At $x = 0$ we have the boundary condition $u(0) = g_0$ and we impose the restriction $w(0) = 0$ on the test functions w because this eliminates the boundary contribution $a(0)du(0)/dx + b(0)u(0)$ from the weak formulation, which we do not want to impose at $x = 0$. The boundary condition $u(0) = g_0$ is called an essential boundary condition.
- At $x = 1$ we can introduce the condition $a(1)du(1)/dx + b(1)u(1) = g_1$ into the weak formulation and there is no restriction on the test functions w at $x = 1$. This boundary condition is called a natural boundary condition because it is directly provided by the weak formulation.

The boundary value problem (2.1.1) can now be formulated in terms of the equivalent weak formulation:
Find a $u \in V_{g_0}$, such that for all $w \in V_0$, the equation:

$$\int_0^1 \Big(-a(x)\frac{dw}{dx}\frac{du}{dx} - b(x)\frac{dw}{dx}u(x) + c(x)w(x)u(x) - s(x)w(x)\Big)dx + w(1)g_1 = 0$$

is satisfied.

An important benefit of the weak formulation is that we now only have to require that u is one time differentiable instead of two times. Also, the functions a, b, c and d only have to be integrable and bounded. The weak formulation provides now the basis for the finite element discretization.

If we introduce the bilinear form $a(u,w) : V_g \times V_0 \to \mathbb{R}$, and the linear form $l(w) : V_0 \to \mathbb{R}$, which are defined as:

$$a(u,w) = \int_0^1 \Big(-a(x)\frac{du}{dx}\frac{dw}{dx} - b(x)u(x)\frac{dw}{dx} + c(x)u(x)w(x)\Big)dx$$
$$l(w) = \int_0^1 s(x)w(x)dx - w(1)g_1,$$

then we can express the weak formulation as:
Find a $u \in V_{g_0}$, such that for all $w \in V_0$, the equation:

$$a(u, w) = l(w)$$

is satisfied.

This abstract formulation is very useful, since many properties of the weak solution and the finite element discretization, such as existence and uniqueness, can be proved without knowing the specific details of the weak formulation.

2.2 Finite element formulation using linear basis functions

The weak formulation is the starting point for the numerical discretization. This is accomplished by introducing discrete spaces $V_{g,h}$, which approximate functions in V_g sufficiently accurate. This can be done with global basis functions, such as Fourier series and several types of orthogonal polynomials, and results in spectral methods. Spectral methods are very accurate with the proper choice of basis functions and when the problem has smooth solutions, but are difficult to apply in domains with a complicated shape and for general boundary conditions.

It is much easier to split the domain Ω into open domains K_j, which are called elements, such that $\cup_{j=1}^{N} \bar{K}_j = \bar{\Omega}$ and $\bar{K}_j \cap \bar{K}_{j'}$ for $j \neq j'$, $(j, j' \in \{1, \cdots, N\})$ is only a vertex. Here the closure $\bar{\Omega}$ of the domain is defined as $\Omega \cup \partial\Omega$, that is, the domain Ω and its boundary $\partial\Omega$. Hence, the elements cover the domain Ω and they do not overlap. The key feature of a finite element method is now the use of local basis functions, which are only non-zero in a small number of elements.

Subdivide the interval $(0, 1)$ into a finite number of N non-overlapping intervals $K_j = (x_{j-1}, x_j)$, $(j = 1, \cdots, N)$. Let $V_{g,h}^k = \{u \in C^0 \,|\, u \in P^k(K_j), \forall K_j \subset \Omega, u(0) = g\} \subset V_g$, with $P^k(0, 1)$ the space of polynomials of degree k on the interval $(0, 1)$. This implies that in each element we use polynomial basis functions of degree k, but we only require that the functions are continuous at the element boundaries. The basis functions ϕ_j are defined as: $\phi_j(x) \in P^k(K_j)$ if $x \in K_j$, and satisfy the condition $\phi_i(x_j) = \delta_{ij}$, with δ_{ij} the Kroncker delta symbol, which is defined as $\delta_{ij} = 1$ if $i = j$, and zero otherwise. This implies that:

$$\phi_j(x) = \begin{cases} 1, & \text{if } x = x_j, \\ 0, & \text{if } x = x_k, \ k \neq j. \end{cases}$$

An example of linear basis functions, see Fig. 2.1a, are:

$$\phi_j(x) = \begin{cases} (x - x_{j-1})/(x_j - x_{j-1}), & \text{if } x_{j-1} \leq x \leq x_j, \\ (x_{j+1} - x)/(x_{j+1} - x_j), & \text{if } x_j \leq x \leq x_{j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

The trial functions $u \in V_{g,h}^k$ and test functions $w \in V_{0,h}^k$ can now be defined as:

$$u(x) = g_0\phi_0(x) + \sum_{j=1}^{N} u_j\phi_j(x) \quad \text{and} \quad w(x) = \sum_{j=1}^{N} w_j\phi_j(x).$$

The contribution $g_0\,\phi_0(x)$ is added to satisfy the boundary condition $u(0) = g_0$, since all other basis functions ϕ_j are zero at the boundary. Since each of the global basis functions ϕ_j is only non-zero in the two elements connecting to the node x_j, it is convenient to introduce the element basis functions ψ_m, $(m = 1, 2)$, which are defined in the interval $(x_{j-1} \leq x \leq x_j)$ as:

$$\psi_1(x) = \frac{x_j - x}{x_j - x_{j-1}}, \quad \text{and} \quad \psi_2 = \frac{x - x_{j-1}}{x_j - x_{j-1}}.$$

Hence, $\psi_1(x) = \phi_{j-1}(x)$ and $\psi_2(x) = \phi_j(x)$ for $x_{j-1} \leq x \leq x_j$.

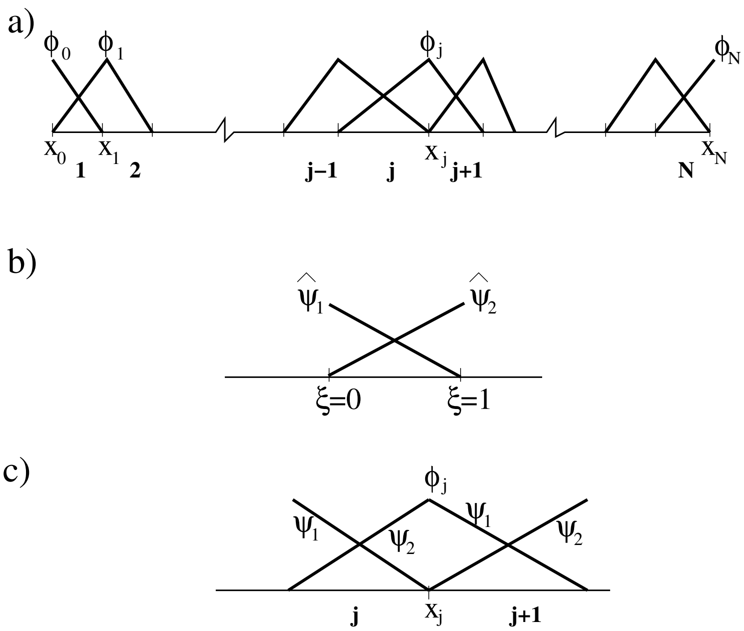

Figure 2.1: a) Linear global basis functions in one dimension are "tent" functions with value one at the base node and zero at the neighbouring node on either side. b) Local basis functions. c) The support of basis function ϕ_j of test function w_j in elements j and $j + 1$.

For more complex basis functions, and also domains in multiple dimensions it is, however, much easier to introduce a reference element $\hat{K} = (0, 1)$, with local coordinates ξ, and use the mapping F_{K_j} which is defined as:

$$F_{K_j} : (0, 1) \to (x_{j-1}, x_j) : \xi \mapsto x = (x_j - x_{j-1})\xi + x_{j-1}. \tag{2.2.1}$$

In local coordinates we define now the basis functions $\hat{\psi}_n(\xi)$, $(n = 1, 2)$, see Fig. 2.1b which are equal to:

$$\hat{\psi}_1(\xi) = 1 - \xi, \quad \text{and} \quad \hat{\psi}_2(\xi) = \xi,$$

and the basis functions $\psi_n(x)$, $(n = 1, 2)$ are now obtained using the relation:

$$\psi_n(x) = \hat{\psi}_n(F_{K_j}^{-1}(x)),$$

with:

$$\xi = F_{K_j}^{-1}(x) = (x - x_{j-1})/(x_j - x_{j-1}).$$

The finite element discretization is now obtained by choosing test functions w which satisfy the condition $w(x) = \phi_j(x)$, $j = 1, \cdots, N$. Since the basis functions ϕ_j are linearly independent and span the the space $V_{0,h}^k$ this will result in N-linearly independent equations for the coefficient u_j.

The finite element discretization is most easily obtained by splitting the integrals in the weak formulation into integrals over each element:

$$\int_0^1 \Big(-a(x)\frac{dw}{dx}\frac{du}{dx} - b(x)\frac{dw}{dx}u(x) + c(x)w(x)u(x) - s(x)w(x)\Big)dx =$$
$$\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j} \Big(-a(x)\frac{dw}{dx}\frac{du}{dx} - b(x)\frac{dw}{dx}u(x) + c(x)w(x)u(x) - s(x)w(x)\Big)dx. \tag{2.2.2}$$

Using the mapping F_{K_j} we can define the following elemental integrals:

$$A_{nm}^j = \int_{x_{j-1}}^{x_j} a(x)\frac{d\psi_n(x)}{dx}\frac{d\psi_m(x)}{dx}dx = \frac{1}{h_j}\int_0^1 a(x(\xi))\frac{d\hat{\psi}_n(\xi)}{d\xi}\frac{d\hat{\psi}_m(\xi)}{d\xi}d\xi, \tag{2.2.3}$$

$$B_{nm}^j = \int_{x_{j-1}}^{x_j} b(x)\frac{d\psi_n(x)}{dx}\psi_m(x)dx = \int_0^1 b(x(\xi))\frac{d\hat{\psi}_n}{d\xi}\hat{\psi}_m(\xi)d\xi, \tag{2.2.4}$$

$$C_{nm}^j = \int_{x_{j-1}}^{x_j} c(x)\psi_n(x)\psi_m(x)dx = h_j\int_0^1 c(x(\xi))\hat{\psi}_n(\xi)\hat{\psi}_m(\xi)d\xi, \tag{2.2.5}$$

$$S_n^j = \int_{x_{j-1}}^{x_j} s(x)\psi_n(x)dx = h_j\int_0^1 s(x(\xi))\hat{\psi}_n(\xi)d\xi \tag{2.2.6}$$

with $h_j = x_j - x_{j-1}$ and $m, n = 1, 2$.

Consider now the test function $w(x) = \phi_j(x)$, $j = 2, \cdots, N-1$. This function is only non-zero in the elements K_j and K_{j+1}, where it is equal to $w(x) = \psi_2(x) = \hat{\psi}_2(F_{K_j}^{-1}(x)) = \hat{\psi}_2(\xi)$ and $w(x) = \psi_1(x) = \hat{\psi}_1(F_{K_{j+1}}^{-1}(x)) = \hat{\psi}_1(\xi)$, respectively. If we introduce the representation of u_h in each element K_j:

$$\begin{aligned} u_h(x) &= u_{j-1}\hat{\psi}_1(F_{K_j}^{-1}(x)) + u_j\hat{\psi}_2(F_{K_j}^{-1}(x)) \\ &= u_{j-1}\hat{\psi}_1(\xi) + u_j\hat{\psi}_2(\xi), \end{aligned}$$

into the element integrals of the weak formulation (2.2.2) then we will only obtain non-zero contributions from the integrals in the elements K_j and K_{j+1}, because in the other elements the test function w is zero, see Fig. 2.1c. The weak formulation for the test function

$w(x) = \phi_j(x)$, $(j = 2, \cdots, N-1)$ is now equal to:

$$\int_{x_{j-1}}^{x_j} \Big(-a(x)\frac{d\psi_2}{dx}\big(u_{j-1}\frac{d\psi_1}{dx} + u_j\frac{d\psi_2}{dx}\big) - b(x)\frac{d\psi_2}{dx}\big(u_{j-1}\psi_1(x) + u_j\psi_2(x)\big) + c(x)\psi_2(x)\big(u_{j-1}\psi_1(x) + u_j\psi_2(x)\big) - s(x)\psi_2(x)\Big)dx + \tag{2.2.7}$$

$$\int_{x_j}^{x_{j+1}} \Big(-a(x)\frac{d\psi_1}{dx}\big(u_j\frac{d\psi_1}{dx} + u_{j+1}\frac{d\psi_2}{dx}\big) - b(x)\frac{d\psi_1}{dx}\big(u_j\psi_1(x) + u_{j+1}\psi_2(x)\big) + c(x)\psi_1(x)\big(u_j\psi_1(x) + u_{j+1}\psi_2(x)\big) - s(x)\psi_1(x)\Big)dx = 0. \tag{2.2.8}$$

If we introduce the integrals (2.2.3-2.2.6) into (2.2.7-2.2.8) then this relation can be simplified as:

$$\big(-A^j_{2,1} - B^j_{2,1} + C^j_{2,1}\big)u_{j-1} + \big(-A^j_{2,2} - B^j_{2,2} + C^j_{2,2} - A^{j+1}_{1,1} - B^{j+1}_{1,1} + C^{j+1}_{1,1}\big)u_j + \big(-A^{j+1}_{1,2} - B^{j+1}_{1,2} + C^{j+1}_{1,2}\big)u_{j+1} = S^j_2 + S^{j+1}_1, \qquad j = 2, \cdots, N-1.$$

For the test function $w(x) = \phi_1(x)$ we obtain slightly different relations, because the trial function u_h in element K_1 must be modified to account for the boundary condition at $x = 0$:

$$\begin{aligned} u_h(x) &= g_0\hat{\psi}_1(F^{-1}_{K_1}(x)) + u_1\hat{\psi}_2(F^{-1}_{K_1}(x)) \\ &= g_0\hat{\psi}_1(\xi) + u_1\hat{\psi}_2(\xi), \end{aligned}$$

If we introduce this expression into the weak formulation then we obtain:

$$\int_{x_0}^{x_1} \Big(-a(x)\frac{d\psi_2}{dx}\big(g_0\frac{d\psi_1}{dx} + u_1\frac{d\psi_2}{dx}\big) - b(x)\frac{d\psi_2}{dx}\big(g_0\psi_1(x) + u_1\psi_2(x)\big) + c(x)\psi_2(x)\big(g_0\psi_1(x) + u_1\psi_2(x)\big) - s(x)\psi_2(x)\Big)dx + \tag{2.2.9}$$

$$\int_{x_1}^{x_2} \Big(-a(x)\frac{d\psi_1}{dx}\big(u_1\frac{d\psi_1}{dx} + u_2\frac{d\psi_2}{dx}\big) - b(x)\frac{d\psi_1}{dx}\big(u_1\psi_1(x) + u_2\psi_2(x)\big) + c(x)\psi_1(x)\big(u_1\psi_1(x) + u_2\psi_2(x)\big) - s(x)\psi_1(x)\Big)dx = 0, \tag{2.2.10}$$

which can be simplified as:

$$\big(-A^1_{2,2} - B^1_{2,2} + C^1_{2,2} - A^2_{1,1} - B^2_{1,1} + C^2_{1,1}\big)u_1 + \big(-A^2_{1,2} - B^2_{1,2} + C^2_{1,2}\big)u_2 = S^1_2 + S^2_1 - \big(-A^1_{2,1} - B^1_{2,1} + C^1_{2,1}\big)g_0.$$

Similarly, we obtain for the $w(x) = \phi_N(x)$ the relation:

$$\int_{x_{N-1}}^{x_N} \Big(-a(x)\frac{d\psi_2}{dx}\big(u_{N-1}\frac{d\psi_1}{dx} + u_N\frac{d\psi_2}{dx}\big) - b(x)\frac{d\psi_2}{dx}\big(u_{N-1}\psi_1(x) + u_N\psi_2(x)\big) + c(x)\psi_2(x)\big(u_{N-1}\psi_1(x) + u_N\psi_2(x)\big) - s(x)\psi_2(x)\Big)dx = -g_1, \tag{2.2.11}$$

which can be simplified into:

$$\big(-A_{2,1}^N - B_{2,1}^N + C_{2,1}^N\big)u_{N-1} + \big(-A_{2,2}^N - B_{2,2}^N + C_{2,2}^N\big)u_N = S_2^N - g_1.$$

Introduce the coefficients:

$$\begin{aligned}
M_j^1 &= -A_{2,1}^j - B_{2,1}^j + C_{2,1}^j, && j = 2,\cdots,N,\\
M_j^2 &= -A_{2,2}^j - B_{2,2}^j + C_{2,2}^j - A_{1,1}^{j+1} - B_{1,1}^{j+1} + C_{1,1}^{j+1}, && j = 1,\cdots,N-1\\
M_N^2 &= -A_{2,2}^N - B_{2,2}^N + C_{2,2}^N,\\
M_j^3 &= -A_{1,2}^{j+1} - B_{1,2}^{j+1} + C_{1,2}^{j+1}, && j = 1,\cdots,N-1,
\end{aligned}$$

and:

$$\begin{aligned}
R_j &= S_2^j + S_1^{j+1}, && j = 2,\cdots,N-1,\\
R_1 &= S_2^1 + S_1^2 + \big(A_{2,1}^1 + B_{2,1}^1 - C_{2,1}^1\big)g_0,\\
R_N &= S_2^N - g_1,
\end{aligned}$$

then we can write the equations for the coefficients u_j for $j = 1,\cdots,N$ as the linear system:

$$\begin{pmatrix}
M_1^2 & M_1^3 \\
M_2^1 & M_2^2 & M_2^3 \\
 & \ddots & \ddots & \ddots \\
 & & M_j^1 & M_j^2 & M_j^3 \\
 & & & \ddots & \ddots & \ddots \\
 & & & & M_{N-1}^1 & M_{N-1}^2 & M_{N-1}^3 \\
 & & & & & M_N^1 & M_N^2
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_j \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix}
=
\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_j \\ \vdots \\ R_{N-1} \\ R_N \end{pmatrix}.$$

This linear system has a tri-diagonal matrix and can be solved using standard numerical linear algebra techniques.

2.3 Finite element discretization for a fourth order ordinary differential equation

Consider the following fourth order differential equation:

$$-\frac{d^4u}{dx^4} + u = 0, \qquad x \in (0,\pi), \tag{2.3.1}$$

with boundary conditions: $u(0) = 1$, $du(0)/dx = 0$, $u(\pi) = 0$, $du(\pi)/dx = u_w$. For this problem we can not directly use the finite element technique discussed in the previous section because a straightforward finite element discretization now requires more smoothness of the basis functions at the element boundaries. In order to define the weak formulation for (2.3.1) we define the spaces:

$$\begin{aligned}
H_0^2(\Omega) &= \big\{u, \frac{du}{dx}, \frac{d^2u}{dx^2} \in L^2(\Omega),\ u(0) = u(\pi) = \frac{du(0)}{dx} = \frac{du(\pi)}{dx} = 0\big\},\\
H_g^2(\Omega) &= \{u, \frac{du}{dx}, \frac{d^2u}{dx^2} \in L^2(\Omega),\ u(0) = 1,\ u(\pi) = \frac{du(0)}{dx} = 0,\ \frac{du(\pi)}{dx} = u_w\}
\end{aligned}$$

with $\Omega = (0, \pi)$ and $L^2(\Omega)$ the space of square (Lebesgue) integrable functions on Ω: $L^2(\Omega) = \{u \mid \int_\Omega |u(x)|^2 dx < \infty\}$. The weak formulation is now obtained if we multiply the differential equation (2.3.1) with arbitrary test functions $w \in H_0^2(\Omega)$:

$$\int_{x=0}^{\pi} w(x)\Big(-\frac{d^4u(x)}{dx^4} + u(x)\Big)dx = 0. \tag{2.3.2}$$

If we integrate by parts the first contribution in (2.3.2) two times then we obtain:

$$\begin{aligned}\int_{x=0}^{\pi} w(x)(-\frac{d^4u(x)}{dx^4})dx = & \int_{x=0}^{\pi} \frac{dw(x)}{dx}\frac{d^3u(x)}{dx^3}dx \\ = & -\int_{x=0}^{\pi} \frac{d^2w(x)}{dx^2}\frac{d^2u(x)}{dx^2}dx,\end{aligned}$$

where all the boundary contributions disappeared due to boundary conditions on w. The weak formulation for (2.3.1) can now be formulated as:

Find a $u \in H_g^2(\Omega)$, such that for all $w \in H_0^2(\Omega)$, the weak formulation:

$$\int_{x=0}^{\pi} \Big(-\frac{d^2w(x)}{dx^2}\frac{d^2u(x)}{dx^2} + w(x)u(x)\Big)dx = 0$$

is satisfied

For the finite element discretization we subdivide the domain Ω into N elements $K_j = (x_{j-1}, x_j)$. An important difference with the weak formulation for the second order differential equation is now that we need basis functions which have an integrable second derivative instead of only an integrable first derivative. Define the spaces:

$$\begin{aligned}V_{0,h}^k &= \{u \in C^1(\Omega) \,|\, u \in P^k(K_j),\, \forall K_j \subset \Omega,\, u(0) = u(\pi) = \frac{du(0)}{dx} = \frac{du(\pi)}{dx} = 0\},\\ V_{g,h}^k &= \{u \in C^1(\Omega) \,|\, u \in P^k(K_j),\, \forall K_j \subset \Omega, u(0) = 1,\, u(\pi) = \frac{du(0)}{dx} = 0,\, \frac{du(\pi)}{dx} = u_w\},\end{aligned}$$

with $P^k(K_j)$ the space of polynomials of degree k on the interval K_j. This type of basis functions can be obtained using Hermite polynomials. Hermite polynomials use both the function value and its derivatives at the nodal points, which automatically ensures that they are continuously differentiable in the whole domain Ω. The simplest Hermite basis functions are the cubic polynomials $p_3 \in P^3([0,1])$, which are given by the relation:

$$p_3(\xi) = \hat{\psi}_1(\xi)f(0) + \hat{\psi}_2(\xi)f(1) + \hat{\psi}_3(\xi)\frac{df(0)}{d\xi} + \hat{\psi}_4(\xi)\frac{df(1)}{d\xi}, \tag{2.3.3}$$

with f and $df/d\xi$ prescribed data at $\xi = 0$ and $\xi = 1$. The polynomial $p_3(\xi)$ must satisfy the conditions:

$$\begin{aligned}p_3(0) &= f(0), & p_3(1) &= f(1),\\ \frac{dp_3(0)}{d\xi} &= \frac{df(0)}{d\xi}, & \frac{dp_3(1)}{d\xi} &= \frac{df(1)}{d\xi},\end{aligned}$$

which can be used to define the basis functions $\hat{\psi}_n$, $(n = 1, \cdots, 4)$. The conditions for the functions $\hat{\psi}_n$, $(n = 1, \cdots, 4)$ are equal to:

$$\begin{aligned}
&\hat{\psi}_1(0) = 1, \quad &&\hat{\psi}_1(1) = 0, \quad &&\frac{d\hat{\psi}_1(0)}{d\xi} = 0, \quad &&\frac{d\hat{\psi}_1(1)}{d\xi} = 0,\\
&\hat{\psi}_2(0) = 0, &&\hat{\psi}_2(1) = 1, &&\frac{d\hat{\psi}_2(0)}{d\xi} = 0, &&\frac{d\hat{\psi}_2(1)}{d\xi} = 0,\\
&\hat{\psi}_3(0) = 0, &&\hat{\psi}_3(1) = 0, &&\frac{d\hat{\psi}_3(0)}{d\xi} = 1, &&\frac{d\hat{\psi}_3(1)}{d\xi} = 0,\\
&\hat{\psi}_4(0) = 0, &&\hat{\psi}_4(1) = 0, &&\frac{d\hat{\psi}_4(0)}{d\xi} = 0, &&\frac{d\hat{\psi}_4(1)}{d\xi} = 1.
\end{aligned}$$

This are necessary and sufficient conditions to define the polynomials $\hat{\psi}_n$, $(n = 1, \cdots, 4)$:

$$\hat{\psi}_1(\xi) = (\xi - 1)^2(2\xi + 1), \tag{2.3.4}$$

$$\hat{\psi}_2(\xi) = \xi^2(3 - 2\xi), \tag{2.3.5}$$

$$\hat{\psi}_3(\xi) = \xi(\xi - 1)^2, \tag{2.3.6}$$

$$\hat{\psi}_4(\xi) = \xi^2(\xi - 1). \tag{2.3.7}$$

The Hermite polynomials are now completely defined with the relation (2.3.3) in combination with the basis functions (2.3.4-2.3.7).

Using the mapping F_{K_j} between the reference element $\hat{K} = (0, 1)$ and the element K_j, given by (2.2.1), we can express the trial functions u_h in each element K_j as:

$$\begin{aligned}
u_h(x) &= u_{j-1}\psi_1(x) + u_j\psi_2(x) + \frac{du_{j-1}}{dx}\psi_3(x) + \frac{du_j}{dx}\psi_4(x), \qquad \text{with } x \in K_j\\
&= u_{j-1}\hat{\psi}_1(\xi) + u_j\hat{\psi}_2(\xi) + \frac{du_{j-1}}{dx}\hat{\psi}_3(\xi) + \frac{du_j}{dx}\hat{\psi}_4(\xi)\\
&= \sum_{n=1}^{4} \hat{u}_n(K_j)\hat{\psi}_n(\xi),
\end{aligned} \tag{2.3.8}$$

with $u_j = u(x_j)$, and $\hat{u}_1(K_j) = u_{j-1}$, $\hat{u}_2(K_j) = u_j$, $\hat{u}_3(K_j) = \frac{du_{j-1}}{dx}$ and $\hat{u}_4(K_j) = \frac{du_j}{dx}$. We also used the relation:

$$\psi_n(x) \equiv \hat{\psi}_n(F_{K_j}^{-1}(x)) = \hat{\psi}_n(\xi), \qquad \text{for } x \in K_j,$$

between the basis functions $\psi_n(x)$ in the element K_j and the basis functions $\hat{\psi}_n(\xi)$ in the reference element $\hat{K}$. The expression for the test functions w_h are identical, only with u replaced with w. The global test functions in the domain Ω can now be written as:

$$w_h(x) = \sum_{j=1}^{N-1} \big(w_j\phi_j^{(0)}(x) + \frac{dw_j}{dx}\phi_j^{(1)}(x)\big),$$

with the global basis functions $\phi_j^{(n)}(x)$, $(n = 0, 1)$ defined as:

$$\begin{aligned}
\phi_j^{(0)} &= \psi_2(x), &&\text{if } x_{j-1} \le x \le x_j \\
\phi_j^{(0)} &= \psi_1(x), &&\text{if } x_j \le x \le x_{j+1} \\
\phi_j^{(1)} &= \psi_4(x), &&\text{if } x_{j-1} \le x \le x_j \\
\phi_j^{(1)} &= \psi_3(x), &&\text{if } x_j \le x \le x_{j+1}
\end{aligned}$$

and zero elsewhere in the domain $[0, \pi]$. See also Figure Fig. 2.2. The expressions for the trial functions in the domain Ω are similar, only the contributions from the boundary must be added:

$$u_h(x) = \sum_{j=0}^{N} \big(u_j \phi_j^{(0)}(x) + \frac{du_j}{dx} \phi_j^{(1)}(x)\big),$$

with $\phi^{(n)}$, $(n = 1, 2)$, at the boundary defined as:

$$\begin{aligned}
\phi_0^{(0)} &= \psi_1(x), &&\text{if } x_0 \le x \le x_1 \\
\phi_0^{(1)} &= \psi_3(x), &&\text{if } x_0 \le x \le x_1 \\
\phi_N^{(0)} &= \psi_2(x), &&\text{if } x_{N-1} \le x \le x_N \\
\phi_N^{(1)} &= \psi_4(x), &&\text{if } x_{N-1} \le x \le x_N,
\end{aligned}$$

and zero elsewhere. For the finite element discretization we do not directly use the global basis functions, but the finite element discretization is constructed with the local basis functions in each element.

Consider now the test function w_h at the internal node points x_j, $(j = 1, \cdots, N)$. The only non-zero basis functions at the point x_j are $\phi_j^{(0)}$ and $\phi_j^{(1)}$, and since both coefficients w_j and $\frac{dw_j}{dx}$ are arbitrary, we obtain now two equations for each node point x_j. The weak formulation for the test functions $w_h(x) = \phi_j^{(0)}(x)$ is now equal to:

$$\int_{x_{j-1}}^{x_j} \big(-\frac{d^2\psi_2}{dx^2}\frac{d^2u_h}{dx^2} + \psi_2(x)u_h(x)\big)dx + \int_{x_j}^{x_{j+1}} \big(-\frac{d^2\psi_1}{dx^2}\frac{d^2u_h}{dx^2} + \psi_1(x)u_h(x)\big)dx = 0,$$

and for $w_h(x) = \phi_j^{(1)}(x)$ we obtain:

$$\int_{x_{j-1}}^{x_j} \big(-\frac{d^2\psi_4}{dx^2}\frac{d^2u_h}{dx^2} + \psi_4(x)u_h(x)\big)dx + \int_{x_j}^{x_{j+1}} \big(-\frac{d^2\psi_3}{dx^2}\frac{d^2u_h}{dx^2} + \psi_3(x)u_h(x)\big)dx = 0.$$

If we transform each element integral to an integral over the reference element $\hat{K}$ using the mapping F_{K_j} (2.2.1) and introduce the polynomial representation of u_h (2.3.8), then we obtain the following set of $2(N-1)$ equations for the unknowns u_j and du_j/dx at each of

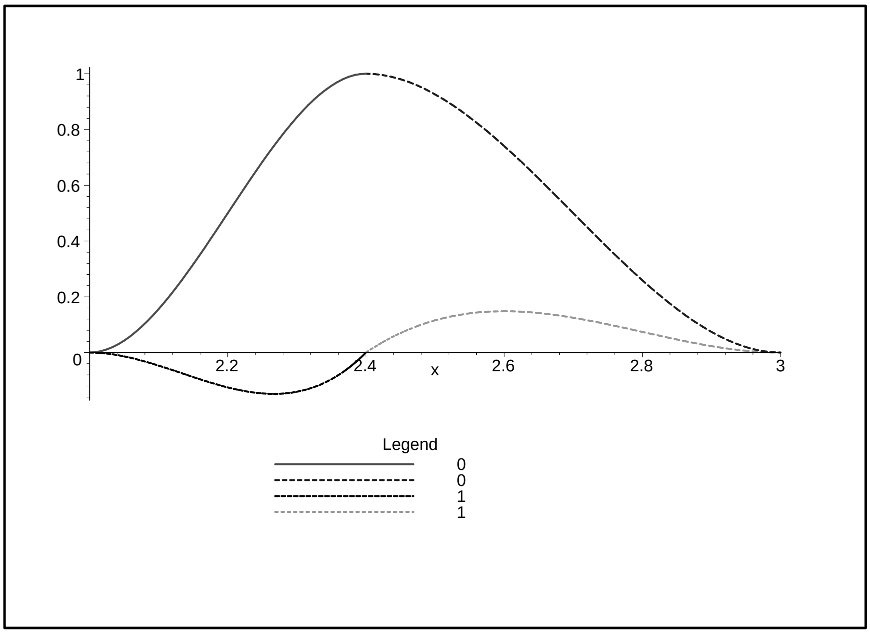

Figure 2.2: Global basis functions $\phi_j^{(0)}$ and $\phi_j^{(1)}$. Here $x_{j-1} = 2.0, x_j = 2.4, x_{j+1} = 3$.

the internal nodal points x_j, $(j = 1, \cdots, N-1)$:

$$\int_0^1 \Big[-\frac{d^2\hat{\psi}_2}{d\xi^2}\frac{d^2}{d\xi^2}\Big(\sum_{n=1}^{4}\hat{u}_n(K_j)\hat{\psi}_n(\xi)\Big)\big(\frac{d\xi}{dx}\big)^2_{K_j} + \hat{\psi}_2(\xi)\Big(\sum_{n=1}^{4}\hat{u}_n(K_j)\hat{\psi}_n(\xi)\Big)\Big]\Big(\frac{dx}{d\xi}\Big)_{K_j} d\xi +$$
$$\int_0^1 \Big[-\frac{d^2\hat{\psi}_1}{d\xi^2}\frac{d^2}{d\xi^2}\Big(\sum_{n=1}^{4}\hat{u}_n(K_{j+1})\hat{\psi}_n(\xi)\Big)\big(\frac{d\xi}{dx}\big)^2_{K_{j+1}} + \hat{\psi}_1(\xi)\Big(\sum_{n=1}^{4}\hat{u}_n(K_{j+1})\hat{\psi}_n(\xi)\Big)\Big]\Big(\frac{dx}{d\xi}\Big)_{K_{j+1}} d\xi = 0$$

$$\int_0^1 \Big[-\frac{d^2\hat{\psi}_4}{d\xi^2}\frac{d^2}{d\xi^2}\Big(\sum_{n=1}^{4}\hat{u}_n(K_j)\hat{\psi}_n(\xi)\Big)\big(\frac{d\xi}{dx}\big)^2_{K_j} + \hat{\psi}_4(\xi)\Big(\sum_{n=1}^{4}\hat{u}_n(K_j)\hat{\psi}_n(\xi)\Big)\Big]\Big(\frac{dx}{d\xi}\Big)_{K_j} d\xi +$$
$$\int_0^1 \Big[-\frac{d^2\hat{\psi}_3}{d\xi^2}\frac{d^2}{d\xi^2}\Big(\sum_{n=1}^{4}\hat{u}_n(K_{j+1})\hat{\psi}_n(\xi)\Big)\big(\frac{d\xi}{dx}\big)^2_{K_{j+1}} + \hat{\psi}_3(\xi)\Big(\sum_{n=1}^{4}\hat{u}_n(K_{j+1})\hat{\psi}_n(\xi)\Big)\Big]\Big(\frac{dx}{d\xi}\Big)_{K_{j+1}} d\xi = 0,$$

where the metrical coefficients $\big(\frac{d\xi}{dx}\big)_{K_j}$ can be computed from the mapping (2.2.1). Introduce the coefficients $A_{mn}(K_j)$, which are defined as:

$$A_{mn}(K_j) = \int_0^1 \Big(-\frac{1}{h_j}\frac{d^2\hat{\psi}_n}{d\xi^2}\frac{d^2\hat{\psi}_m}{d\xi^2} + h_j\hat{\psi}_n(\xi)\hat{\psi}_m(\xi)\Big)d\xi,$$

and combine the coefficients u_j and $\frac{du_j}{dx}$ in $\hat{u}_n(K_j)$ and $\hat{u}_n(K_{j+1})$, then we obtain the following set of algebraic equations for the coefficients u_j, and $\frac{du_j}{dx}$, $(j = 1, \cdots, N-1)$:

$$A_{2,1}(K_j)u_{j-1} + A_{2,3}(K_j)\frac{du_{j-1}}{dx} + \big(A_{2,2}(K_j) + A_{1,1}(K_{j+1})\big)u_j + \big(A_{2,4}(K_j) + A_{1,3}(K_{j+1})\big)\frac{du_j}{dx} + A_{1,2}(K_{j+1})u_{j+1} + A_{1,4}(K_{j+1})\frac{du_{j+1}}{dx} = 0 \tag{2.3.9}$$

$$[12pt] A_{4,1}(K_j)u_{j-1} + A_{4,3}(K_j)\frac{du_{j-1}}{dx} + \big(A_{4,2}(K_j) + A_{3,1}(K_{j+1})\big)u_j + \big(A_{4,4}(K_j) + A_{3,3}(K_{j+1})\big)\frac{du_j}{dx} + A_{3,2}(K_{j+1})u_{j+1} + A_{3,4}(K_{j+1})\frac{du_{j+1}}{dx} = 0. \tag{2.3.10}$$

The finite element discretization is now complete, except for the boundary conditions. The coefficients u_j and $\frac{du_j}{dx}$ for $j = 0$ and $j = N$ are given by the boundary conditions $u(0) = 1$, $\frac{du(0)}{dx} = 0$, $u(\pi) = 0$, $\frac{du(\pi)}{dx} = u_w$. If we bring these coefficients to the righthand side of (2.3.9-2.3.10) then we obtain for $j = 1$ the equations:

$$\big(A_{2,2}(K_1) + A_{1,1}(K_2)\big)u_1 + \big(A_{2,4}(K_1) + A_{1,3}(K_2)\big)\frac{du_1}{dx} + A_{1,2}(K_2)u_2 + A_{1,4}(K_2)\frac{du_2}{dx} = -A_{2,1}(K_1)$$

$$\big(A_{4,2}(K_1) + A_{3,1}(K_2)\big)u_1 + \big(A_{4,4}(K_1) + A_{3,3}(K_2)\big)\frac{du_1}{dx} + A_{3,2}(K_2)u_2 + A_{3,4}(K_2)\frac{du_2}{dx} = -A_{4,1}(K_1)$$

and for $j = N - 1$:

$$A_{2,1}(K_{N-1})u_{N-2} + A_{2,3}(K_{N-1})\frac{du_{N-2}}{dx} + A_{2,2}(K_{N-1}) + A_{1,1}(K_N)u_{N-1} + \big(A_{2,4}(K_{N-1}) + A_{1,3}(K_N)\big)\frac{du_{N-1}}{dx} = -A_{1,4}(K_N)u_w$$

$$A_{4,1}(K_{N-1})u_{N-2} + A_{4,3}(K_{N-1})\frac{du_{N-2}}{dx} + \big(A_{4,2}(K_{N-1}) + A_{3,1}(K_N)\big)u_{N-1} + \big(A_{4,4}(K_{N-1}) + A_{3,3}(K_N)\big)\frac{du_{N-1}}{dx} = -A_{3,4}(K_N)u_w.$$

The finite element discretization for the fourth order ordinary differential equation (2.3.1) can now be summarized as follows:

Introduce the matrices $M^1(K_j), M^2(K_j), M^3(K_j) \in \mathbb{R}^{2\times 2}$ and the vector $R(K_j) \in R^2$, which are defined as:

$$
\begin{aligned}
&M^1_{1,1}(K_j) = A_{2,1}(K_j), && M^1_{1,2}(K_j) = A_{2,3}(K_j) && \\
&M^1_{2,1}(K_j) = A_{4,1}(K_j), && M^1_{2,2}(K_j) = A_{4,3}(K_j), && j = 2, N-1 \\
&M^2_{1,1}(K_j) = A_{2,2}(K_j) + A_{1,1}(K_{j+1}), && M^2_{1,2}(K_j) = A_{2,4}(K_j) + A_{1,3}(K_{j+1}) && \\
&M^2_{2,1}(K_j) = A_{4,2}(K_j) + A_{3,1}(K_{j+1}), && M^2_{2,2}(K_j) = A_{4,4}(K_j) + A_{3,3}(K_{j+1}), && j = 1, N-1 \\
&M^3_{1,1}(K_j) = A_{1,2}(K_{j+1}), && M^3_{1,2}(K_j) = A_{1,4}(K_{j+1}) && \\
&M^3_{2,1}(K_j) = A_{3,2}(K_{j+1}), && M^3_{2,2}(K_j) = A_{3,4}(K_{j+1}), && j = 1, N-2
\end{aligned}
$$

$$
\begin{aligned}
&R_1(K_1) = -A_{2,1}(K_1) \\
&R_2(K_1) = -A_{4,1}(K_1) \\
&R_1(K_j) = 0 \\
&R_2(K_j) = 0, \qquad j = 2, N-2 \\
&R_1(K_{N-1}) = -A_{1,4}(K_N)u_w \\
&R_2(K_{N-1}) = -A_{3,4}(K_N)u_w
\end{aligned}
$$

The finite element discretization then can be expressed in terms of the following linear system of equations for the coefficients u_j and $\frac{du_j}{dx}$:

$$MV = R$$

with the matrix $M \in \mathbb{R}^{2(N-1)\times 2(N-1)}$ defined as:

$$
M = \begin{pmatrix}
M^2(K_1) & M^3(K_1) & & & & & \\
M^1(K_2) & M^2(K_2) & M^3(K_2) & & & & \\
 & \ddots & \ddots & \ddots & & & \\
 & & M^1(K_j) & M^2(K_j) & M^3(K_j) & & \\
 & & & \ddots & \ddots & \ddots & \\
 & & & M^1(K_{N-2}) & M^2(K_{N-2}) & M^3(K_{N-2}) & \\
 & & & & M^1(K_{N-1}) & M^2(K_{N-1}) &
\end{pmatrix}.
$$

The vector of coefficients $V \in \mathbb{R}^{2(N-1)}$ is defined as

$$
V = \begin{pmatrix}
v(K_1) \\
v(K_2) \\
\vdots \\
v(K_j) \\
\cdots \\
v(K_{N-2}) \\
v(K_{N-1})
\end{pmatrix}
$$

with $v_j = (u_j, \frac{du_j}{dx}) \in \mathbb{R}^2$, and the righthand side $R \in \mathbb{R}^{2(N-1)}$ is defined as:

$$R = \begin{pmatrix} R(K_1) \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ R(K_{N-1}) \end{pmatrix}$$

This linear system has a block-tridiagonal matrix and can be solved using standard numerical linear algebra techniques.

2.4 Discontinuous Galerkin methods

2.4.1 Model hyperbolic problem

Consider the following scalar model hyperbolic equation

$$\partial_t u + \partial_x F = S \tag{2.4.1}$$

with $u = u(x,t)$, flux $F = F(u(x,t), x, t)$ and an additional "source" term $S = S(u(x,t), x, t)$. Hence, F and S can depend implicitly on x and t through their dependence on u, and explicitly on x and t . When S and F do not depend explicitly on x and t, we can write $F = F(u)$ and $S = S(u)$.

2.4.2 Finite elements

We define a tessellation $\mathcal{T}_h$ of N elements K_k in the open spatial flow domain $\Omega \in \mathcal{R}^{n_d}$ with dimension n_d and boundary $\partial\Omega$:

$$\mathcal{T}_h = \{K_k | \bigcup_{k=1}^{N} K_k = \Omega \text{ and } K_k \cap K_{k'} = \emptyset \text{ if } k \neq k', 1 \leq k, k' \leq N\}. \tag{2.4.2}$$

We consider finite element discretizations of (2.4.1) with approximations U_h and v_h to the variable $u(x,t)$ and basis function $v(x)$, to be introduced shortly, respectively. These are such that U_h and v_h belong to the space

$$V_h = \{v \in L^1(0,1) : v|_{K_k} \in P^{d_P}(K_k), k = 1, \ldots, N\}, \tag{2.4.3}$$

in which $P^{d_P}(K_k)$ denotes the space of polynomials in K_k of degree d_P, and $L^1(0,1)$ the space of Lebesque integrable functions (Brenner and Scott, 1994).

2.4.3 Polynomial basisfunctions

In one dimension, the bounded interval $\Omega := [a,b] \subset \mathcal{R}$ is partitioned by $N+1$ "regular" faces (points in one dimension) $\mathcal{E} := \{x_k\}_{k=0}^{N}$ and into N "regular" elements, see for one dimension Fig. 2.3. It is convenient to introduce a reference element in one dimension,

$\hat{K} = [-1, 1]$, and define the mapping $F_K : \mathcal{R} \to \mathcal{R}$ between the reference element $\hat{K}$ and element K_k as follows

$$x = F_{K_k}(\zeta) = \sum_{m=1}^{2} x_{k,m}\,\chi_m(\zeta) = \hat{x}_k + |K_k|\,\zeta/2, \tag{2.4.4}$$

where $x_{k,1} = x_{k,L}$ and $x_{k,2} = x_{k,R}$ are the left and right end points of element $K_k = (x_{k,L}, x_{k,R}) = (x_k, x_{k+1})$. The shape functions are $\chi_1(\zeta) = (1-\zeta)/2$, $\chi_2(\zeta) = (1+\zeta)/2$; $\hat{x}_k = (x_{k,L} + x_{k,R})/2$, and $|K_k| = (x_{k,R} - x_{k,L})$. In the basis element $\hat{K}$ we define basis functions

$$\hat{\varphi}_0(\zeta) = 1, \quad \text{and} \quad \hat{\varphi}_m(\zeta) = \zeta^m \quad \text{for} \quad m = 1, \ldots, d_P \tag{2.4.5}$$

with d_P the maximum degree of the polynomials used. Finally we relate the local basis functions in $\hat{K}$ to the basis functions in K_k as follows

$$\hat{\varphi}_n(\zeta) = \hat{\varphi}_n(F_{K_k}^{-1}(x)) = \varphi_{n,k}(x) \quad \text{for} \quad n = 0, \ldots, d_P. \tag{2.4.6}$$

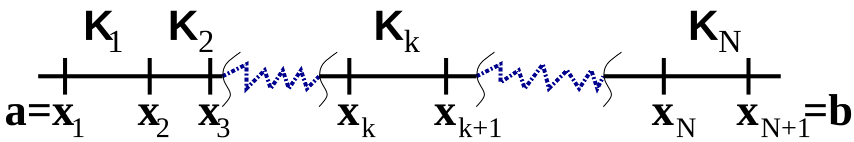

Figure 2.3: A sketch of the finite element space Ω and some of the elements.

The variable u and test functions v are approximated in each element K_k by their polynomial approximations U_h and v_h, respectively, as follows

$$U_h(x,t) = \sum_{m=0}^{d_P} \hat{U}_m(K_k,t)\,\psi_m(x) \qquad \text{and} \qquad v_h(x) = \sum_{m=0}^{d_P} \hat{v}_m(K_k)\,\psi_m(x) \tag{2.4.7}$$

with polynomial basis functions $\psi_m(x)$'s $\in P^{d_P}(K_k)$. These are chosen such that $\hat{U}_0 = \bar{U}$ represents the mean and $\hat{U}_m$ with $m > 0$ the higher-order projections of u on U_h, as follows

$$\hat{U}_0 = \bar{U}(K_k,t) = \int_{K_k} u(x,t)\,\mathrm{d}x/|K_k| \quad \text{and}$$
$$\psi_{m,k}(x) = \left\{ \begin{array}{ll} 1 & \text{if } m = 0 \\ \varphi_{m,k}(x) - \int_{K_k} \varphi_{m,k}(x)\,\mathrm{d}x/|K_k| & \text{if } m \geq 1 \end{array} \right\}. \tag{2.4.8}$$

2.4.4 Weak formulation and finite element discretization

A weak discrete formulation is found by multiplying (2.4.1) by an arbitrary smooth function $v_h = v_h(K_k) \in C^1$ in each element K_k and integrating over each element K_k. Note that v_h is discontinuous across element boundaries. A global weak formulation is then obtained after summing the local weak formulation over all elements

$$\sum_{k=1}^{N} \Big\{ \int_{K_k} v_h\,\partial_t U_h\,\mathrm{d}x + [F(x_{k,R}^-)\,v_h(x_{k,R}^-) - F(x_{k,L}^+)\,v_h(x_{k,L}^+)] -$$
$$\int_{K_k} F\,\partial_x v_h\,\mathrm{d}x - \int_{K_k} S\,v_h\,\mathrm{d}x \Big\} = 0, \tag{2.4.9}$$

where $v_h(x_{k,R}^-) = \lim_{x \uparrow x_{k,R}} v_h(x,t)$ and $v_h(x_{k,L}^+) = \lim_{x \downarrow x_{k,L}} v_h(x,t)$, etcetera. Note that the flux $F(x_{k,R}^-)$ on the right and flux $F(x_{k,L}^+)$ on the left are evaluated at the inside of the element K_k. (We only denote dependencies explicitly when confusion may arise.) Sofar, the fluxes immediately to the left and right of an element face are not necessarily the same, due to the discontinuous nature of the polynomial representation in each element. To ensure a continuous flux at each face, a numerical flux $H(\cdot,\cdot)$ will be constructed as a function of the values of U_h immediately left and right of each face at x_k.

Taking $d_P = 1$ in (2.4.7), we approximate U by its mean and its slope, and also its quadratic approximation at edge elements, while v_h is approximated by its mean and slope

$$U_h(x,t) = \bar{U}_k + \hat{U}_k\, \psi_{1,k}(x) \quad \text{and} \quad v_h(x) = \bar{W}_k + \hat{W}_k\, \psi_{1,k}(x) \tag{2.4.10}$$

with $\bar{U}_k = \bar{U}(K_k,t)$, etcetera. Since $\bar{W}(K_k)$ and $\hat{W}(K_k)$ are arbitrary and $\int_{K_k} \psi_{1,k}\,\mathrm{d}x = 0$, we obtain the following equations for the mean and fluctuating part

$$\begin{aligned}
&|K_k|\,\frac{d\bar{U}}{dt} + [F(x_{k,R}^-) - F(x_{k,L}^-)] - \int_{K_k} S\,\mathrm{d}x = 0 \qquad \text{and}\\
&\frac{d\hat{U}}{dt}\int_{K_k} \psi_{1,k}^2\,\mathrm{d}x + [F(x_{k,R}^-) - F(x_{k,L}^-)] -\\
&\qquad \int_{K_k} F\,\frac{d\psi_{1,k}}{dx}\,\mathrm{d}x - \int_{K_k} S\,\psi_{1,k}\,\mathrm{d}x + D(U_h)\,\hat{U}_k \int_{K_k}\left(\frac{d\psi_{1,k}}{dx}\right)^2 \mathrm{d}x = 0,
\end{aligned} \tag{2.4.11}$$

where we have used $\psi_{1,k}(x_{k,R}^-) = 1$ and $\psi_{1,k}(x_{k,L}^+) = -1$, and introduced a friction term in the slope equation with dissipation $D(U_h) > 0$ intended to stabilize the integration of $\hat{U}_k$. The elemental integrals in (2.4.11) are

$$\begin{aligned}
&\int_{K_k} \psi_{1,k}^2\,\mathrm{d}x = \int_{K_k} \varphi_{1,k}^2\,\mathrm{d}x - \frac{1}{|K_k|}\left(\int_{K_k} \varphi_{1,k}(x)\,\mathrm{d}x\right)^2 = \frac{|K_k|}{3},\\
&\int_{K_k} (d\psi_{1,k}/dx)^2\,\mathrm{d}x = 4/|K_k|.
\end{aligned} \tag{2.4.12}$$

By combining (2.4.11) and (2.4.12) and introducing a coontinuous numerical flux $H(\bar{U}_{k-1} + \hat{U}_{k-1}, \bar{U}_k - \hat{U}_k)$ at each face x_k, we obtain a spatial discretization of the scalar model hyperbolic problem (2.4.1)

$$\begin{aligned}
\frac{d\bar{U}_k}{dt} &= -\frac{1}{|K_k|}\Big(H(\bar{U}_k + \hat{U}_k, \bar{U}_{k+1} - \hat{U}_{k+1}) - H(\bar{U}_{k-1} + \hat{U}_{k-1}, \bar{U}_k - \hat{U}_k)\Big) +\\
&\quad \frac{1}{2}\int_{-1}^{1} S\,\mathrm{d}\zeta\\
\frac{d\hat{U}_k}{dt} &= -\frac{3}{|K_k|}\Big(H(\bar{U}_k + \hat{U}_k, \bar{U}_{k+1} - \hat{U}_{k+1}) + H(\bar{U}_{k-1} + \hat{U}_{k-1}, \bar{U}_k - \hat{U}_k)\Big) +\\
&\quad \frac{3}{|K_k|}\int_{-1}^{1} F\,\mathrm{d}\zeta + \frac{3}{2}\int_{-1}^{1} S\,\zeta\,\mathrm{d}\zeta - \frac{12}{|K_k|^2}\,D(U_h)\,\hat{U}_k.
\end{aligned} \tag{2.4.13}$$

The dissipation factor $D(U_h)$ in each element is usually chosen to be the sum of the square or modulus of the jump in U_h at each face times a small dissipation constant ϵ_d. The value of ϵ_d can be chosen judiciously. A more rational, mathematical choice of stabilizing but minimal dissipation remains a topic of active research.

2.4.5 Numerical flux

The approximate solution is discontinuous at x_k and, hence, the flux $F(x_k)$ is not well defined. The flux F at x_k is therefore replaced by a numerical flux $H(U_-, U_+)$ with $U_- = \lim_{x\uparrow x_k} U$ and $U_+ = \lim_{x\downarrow x_k} U$ such that it is i) locally Lipschitz which implies that there is a constant $K \geq 0$ such that:

$$|H(v,w) - F(u)| \leq K \max(|v-u|, |w-u|)$$

for all v, w with $|v-u|$ and $|w-u|$ sufficiently small; ii) consistent, that is, $H(U,U) = F(U)$; iii) a non-decreasing function of its first argument; and iiv) a non-increasing function of its second argument.

A straightforward choice would be to take the average of the value left and right of x_k, giving $H(U_-, U_+) = (U_- + U_+)/2$. However, this choice generally leads to instability and several alternatives fluxes introduce either numerical dissipation or exactly calculate the numerical flux based on a local analysis of a simplified initial-value problem, such as the Riemann problem, or both.

The Rusanov flux (Batten *et al.*, 1997) is a popular and simple flux approximation at x_R^k

$$H(U_-, U_+) = H_{Rusanov} = \frac{1}{2}\Big(F(U_-) + F(U_+)\Big) - \frac{|\lambda_m|}{2}|U_+ - U_-| \tag{2.4.14}$$

with $\lambda_m = \partial F/\partial u$ the maximum signal speed in the flow. It is, however, quite dissipative (Batten *et al.*, 1997).

The two-state HLLC flux approximation by Toro, Spruce and Spears (see, e.g., Toro 1999) and the Engquist–Osher flux (Osher and Chrakravarthy, 1983) are more accurate. For the Burgers equation with $F(u) = u^2/2$ the Engquist-Osher flux reads

$$H(U_-, U_+) = \begin{cases} \frac{1}{2}U_+^2 & \text{if} \quad U_\pm < 0 \\ \frac{1}{2}U_-^2 & \text{if} \quad U_\pm > 0 \\ 0 & \text{if} \quad U_- < 0 < U_+ \\ \frac{1}{4}(U_-^2 + U_+^2) & \text{if} \quad U_- > 0 > U_+. \end{cases} \tag{2.4.15}$$

2.4.6 Time integration

Writing the finite element discretization abstractly as the dynamical system $dU(t)/dt = R(U)$ a total variation diminishing (TVD) third-order Runge-Kutta discretization (see, for the definition of TVD, Shu and Osher, 1989) is

$$\begin{aligned} U^{(1)} &= U^n + \Delta t\, R(U^n) \\ U^{(2)} &= \frac{1}{4}\Big(3\,U^n + U^{(1)} + \Delta t\, R(U^{(1)})\Big) \\ U^{n+1} &= \frac{1}{3}\Big(U^n + 2\,U^{(2)} + 2\,\Delta t\, R(U^{(2)}\Big) \end{aligned} \tag{2.4.16}$$

with $U^n = U(t)$ and $U^{n+1} = U(t + \Delta t)$. The use of a TVD Runge-Kutta time integration scheme is essential to avoid severe stability restrictions.

Chapter 3

Finite Elements Methods for Two-Dimensional Elliptical Equations

3.1 Model Equation

Consider the following model problem

$$\boldsymbol{\nabla} \cdot (A \boldsymbol{\nabla} u) + \boldsymbol{\nabla} \cdot (\mathbf{B} u) + \mathbf{C} u = \mathbf{S} \tag{3.1.1}$$

with $u = u(x, y)$, $A = A(x, y)$, $\mathbf{B} = \mathbf{B}(x, y)$, $\mathbf{C} = \mathbf{C}(x, y)$, and $\mathbf{S} = \mathbf{S}(x, t, u)$.

3.2 Weak Formulation

Keywords: function spaces, energy minimization/variational principle, basis functions, stiffness, assemblage.

Lucquin & Pironneau §2.1–2.3, §3.1–3.2.1 (Laplace's equation with Dirichlet boundary conditions); §5.1, 5.2, 5.4 (Laplace's equation with Robin boundary conditions); 6.1 (Non-symmetric problems).

3.3 Mesh & Master Elements

Keywords: tesselation, triangular and quadrilateral elements, higher-order elements.

Lucquin & Pironneau §2.4–2.5.1 (triangulation and quadrature); 5.3 (second-order finite elements).

3.4 Data Structures

Lucquin & Pironneau §3.2.2 (band storage), 3.2.5 (optimizing band storage: Cuthill and McKee algorithm), 3.3 (numbering of nodes and elements), 3.4 (compressed spare storage).

3.5 Solution Techniques

Keywords: direct methods, LU decomposition, Choleski decomposition, preconditioned gradient methods, GMRES.

Lucquin & Pironneau §3.2.3 (Choleski decomposition & solution), 3.4.7 (Choleski factorization), 4.1 (preconditioned conjugate gradient), 6.2 (GMRES)

Chapter 4

Exercises

i. Consider a horizontal channel periodic in y with slip flow at the walls at $x = 0, L_x$ filled with water with a rest depth D. The domain is $L_x \times L_y$ with $L_x = 1$. The linear nondimensional non-rotating shallow water equations are

$$\partial_t u = -\partial_x \eta, \quad \partial_t v = -\partial_y \eta, \quad \partial_t \eta + \boldsymbol{\nabla} \cdot (D\,\mathbf{v}) = 0 \quad \text{with} \quad u|_{x=0,Lx} = 0 \tag{4.0.1}$$

with η the (small) deviation from the free surface and a typical mean rest depth H. The non-rotating shallow water equations (4.0.1) can for topography $D = D(x, y)$ be reduced to

$$\partial_{tt}\eta - \boldsymbol{\nabla} \cdot (D\,\boldsymbol{\nabla}\eta) = 0 \qquad \text{with} \qquad \partial_x \eta\Big|_{x=0,L_x} = 0 \tag{4.0.2}$$

(effectively taking the gravitational acceleration $g = 1$). Consider normal mode solutions for the simplified case with $D = D(x)$

$$\eta(x, y, t) = \sum_{m,n} \hat{\eta}_m \Big(A_{mn} \cos(\lambda_n\, y + \mu_{mn}\, t) + B_{mn} \sin(\lambda_n\, y + \mu_{mn}\, t) \Big). \tag{4.0.3}$$

Substitution of (4.0.3) into (4.0.2) for the simplified case $D = D(x)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(D\,\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x} \right) + (\mu^2 - D\,\lambda^2)\,\hat{\eta} = 0 \qquad \text{with} \qquad \frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\Big|_{x=0,L_x} = 0. \tag{4.0.4}$$

We will solve (4.0.4) with a continuous Galerkin finite element method brute force at high resolution using linear test and basis functions. The weak formulation follows by multiplying (4.0.4) with a test function w, integration by parts and using the boundary conditions. As w is arbitrary, we choose $w = \phi_i$ and expand $\hat{\eta}_h = \sum_{i=1}^{N_{el}} \eta_j\, \phi_j$ on the N_{el} elements using global, piecewise linear top-hat basis functions ϕ_j. Hence, we obtain the matrix system

$$R_{ij}\eta_j = \frac{1}{\mu^2}\, A_{ij}\, \eta_j \tag{4.0.5}$$

$$A_{ij} = C_{ij} + E_{ij} \tag{4.0.6}$$

for appropriate matrices A, C, E and R. Hence, we have a generalized eigenvalue problem. Usually linear algebra routines solving this problem $R\,\eta(1/\mu^2) = A\,\eta$ calculate a few of the largest eigenvalues and -vectors $1/\mu^2$, while we need the smallest values of μ^2. Each global basis function ϕ_i consist of the sum of two local basis functions $\phi_i^{(2)} = (1-\zeta)/2$ and $\phi_{i-1}^{(1)} = (1+\zeta)/2$ residing in the adjacent elements i and $i-1$, and with $\zeta \in [-1,1]$ the local reference coordinate. In element K_k, $\zeta = 1$ occurs at node x_{k+1} and $\zeta = -1$ at node x_k. So element K_k has length $|K_k| = x_{k+1} - x_k$.

(a) Check the above manipulations.

(b) Derive the weak formulation. Provide full detail about the function spaces used, integration by parts and use of the boundary conditions.

(c) Derive the discretized weak formulation in terms of an expansion in global basis functions. Describe how you use the local basis functions in an implementation of your discretization. Use piecewise linear basis or trial functions. Show how an assembly of your global matrices which can also be used in higher dimensions. Provide full detail.

(d) Use a numerical (Matlab) routine to solve the generalized eigenvalue problem, for unit depth $D = 1$ for which

$$\mu_{mn,s=\pm 1} = s\,\pi\,\sqrt{m^2 + l_n^2} \tag{4.0.7}$$

for $m = \ldots, -1, 0, 1, \ldots$ with $\lambda_n = 2\,n\,\pi/L_y = \pi\,l_n$. The solution is then

$$\eta = \sum_{m,n,s=\pm 1} \cos(\pi\,m\,x)\,\big(A_{mns}\,\cos(\lambda_n\,y + \mu_{mn,s}\,t) + B_{mns}\,\sin(\lambda_n\,y + \mu_{mn,s}\,t)\big). \tag{4.0.8}$$

Note that indeed $\mathrm{d}\left(\cos(\pi\,m\,x)\right)/\mathrm{d}x|_{x=0,1} = 0$. Show that the above is an exact solution. Verify your numerical solution in detail by comparing with a few exact solutions.

(e) Choose a variable depth $D = D(x)$ of your choice and plot the first four eigenvalues and eigenvectors. Normalize the eigenvectors by setting $\eta(0) = 1$. Motivate why your solution is trustworthy.

ii. Consider the following non-dimensional differential equation for the prediction of coastal waves over coastal topography:

$$\nabla\cdot\left(\frac{1}{H(x)}\nabla\psi_t\right) + J\left(\psi, \frac{f}{H(x)}\right) = 0 \tag{4.0.9}$$

with $0 < x < L$, a transport stream function ψ, simplified boundary conditions $\psi(x=0) = \psi(x=L) = 0$, Jacobian $J(a,b) \equiv a_x\,b_y - a_y\,b_x$, a Coriolis parameter f which incorporates effects due to the Earth's rotation, time t and variable depth $H = H(x)$, see also Fig. 4.1. Partial derivatives of $\psi(x,y,t)$ with respect to x, y and t are written as ψ_x, ψ_y and ψ_t, respectively. The streamfunction is such that the velocities off and along the coast are

$$u = -\frac{1}{H(x)}\,\psi_y \qquad \text{and} \qquad v = \frac{1}{H(x)}\,\psi_x, \tag{4.0.10}$$

in the x- and y-direction, respectively.

Introducing a normal-mode ansatz $\psi = \hat{\psi} \exp\left[i\,(l\,y + \omega\,t)\right]$, with $i^2 = -1$, alongshore wavenumber l and frequency ω, into (4.0.9) we find, after dropping the carets, that (4.0.9) becomes:

$$\omega \left(\frac{d}{dx}\left[\frac{1}{H(x)}\frac{d\psi}{dx}\right] - \frac{l^2}{H(x)}\,\psi \right) - f\,l\,\psi \frac{d}{dx}\left[\frac{1}{H(x)}\right] \quad = \quad 0. \qquad (4.0.11)$$

We assume that $H = H(x)$ for $0 < x < W$ and that $H = H_0$ for $W < x < L$. Note that wavenumber l, frequency ω, and f are independent of x.

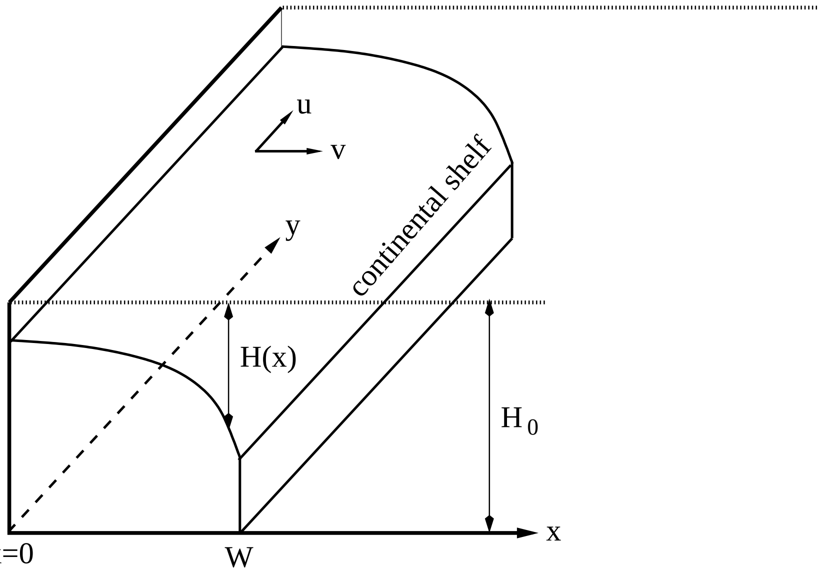

Figure 4.1: Definition sketch of the coastal area.

- a) Determine the weak formulation of (4.0.11).
  b) Specify explicitly which boundary conditions on ψ and the test function you have used.
- Introduce a local coordinate system in each element with local coordinate $\xi \in [-1, 1]$.
- a) Use linear basis functions to discretize the weak formulation.
  b) Make a sketch of the local and global basis functions versus ξ with special attention to boundary areas.
- Use the weak formulation and the linear basis functions to derive the finite element discretization for (4.0.11). In order to simplify notation introduce the following symbols for the elemental stiffness matrices in element Ω_k:

$$C_{ij}^{(k)} = \int_{\Omega_k} \frac{1}{H(x)} \frac{d\hat{\phi}_i(x)}{dx} \frac{d\hat{\phi}_j(x)}{dx}\,\mathrm{d}x, \quad D_{ij}^{(k)} = \int_{\Omega_k} \frac{1}{H(x)}\,\hat{\phi}_i(x)\hat{\phi}_j(x)\,\mathrm{d}x,$$

$$E_{ij}^{(k)} = \dots$$

for the elemental or local basis functions $\hat{\phi}_i$ with $i, j \in \{0, 1\}$.

- Give the expressions for the element integrals $C_{ij}^{(k)}$, $D_{ij}^{(k)}$ and $E_{ij}^{(k)}$ using the local coordinate system and Simpson's rule[1].
- Give a sketch of the global matrix structure for the finite element discretization in terms of the elements of the elemental matrices.
- The exact solution of (4.0.11) can be found when we take $H = H_1 < H_0$ for $0 < x < W$. We have to match the solutions for $x < W$ and $x > W$ at $x = W$. That is: ψ should be continuous at $x = W$ and the pressure should be continuous. Continuity of pressure can be shown to result in the following:

$$\frac{\omega}{f}\left[\frac{1}{H}\frac{d\psi}{dx}\right]_{x=W_-}^{x=W_+} - l\left[\psi\,\frac{1}{H}\right]_{x=W_-}^{x=W_+} = 0,$$

which can also be seen as an integrated form of (4.0.11).

a) Verify that the dispersion relation is

$$\sigma = \frac{\omega}{f} = \frac{\sinh(l\,W)\,\sinh l\,(L-W)\,[1 - H_1/H_0]}{H_1\,\sinh(l\,W)\,\cosh l\,(L-W)/H_0 + \cosh(l\,W)\,\sinh l\,(L-W)} \tag{4.0.12}$$

and plot ω versus l for $f = 1, H_0 = 1, H_1 = 0.1, W = 1$.

b) Denote the frequencies with $l = 0.2, 0.5, 1, 2$ also separately in this plot and put them in a table.

c) Verify that the streamfunction is

$$\psi \quad = \quad \left\{ \begin{array}{ll} \psi_c = a_l\,\sinh(l\,x)\,\sinh l\,(W-L) & 0 < x < W \\ \psi_s = a_l\,\sinh(l\,W)\,\sinh l\,(x-L) & W < x < L \end{array} \right\} \tag{4.0.13}$$

and plot the stream function $\psi = \hat{\psi(x)}$ for $l = 0.2, 0.5, 1, 2$ with $f = 1, H_0 = 1, H_1 = 0.1, W = 1$. Choose a_l such that the $\int_0^L \psi\,\mathrm{d}x = 1$. (In dimensional form we can for example take $f^* = 10^{-4}\,s^{-1}, H_0^* = 2000\,m, H_1^* = 200\,m, W^* = 10\,km$ with the asterisks denoting the dimensional quantities.)

- How do we incorporate a discontinuity in $H = H(x)$ at $x = W$ in the finite-element and its weak formulation?
- The discretized linear system you have found is a generalized eigenvalue problem of the form $\mathbf{A}\,\boldsymbol{\psi} = \omega\,\mathbf{B}\boldsymbol{\psi}$ for the appropriate matrices $\mathbf{A}, \mathbf{B}$ and vector of nodal points $\boldsymbol{\psi}$.

a) Identify $\mathbf{A}, \mathbf{B}$.

b) Use Matlab to solve this linear system for $f = 1, H_0 = 1, H_1 = 0.1, W = 1$, the selected values of $l = 0.2, 0.5, 1, 2$ and $L = 10\,W$. Choose the right eigenvalue and plot the relevant eigenvector, such that $\int_0^L \psi\,\mathrm{d}x = 1$.

c) Calculate the solution for 25, 100 and 200 elements. It may be useful to experiment with using an irregular discretization. Mention the elemental distribution you use.

d) Put these selected outcomes in a table.

[1]Hint: use Maple throughout this exercise to check the calculations

e) Explain the difference between the numerical and exact solutions for ω.

Linear algebra routines in other languages can be found at www.netlib.org.

- a) Plot $\ln(\max_{x\in(0,L)} ||\psi_{exact}(x)| - |\psi(x)||)$ versus the number of nodes for $L = 10\,W$ and $l = 0.2, 0.5, 1, 2$ for 200 uniformly distributed elements.

  b) What is the order of accuracy of the method?
- Plot the numerical and exact solutions for ψ together on the finest mesh for the four selected values of l and $L = 10\,W$. Mention your elemental distribution.
- The answers to the above questions and an outline of the structure of your program should be given clearly and separately from the numerical program, which you should attach as well.

iii. Consider a finite layer of rock of density ρ under the influence of gravity $g = 9.81\,m/s^2$ and atmospheric pressure $p_a = 0.1\,MPa$. Treating the rock layer as an elastic solid, the vertical displacement $w(z,t)$ is modeled with Navier's equation and boundary conditions

$$\begin{aligned} &\rho\,\partial_{tt}^2 w = \partial_z((\lambda + 2\,\mu)\partial_z w) - \rho\,g \qquad \text{with} \quad z \in \Omega = [0, L], \\ &w(0,t) = A\,\cos\sigma\,t, \quad \text{and} \quad (\lambda + 2\,\mu)\partial_z w = p_a \quad \text{at} \quad z = L \end{aligned} \tag{4.0.14}$$

with vertical coordinate z, time t, and the Lamé constants λ, μ (N/m^2).

- Show that w is the sum of a steady state part $\bar{w}(z)$ and a time-dependent part $w'(z,t)$, i.e. $w = \bar{w} + w'$, where the steady state $\bar{w}$ satisfies

$$\begin{aligned} 0 &= \partial_z((\lambda + 2\,\mu)\partial_z \bar{w}) - \rho\,g \qquad \text{with} \quad z \in [0, L], \\ \bar{w}(0) &= 0, \quad \text{and} \quad (\lambda + 2\,\mu)\partial_z \bar{w} = p_a \quad \text{at} \quad z = L. \end{aligned} \tag{4.0.15}$$

- Find the weak formulation of (4.0.15) and clearly denote the boundary conditions required for $\bar{w}$ and the test function v for general $\lambda(z), \mu(z)$.
- Use linear basis functions to discretize the weak formulation into a finite element discretization. Introduce a local coordinate $\xi \in [-1, 1]$ in each element. Make a sketch of the local and global basis functions versus ξ with special attention to boundary areas.
- Give a sketch of the global matrix structure for the finite element discretization in terms of the elements of the elemental stiffness matrices in element Ω_k. Describe how you can assemble the global mass matrix from these elemental matrices. Find the expressions for the elemental matrices using the local coordinate system and use two-point Gauss-Legendre quadrature

$$\int_{-1}^{1} f(\zeta)d\zeta \approx f\Big(-\sqrt(1/3)\Big) + f\Big(\sqrt(1/3)\Big) \tag{4.0.16}$$

  in your numerical implementation.
- Identify $\mathbf{A}, \mathbf{b}$ in the discretized system $\mathbf{A}\,\boldsymbol{w} = \mathbf{b}$ of (4.0.15) and solve this system (using linear algebra routines in e.g. Matlab). Take 10, 50 and 100 uniform elements and use the parameter values $\rho = 2500\,kg/m^3, L = 3\,km, \lambda = 1.85 \times 10^4\,MPa, \mu = 1.30 \times 10^4\,MPa$.

- Calculate the exact solution. Compare the exact and numerical solution by plotting them in one figure, and plot the spatial integral of the absolute value of their difference as function of the resolution.

iv. Consider the dimensionless system of equations for a rotor blade[2]

$$m\,\partial_{\tau\tau}w + m\,e\partial_{\tau\tau}\theta + \partial_{xx}(a_1\,\partial_{xx}w) - \partial_x(a_2\,\partial_x w) - \partial_x(m\,x\,e\,\theta) = l(x,\tau) \quad (4.0.17)$$
$$b_3\,\partial_{\tau\tau}\theta + m\,e\,\partial_{\tau\tau}w + b_3\,\theta - \partial_x\left((b_1+b_2)\,\partial_x\theta\right) + m\,x\,e\,\partial_x w = q(x,\tau) \quad (4.0.18)$$

with variables $w = w(x,\tau)$ the displacement in the vertical and $\theta = \theta(x,\tau)$ the rotation angle of the blade, given loads $l = l(x,\tau)$ and torsion load $q = q(x,\tau)$, in a domain $x \in [0, L]$ with the fixed end at $x = 0$ and the unclamped end at $x = L$, time τ, the functions

$$a_2 = 0.5\,(1-x^2) \qquad b_1 = 5 * 10^{-5}\,(1-x^2), \quad (4.0.19)$$

constants

$$e = 0.01 \quad m = 1.0 \quad a_1 = 0.2404875914 \quad b_2 = 0.1202437957 \quad b_3 = 10^{-2} \quad L = 1, \quad (4.0.20)$$

and partial derivatives denoted as $\partial_{\tau\tau} = \partial^2/\partial\tau^2, \partial_x = \partial/\partial x, \partial_{xx} = \partial^2/\partial x^2$.

System (4.0.17) is subject to the following boundary conditions

$$w(x=0) = 0 \quad (\partial_x w)(x=0) = 0 \quad \theta(x=0) = 0 \quad (4.0.21)$$
$$(a_1\,\partial_{xx}w)(x=L) = 0 \quad \partial_x(a_1\partial_{xx}w)(x=L) = 0 \quad (b_2\,\partial_x\theta)(x=L) = 0. \quad (4.0.22)$$

- Consider the steady state version of (4.0.17) with no time dependence till noted otherwise. Give the function space for the test and basis functions w, u and θ, v, respectively, for the (steady-state) coupled rotor blade system (4.0.17).
- Give the weak formulation the (steady-state) coupled rotor blade system (4.0.17) including all considerations of the boundary conditions (4.0.21).
- Identify the local and global Hermite basis and test functions for w, u; and linear basis functions for θ, v, respectively, both in global and reference coordinates. Sketch them.
- Determine the matrix-vector system for the uncoupled rotary wing model, *i.e.* the equation for θ with no coupling terms to the displacement equation for w.
- Check the trial solution of the uncoupled rotary wing equation

$$\theta(x) = 0.1\,(2\,x - x^2) \qquad q(x) = 0.02405875914 + 0.00202\,x - 0.00103\,x^2. \quad (4.0.23)$$

- Implement the matrix-vector system and find the solution for several resolutions, *e.g.* 10, 20, 40, 80 elements. Plot the results including the exact solution (4.0.23) for $\theta(x)$.
- Determine the L_2–error (see book or internet) between exact and numerical solutions, put them in a table, and determine the order of accuracy.

[2]Tjahjanto, D.D. (2003) *A four-dimensional, time-accurate, aero-structure coupling method for simulation of helicopter rotors in forward flight.* M.Sc. Thesis, University of Twente.

- Integrate the rotary-wing equation for another acceptible torsion load $q(x)$ (checking convergence), display and interpret the solution.

v. Consider two-dimensional electromagnetic (E and M) waveguide problems with closed boundaries. Given an electric field $\mathbf{E}$ and magnetic field $\mathbf{M}$ we can under certain assumptions reduce the problem to a scalar problem with a potential ψ such that $\mathbf{E} = \boldsymbol{\nabla}\hat{\psi}$ and $\mathbf{M} = \boldsymbol{\nabla}\tilde{\psi}$, where $\boldsymbol{\nabla} = (\partial/\partial x, \partial/\partial y)^T$ and spatial coordinates $\mathbf{x} = (x, y)^T \in \Omega$. The resulting fields are governed by the Helmholtz equation with wavenumber k

$$\nabla^2\psi + k^2\,\psi = 0,$$

with $\psi = \hat{\psi}$ or $\psi = \tilde{\psi}$, and boundary conditions

$$\mathbf{n}\cdot\boldsymbol{\nabla}\hat{\psi} = 0 \quad \text{or} \quad \tilde{\psi} = 0 \quad \text{at } \partial\Omega$$

for the E and M case, respectively, with $\mathbf{n}$ the normal of the boundary $\partial\Omega$.

- Find the minimization problem for the Helmholtz problem for both cases. Clearly denote the function spaces you use for ψ and the basis functions.
- Formulate the weak formulation of the Helmholtz problem for both cases. (Hint: use Green's formula, the relation $\boldsymbol{\nabla}(w\,\boldsymbol{\nabla}\phi) = w\,\nabla^2\phi + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}w$, and Gauss' theorem.)

vi. We are given the two-dimensional linearized shallow water equations

$$\partial_t\mathbf{v} = -g\,\boldsymbol{\nabla}\eta$$
$$\partial_t\eta + \boldsymbol{\nabla}\cdot[H(\mathbf{x})\,\mathbf{v}] = 0 \quad \text{with} \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{at } \partial\Omega,$$

where the horizontal velocity $\mathbf{v}(\mathbf{x}, t) = (u, v)^T$ in the x, y-direction, respectively; spatial coordinates $\mathbf{x} = (x, y)^T \in \Omega$; time t; $\partial_t = \partial/\partial t$; the acceleration of gravity g; and the total water depth $h = H + \eta$ with the rest depth $H(\mathbf{x})$ and the free surface $\eta(\mathbf{x}, t)$. The flow is at rest when $\mathbf{v} = \eta = 0$. Consider these equations with a tidal forcing such that $\mathbf{v} = \mathbf{v}_0(\mathbf{x})e^{i\,\sigma\,t}$, $\eta = \eta_0(\mathbf{x})e^{i\,\sigma\,t}$, $i^2 = -1$, where σ is a typical tidal frequency.

- Find the weak formulation for this linearized shallow water problem.
- Explain how we can deal with discontinuous changes in the rest depth $H(\mathbf{x})$ in the weak formulation.

vii. We are interested[3] in solving the equation for the generalized vorticity, $\omega = \omega(x, y, t)$, with a generalized elliptical equation for the stream function $\psi = \psi(x, y, t)$ in a two-dimensional domain Ω. The elliptical equation is assumed to be linear in ψ. Spatial coordinates are denoted by x and y, and t is time.

This coupled system in a domain $\Omega \subset \mathbb{R}^2$ is defined as follows

$$\partial_t\omega + \nabla\cdot(\omega\vec{u}) = 0 \qquad (4.0.24)$$
$$\vec{u} = A\nabla^{\perp}\psi \qquad (4.0.25)$$
$$\nabla\cdot(A\nabla\psi) - B\psi + C = \omega \qquad (4.0.26)$$

[3] Bernsen, E. (2003) *Discontinuous finite-element approximations of two-dimensional vorticity equations with linear elliptic inversions and circulation.*. M.Sc. Thesis, University of Twente

with $A(x, y) > 0$ a continuous function, $B(x, y) \geq 0$, $C(x, y) \in \mathbb{R}$, $\partial_t = \partial/\partial t$, $\nabla = [\partial_x, \partial_y]^T$ and the two dimensional curl operator $\nabla^\perp = [-\partial_y, \partial_x]^T$. These equations describe the motion of a fluid in a 2D domain. The velocity of the fluid is represented by $\vec{u} = [u, v]^T$ and the generalized vorticity is given by ω. We use the term generalized vorticity to emphasize that ω is defined by (4.0.26) which is different from the usual definition of vorticity $\partial_x v - \partial_y u = \nabla \cdot (A \nabla \psi)$. This generalized vorticity-stream function formulation has the advantage to include at least three (geophysical) systems of interest in atmospheric and ocean dynamics:

- the 2D time-dependent incompressible Euler equations in vorticity stream-function formulation (Landau and Lifschitz, 1959),
- the quasi-geostrophic equations (Pedlosky, 1979),
- the rigid lid equations (Leblond and Mysak, 1978)

for suitable choices of specified functions A, B and C.

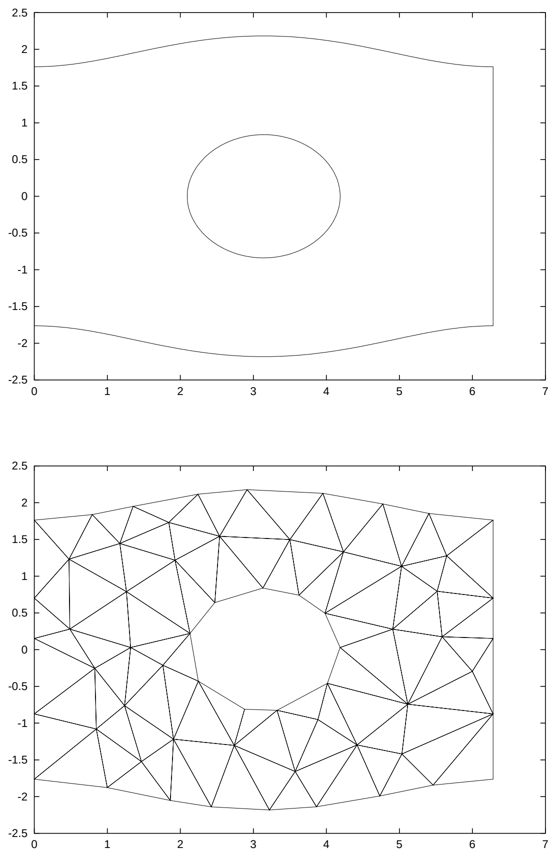

Figure 4.2: Example of domain (top) and a triangulation (bottom) for the steady state Stuart vortex with an island in the middle. The curved top, bottom, and island boundaries are solid walls, while the vertical left ($x = 0$) and right boundary ($x = 2\pi$) is periodic.

We consider closed domains which may be multiple connected, *i.e.*, contain islands, possibly combined with a periodic boundary in the x-direction, see Fig. 4.2. The boundary $\partial\Omega$ may therefore be split into solid walls $\partial\Omega_S$ and periodic parts $\partial\Omega_P$ such that $\partial\Omega = \partial\Omega_S + \partial\Omega_P$. At a solid wall the velocity $\vec{u}$ normal to the wall is zero

$$\vec{u} \cdot \hat{n} = A\,\boldsymbol{\nabla}\psi \cdot \hat{\tau} = 0. \tag{4.0.27}$$

with $\hat{n}$ the outward normal and $\hat{\tau}$ the tangential unit vector in the counterclockwise direction. Hence

$$\psi|_{\partial\Omega_{S_i}} = \mathcal{C}_i(t) \tag{4.0.28}$$

with the time-dependent constant $\mathcal{C}_i(t)$ chosen such that

$$\frac{d}{dt}\int_{\partial\Omega_{S_i}} \vec{u} \cdot \hat{\tau}\, \mathrm{d}\Gamma = 0 \quad \Longleftrightarrow \quad \int_{\partial\Omega_{S_i}} \vec{u} \cdot \hat{\tau}\, \mathrm{d}\Gamma = \mathcal{K}_i \tag{4.0.29}$$

with the circulation $\mathcal{K}_i$ a given constant (in space and time) on boundary piece $\partial\Omega_{S_i}$.

- Define the space of test and basis functions, $H^1(\Omega)$, for the elliptic equation (4.0.26) for given A, B, C and vorticity ω at a given time t.
- Give the weak formulation for the elliptic equation (4.0.26) for given A, B, C and vorticity ω at a given time t. Use a boundary with a curved top and bottom boundary, islands in the middle, and a periodic boundary in the x-direction. Clearly derive your result.
- Find the minimization problem for the elliptic equation (4.0.26).

viii. **Test problem: Laplace equation** Consider Laplace' equation

$$\nabla^2\phi = 0 \qquad \text{on a domain } D: \quad x \in [0, L_x], y \in [0, Ly] \qquad \text{with} \tag{4.0.30a}$$

$$\phi(0, y) = 0, \phi(L_x, y) = L_x \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = 0 \text{ and on } y = L_y, \tag{4.0.30b}$$

and general combinations of Neuman and Dirichlet boundary conditions.

(a) Check that the exact solution is $\phi = x$.

(b) Give the weak formulation: multiply Laplace's equation by a trial or test function, integrate over the domain, use Gauss' theorem and the boundary conditions.

(c) Introduce the finite element expansion of ϕ as follows: $\phi = \sum_{k=1}^{N} \phi_j \varphi_j(x, y)$ over the N nodes, where ϕ_j are the scalar expansion coefficients and $\varphi(x, y)_j$ the global basis functions with compact support. Choose the test function, which is arbitrary, to be φ_i for all $i = 1, \ldots, N$ in turn. Usually, $\varphi_j(x_l, y_l) = 0$ if $l \neq j$ and $\varphi_j(x_l, y_l) = 1$ if $l = j$. Substitute this expansion to obtain the discretized weak formulation.

(d) Express the discretized weak formulation as a linear system $\tilde{A}\,\vec{x} = \vec{b}$ with matrix $\tilde{A}$ and vectors $\vec{x}, \vec{b}$. Determine $\tilde{A}, \vec{x}$ and $\vec{b}$.

(e) Define a quadrilateral mesh on the domain. Make at least three meshes by using refinement. Define global node numbers and the coordinates of each node. Each element has a number, the associated four global node numbers, four local node numbers. The resulting mesh file consists usually of:

- an integer expressing the total number of nodes,
- followed by a list of x and y coordinates of a node,
- an integer expressing the total number of elements,
- followed by a list with the global node numbers associated with the four local node numbers,
- an integer expressing how many boundary nodes there are of a certain type,
- followed for each boundary type by a list of these global node numbers.

Alternatively, one may mark the boundary nodes with a Dirichlet condition by giving them a negative number or define another way of denoting the boundary nodes and their type.

(f) Define global test and basis functions on this quadrilateral mesh. Introduce a square reference element with a local coordinate system.

(g) Identify the four local test and basis functions.

(h) Describe the global matrix assembly using the local basis functions. This involves a loop over all elements. Clearly indicate how you deal with the Dirichlet and Neumann boundary conditions, and how you build $\tilde{A}$ and $\vec{b}$.

(i) Implement your numerical scheme and test it. Verify the order of accuracy by analyzing the error. *Optional: Implement your numerical scheme and test it using inbuild linear algebra routine, Choleski factorization, and conjugate gradients. Why are the Choleski and the preconditioned conjugate gradient method applicable? Ensure that the three solutions coincide on the different meshes.*

(j) Try another ϕ and construct the corresponding boundary conditions. Alter the numerical discretization and implementation accordingly. Verify the order of accuracy by analyzing the error.

ix. Consider the Poisson equation:

$$\nabla \cdot \Big(H(x,y)\,\nabla\psi(x,y)\Big) = f(x,y), \qquad \forall\, \mathbf{x} \in \Omega, \tag{4.0.31}$$

$$\psi(x,y) = h(x,y), \qquad \forall\, \mathbf{x} \in \partial\Omega_u \tag{4.0.32}$$

$$\hat{\mathbf{n}}_i \cdot \nabla\psi(x,y) = g_i(x,y), \qquad \forall\, \mathbf{x} \in \partial\Omega_i \qquad i = 1,2,3 \tag{4.0.33}$$

with $\hat{\mathbf{n}}_i$ the outward normal at boundary $\partial\Omega_i$. Determine the numerical solution for the following case:

$$U = 1, \quad k = 2, \quad \epsilon = 0.8, \quad \alpha = 0.1, \quad \gamma = 0.1 \tag{4.0.34}$$

$$H(x,y) = e^{\alpha\,(x+y)}, \tag{4.0.35}$$

$$\Omega = \Big\{ (x,y) \in R^2 | -\pi/k \le x < \pi/k \wedge \{\cosh(k\,y) - \epsilon\,\cos(k\,x) < \cosh k - \epsilon\} \wedge y > \gamma \Big\}, \tag{4.0.36}$$

$$f(x,y) = U\,e^{\alpha\,(x+y)} \left\{ \frac{k\,(1-\epsilon^2)}{[\cosh(k\,y) - \epsilon\,\cos(k\,x)]^2} + \frac{\alpha\,[\sinh(k\,y) + \epsilon\,\sin(k\,x)]}{[\cosh(k\,y) - \epsilon\,\cos(k\,x)]} \right\}, \tag{4.0.37}$$

$$\partial\Omega_u = \{ y > \gamma \wedge \cosh(k\,y) - \epsilon\,\cos(k\,x) = \cosh k - \epsilon \}$$

$$h(x,y) = (U/k)\,\ln(\cosh k - \epsilon), \tag{4.0.38}$$

$$\partial\Omega_1 = \{ x = -\pi/k \wedge \cosh(k\,y) - \epsilon\,\cos(k\,x) < \cosh k - \epsilon \}$$

$$g_1(x,y) = 0 \tag{4.0.39}$$

$$\partial\Omega_2 = \{ x = \pi/k \wedge \cosh(k\,y) - \epsilon\,\cos(k\,x) < \cosh k - \epsilon \}$$

$$g_2(x,y) = 0 \tag{4.0.40}$$

$$\partial\Omega_3 = \{ x \in [-\pi/k, \pi/k] \wedge y = \gamma \}$$

$$g_3(x,y) = -\frac{U\,\sinh(k\,y)}{\cosh(k\,y) - \epsilon\,\cos(k\,x)}. \tag{4.0.41}$$

- Find weak formulations of the Poisson equation. *Hint: first find the weak formulation without substituting the specific values of* f, h *and* $g_{1,2,3}$.
- Give the finite element discretization using the weak formulation for the Poisson equation. Use triangular or quadrilateral elements with linear basis functions, see Morton (1996). Clearly show the different steps you make to construct the finite element model. Implement the boundary conditions. Devise your own triangulation in a separate routine by dividing the domain in regular intervals in the x– and y–direction.
- Describe how you obtain the matrix elements for the linear system $A\Psi = F$ in the finite element discretization. What type of properties does the matrix A have? Make a routine which uses the Choleski factorization and use it to solve the linear system. Note, you do not have to optimize the matrix storage (see, *e.g.*, Press *et al.*, 1997).
- Plot the solution of the Poisson equation with contour levels at $\psi = -0.8, -0.6, \ldots, 0.4, (U/k)\,\ln(\cosh k - \epsilon)$.
- Make a routine which uses a preconditioned conjugate gradient method to solve the linear system iteratively. Use a diagonal preconditioner. What can you say about the performance of the iterative and direct solution technique?

x. Consider the elliptic equation (4.0.26), *i.e.*,

$$\nabla \cdot (A\nabla\psi) - B\psi + C = \omega \tag{4.0.42}$$

for given functions

$$A(x,y) = 1, \quad B(x,y) = C(x,y) = 0, \tag{4.0.43}$$

and vorticity

$$\omega(x,y) = 1/\left(a\cosh y + \sqrt{a^2-1}\cos x\right) \tag{4.0.44}$$

with $a = 3/2$ on the domain Ω in Fig. 4.2.

The domain Ω has periodic East and West bpoundaries at $x = 0$ $x = 2\pi$. The upper, lower and middle boundaries $\partial\Omega_{\mathrm{up}}, \partial\Omega_{\mathrm{down}}, \partial\Omega_{\mathrm{mid}}$ are defined by

$$a\cosh y + \sqrt{a^2-1}\cos x = a\,c + a + \sqrt{a^2-1} \tag{4.0.45}$$

with $c = 2$ for the upper and lower boundary, and $c = -\sqrt{5}/2$ for the inner boundary or "island". On the island a Dirichlet boundary is used with

$$\psi = -0.06083019946945424 \quad \text{on} \quad \partial\Omega_{\mathrm{mid}} \tag{4.0.46}$$

Onthe upper and lower boundary the following boundary conditions are applied

$$\int_{\partial\Omega_{\mathrm{up,down}}} \nabla\psi\cdot\hat{n}\,d\Gamma = 6.180637249213425, \quad \nabla\psi\cdot\hat{\tau} = 0. \quad \text{on} \quad \partial\Omega_{\mathrm{up,down}} \tag{4.0.47a}$$

Alternatively, in testing the numerical implementation use the Dirichlet boundary conditions

$$\psi = log(a\,x + c\,\sqrt{a^2-1}) \tag{4.0.48}$$

with $a = 3/2$ and $c = -\sqrt{5}/2$ on the island and $a = 3/2$ and $c = 2$ on the upper and lower boundary.

- Provide a detailed finite element discretization for a simple Poisson equation with a known nontrivial exact solution of your choice. Use quadrilaterals. Make your own grid. Use this simpler discretization to test and slowly build up your numerical routines before advancing to implement the discretization of (4.0.42).
- Formulate the weak formulation for the streamfunction (4.0.42) for a simplified domain, boundary conditions and the vorticity, all of your choice.
- Formulate the weak formulation of (4.0.42) for (i) Dirichlet and periodic boundary conditions; and (ii) the circulation, no-flow, and periodic boundary conditions, given above.
- Formulate the discretized weak formulation for (4.0.42) using local and global basis functions on a quadrilateral mesh for cases (i) and (ii).
- Make a routine which uses the Choleski factorization and use it to solve the linear system of problem (4.0.42) for both cases. Note, you do not have to optimize the matrix storage (see, *e.g.*, Press *et al.*, 1997).
- Make a routine which uses a preconditioned conjugate gradient method to solve the linear system iteratively. Use a diagonal preconditioner.
- Test your numerical algorithm first using the simplified case on a simplified grid.

- Determine solutions of (4.0.42) for the given boundary conditions for the two unstructured meshes provided. The mesh format is described as well in a third file. The fine mesh and the vorticity are shown Fig. 4.3.
- What can you say about the performance of the iterative and direct solution technique?
- What can you say about the order of convergence?

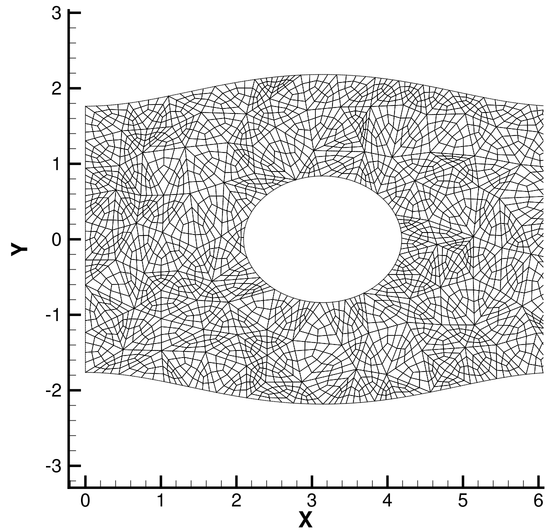

Figure 4.3: The fine mesh for the Stuart vortex as well as the vorticity $\omega(x, y)$.

xi. **Linear potential flow for free surface waves.**

In the three dimensional wave basin at the Maritime Research Institute Netherlands (MARIN), the impact of waves on floating and offshore structures is investigated experimentally. Such laboratory investigations are valuable but expensive. The use of a numerical wave tank is therefore of complimentary value.

We will consider linear waves in a simpler two dimensional wave basin in a vertical

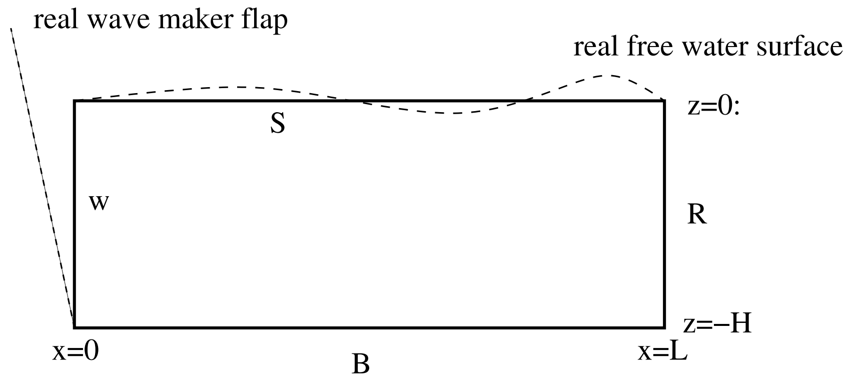

Figure 4.4: Sketch of the wave basin and its boundaries.

cross section for linear potential flow under gravity. The velocity $\vec{u} = (u, w)^T = \boldsymbol{\nabla}\phi$ is expressed in terms of velocity potential ϕ with gradient operator $\boldsymbol{\nabla} = (\partial/\partial x, \partial/\partial z)^T$ in the horizontal x- and vertical z-direction. The governing equations are

$$\nabla^2\phi = 0 \tag{4.0.49a}$$

$$\partial_t\phi + g\,\eta = 0 \quad \text{at} \quad \Gamma_S : z = 0 \tag{4.0.49b}$$

$$\partial_t\eta = \partial_z\phi \quad \text{at} \quad \Gamma_S : z = 0 \tag{4.0.49c}$$

$$\hat{\mathbf{n}}_b \cdot \boldsymbol{\nabla}\phi = 0 \quad \text{at} \quad \Gamma_B \text{ and } \Gamma_R \tag{4.0.49d}$$

$$\partial_x\phi = \partial_t X_w \quad \text{at} \quad \Gamma_w : x = 0 \tag{4.0.49e}$$

with free surface deviation η, gravitational acceleration g, $\hat{\mathbf{n}}_b$ the outward normal at the (bottom and right) solid boundaries Γ_B and Γ_R, abbreviation $\partial_t = \partial/\partial t$ and so forth. The linearized condition for continuity of pressure at the linearized free surface $\Gamma_s : z = 0$ is given by (4.0.49b). The linearized kinematic condition at Γ_S is given (4.0.49c). Slip flow at the solid boundary is given by condition (4.0.49d) with normal $\hat{\mathbf{n}}_b$ at the bottom and right boundary Γ_B and Γ_R, and the linearized wave maker condition at $\Gamma_w : x = 0$ is (4.0.49e). The wave maker is after linearization specified at its mean position $x = 0$.

(a) Give the weak formulation of the Laplace equation (4.0.49a), $\nabla^2\phi = 0$, subject to the boundary conditions (4.0.49b)–(4.0.49e). Either eliminate η or discretize (4.0.49b) separately at the free surface with another test function at the free boundary Γ_S. Do not discretize time yet.

(b) Give the discretized weak formulation by expanding the variables as a sum over global basis functions:

$$\phi(x, z, t) = \sum_{j=1}^{N} \varphi_j(x, z)\, \phi_j(t) \tag{4.0.50}$$

with global basis function $\varphi_j(x,z)$, and N the number of boundary and interior nodes. At the free surface Γ_S:

$$\phi_s(x,t) = \phi(x, z=0, t) = \sum_{\alpha=1}^{N_s} \varphi_\alpha(x, z=0)\, \phi_\alpha(t) \tag{4.0.51}$$

with α a node on the free surface, and $N_s < N$ the number of nodes at this free surface. Subsequently, describe how the integrals can be evaluated using local basis functions. Use linear basisfunctions on quadrilateral elements. Denote how the global matrices are assembled.

(c) Consider forced oscillations with $X_w = \tilde{X}_w \sin(\sigma t)$ and $\phi = \tilde{\phi} \cos(\sigma t)$ with forcing frequency σ. Write down the system in terms of $\tilde{X}_w$, $\tilde{\phi}$ and their spatial discretizations. Then drop the tildes.

(d) For free waves there is no wavemaker: $u = \partial_x \phi = 0$ at $\Gamma_w : x = 0$. Derive or check the following exact solution for free waves in a rectangular basin with $x \in [0, L]$ and $z \in [-H, 0]$

$$\phi = \sum_{n=0}^{\infty} \sum_{s=-1}^{1} \big(a_{ns} \cos(\omega_s t) + b_{ns} \sin(\omega_s t)\big)\, cos(\lambda_n x)\, \cosh\big(\lambda_n (z+H)\big) \tag{4.0.52a}$$

$$\lambda_n = \frac{\pi n}{L} \tag{4.0.52b}$$

with integer n and frequency

$$\omega_{s=\pm 1} = s \sqrt{g \lambda_n \tanh(\lambda_n H)}. \tag{4.0.53}$$

Note that the free surface boundary conditions can be combined to

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = 0. \tag{4.0.54}$$

Use for example one mode with one $n \neq 0$ and one s, but low n, and either a chosen $a_{ns} = 0, b_{ns} \neq 0$ or vice versa, as exact solution for comparison with the numerics. Of course, in principle any linear combination of waves with varying amplitudes a_{ns}, b_{ns} can be chosen.

(e) Make a quadrilateral grid on $x \in [0, L]$ and $z \in [-H, 0]$.

(f) Implement the discretization for the case with a wavemaker numerically for $X_w = \tilde{X}_w \sin(\sigma t) = A (z + H) \sin(\sigma t)$. Hence, choose $\tilde{\phi} = \tilde{\phi} \cos(\sigma t)$. The weak formulation then becomes something like

$$\int_D \nabla \tilde{\phi} \cdot \nabla v \, \mathrm{d}x\, \mathrm{d}y - \int_{\Gamma_S} \frac{\sigma^2}{g} v\, \tilde{\phi}\, \mathrm{d}\Gamma = \int_{\Gamma_w} v\, \sigma\, \tilde{X}_w \mathrm{d}\Gamma \tag{4.0.55}$$

with test function v. After the introduction of basis and test functions, this reduces again to an algebraic system of the form $\tilde{A}\vec{x} = \vec{b}$ for appropriately defined $\tilde{A}, \vec{x}, \vec{b}$. In order to evaluate the boundary integrals, it is convenient to number these in an ordered fashion. Also indicate in the mesh file the two types of boundary nodes.

(g) Solve the resulting algebraic system using a Matlab solver, a self-developed Choleski factorization, and a preconditioned conjugate gradient method with a diagonal preconditioner. Solve the system for several values of σ around a resonant frequency ω but such that they do not coincide. When the forcing frequency approaches an ω, the solution for ϕ and η must resemble the exact solution for free waves. Normalize the numerical solution by letting a point value coincide at a corner point. Take $g = 1$, which will be valid after an appropriate scaling of the equations, and $L = 1, H = 1$. Compare the solvers.

(h) *Bonus question* Discretize time for example as follows: $\mathrm{d}^2\phi_j/\mathrm{d}t^2 = (\phi_j^{n+1} - 2\,\phi_j^n + \phi_j^{n-1})/\Delta t^2$ and solve the resulting system in time using the above solvers. Initialize the scheme with an exact free wave solution and verify the numerical implementation. Indicate how to implement the initial condition for ϕ and η at the free boundary Γ_S.

xii. Consider the linearly damped advection equation

$$\partial_t u + u\,\partial_x u = -\kappa_d\, u \quad \text{on} \quad x \in \Omega = [0, L] \tag{4.0.56}$$

with $u = u(x,t)$ and damping $\kappa_d > 0$. Consider a tesselation $\mathcal{T}_h$ of the domain in which h is the minimum size of the element.

- Write (4.0.56) in a conservative form using an integrating factor to include the damping term $-\kappa_d\, u$ in a flux term. That is, take $u(x,t) = u_0(x,t)\, e^{-\kappa_d\, t}$.
- Introduce a local coordinate system in the master element. Introduce basis functions of order $d_P = 1$ such that $U_h = \bar{U}_k + \hat{U}_k \psi_1$ with $\int_{K_k} \psi_1\, \mathrm{d}x = 0$ in cell K_k of element k with $k = 1, \ldots, N$.
- Formulate the weak formulation of this conservative form of (4.0.56) using arbitrary smooth test functions $v(x)$. Write down the weak formulation of the discrete problem using discontinuous basis functions $v \in V_h^{d_P}$ with

$$V_h^{d_P} = \{v : v_K \in P^{d_P}, \forall K \in \mathcal{T}_h\}, \tag{4.0.57}$$

  where $P^{d_P}(K)$ is the set of polynomials in cell K.
- Write down the finite-element discretization.
- Determine the flux function using the Engquist-Osher flux or the flux derived from the solution of the Riemann problem for Burgers equation.
- Introduce an artifical damping term for stability.
- Implement inflow- and outflow boundary conditions by specifying the flux at the boundary where necessary and by using the Riemann invariant. That is, when $u > 0$ at the left boundary then impose a flux based on an imposed condition, and when $u < 0$ extrapolate the interior value to the left; and likewise at the other boundary.
- Use a third-order total variation diminishing Runge-Kutta time discretization.
- Determine the numerical solution for an initial condition

$$u(x,0) = \begin{cases} 1 & x < x_0 \\ 1/2 & x \geq x_0 \end{cases} \tag{4.0.58}$$

with $L = 1.5, \kappa_d = 0.1, x_0 = 0.25$ and $N_k = 10, 50$ uniformly distributed elements. Show the numerical solution for all $x \in [0, 1.5]$ at $t = 0, 0.1, \cdots, 1$. Determine the influence of the artificial damping term by determining the minimum value of the damping coefficient for each flux scheme which gives stable results.

- Compare the numerical solution with the exact solution.

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